Nihonkai Math. J.
Vol.12(2001), 135-143

STRUCTURE OF GROUP C^*-ALGEBRAS OF SEMI-DIRECT PRODUCTS OF $\mathbb{C}^n$ BY $\mathbb{Z}$

TAKAHIRO SUDO

ABSTRACT. We consider the structure of group C^*-algebras of semi-direct products of $\mathbb{C}^n$ by $\mathbb{Z}$. As an application we estimate the stable rank and connected stable rank of these C^*-algebras, and treat the case of semi-direct products of $\mathbb{R}^n$ by $\mathbb{Z}$ similarly.

§0. INTRODUCTION

Group C^*-algebras have played important roles in the progress of the theory of C^*-algebras. In particular, their structure for Lie groups has been investigated (cf.[Dx], [Rs], [Gr1,2], [Pg], [Wg], etc). On the other hand, the stable rank for C^*-algebras was introduced by M.A. Rieffel [Rf1] as a noncommutative analogue of the covering dimension for topological spaces, and he raised an interesting problem such as describing the stable rank of group C^*-algebras of Lie groups in terms of groups. On this problem some partial answers were obtained by [Sh],[ST1,2] and [Sd1-4]. In particular, in [Sd4] the author investigated the structure of group C^*-algebras of Lie semi-direct products of $\mathbb{C}^n$ by $\mathbb{R}$, and estimated their stable rank and connected stable rank.

In this paper we obtain finite composition series of group C^*-algebras of the semi-direct products of $\mathbb{C}^n$ by $\mathbb{Z}$, by analyzing their subquotients explicitly using some methods of [Sd4] similarly. Using this result we give the rank estimations of these group C^*-algebras, and especially that of semi-direct products of $\mathbb{R}^n$ by $\mathbb{Z}$. These are disconnected solvable (Lie) groups, and contain the discrete Mautner group studied by L. Baggett [Bg] to construct some unitary representations of the Mautner group through Mackey machine. We emphasize that this paper will be the first step to explore the algebraic structure of C^*-algebras of general disconnected solvable Lie groups.

We now prepare some notations. Let $C^*(G)$ be the (full) group C^*-algebra of a locally compact group G (cf.[Dx, Part II],[Pd, Chapter 7]). We denote by $\hat{G}_1$ the space of all 1-dimensional representations of G. Let $C_0(X)$ be the C^*-algebra of all complex valued continuous functions on a locally compact Hausdorff space X vanishing at infinity. When X is compact, we set $C_0(X) = C(X)$. Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. For a C^*-algebra $\mathfrak{A}$, we denote by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ its stable rank, connected stable rank respectively ([Rf1]).

2000 Mathematics Subject Classification: Primary 46L05, 46L80, 22D25, 19K56.
Key words and phrases: Group C^*-algebras, Semi-direct products, Lie groups, Stable rank
Research partially supported by Japan Society for the Promotion of Science

By definition, $\mathrm{sr}(\mathfrak{A}), \mathrm{csr}(\mathfrak{A}) \in \{1, 2, \cdots, \infty\}$. We review some formulas of these stable ranks used later as follows:

(F1): For an exact sequence of C^*-algebras: $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$, we have that

$$\mathrm{sr}(\mathfrak{J}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{J}) \leq \mathrm{sr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{J}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{J}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{J}), \quad \mathrm{csr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{J}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{J}),$$

where $\vee$ is the maximum (See [Rf1, Theorem 4.3, 4.4 and 4.11], [Sh, Theorem 3.9]).

(F2): By [Rf1, Proposition 1.7] and [Ns1], for X a locally compact Hausdorff space,

$$\mathrm{sr}(C_0(X)) = [\dim X^+/2] + 1, \quad \mathrm{csr}(C_0(X)) \leq [(\dim X^+ + 1)/2] + 1$$

where X^+ means the one-point compactification of X, $\dim X^+$ is the covering dimension of X^+, and $[x]$ means the maximum integer $\leq x$. We set $\dim_{\mathbb{C}} X = [\dim X/2] + 1$.

(F3): For the $n \times n$ matrix algebra $M_n(\mathfrak{A})$ over a C^*-algebra $\mathfrak{A}$, by [Rf1, Theorem 6.1] and [Rf2, Theorem 4.7],

$$\mathrm{sr}(M_n(\mathfrak{A})) = \{(\mathrm{sr}(\mathfrak{A}) - 1)/n\} + 1, \quad \mathrm{csr}(M_n(\mathfrak{A})) \leq \{(\mathrm{csr}(\mathfrak{A}) - 1)/n\} + 1$$

where $\{x\}$ means the least integer $\geq x$.

(F4): For a C^*-algebra $\mathfrak{A}$,

$$\mathrm{sr}(\mathfrak{A} \otimes \mathbb{K}) = \mathrm{sr}(\mathfrak{A}) \wedge 2, \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbb{K}) \leq \mathrm{csr}(\mathfrak{A}) \wedge 2$$

where $\wedge$ is the minimum. See [Rf1, Theorem 3.6 and 6.4], ([Sh, Theorem 3.10], [Ns1]).

§1. Group C^*-algebras of semi-direct products of $\mathbb{C}^n$ by $\mathbb{Z}$

Let $G = \mathbb{C}^n \rtimes_\alpha \mathbb{Z}$ be a semi-direct product with α an automorphic action of $\mathbb{Z}$ on $\mathbb{C}^n$, in other words, $\alpha_t \in GL_n(\mathbb{C})$ for $t \in \mathbb{Z}$. By definition of C^*-crossed products (cf.[Pd, Chapter 7]) and using the Fourier transform, we have the isomorphisms:

$$C^*(G) \cong C^*(\mathbb{C}^n) \rtimes_\alpha \mathbb{Z} \cong C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}$$

where $\hat{\alpha}$ is defined by the equation of the inner product: $\langle \alpha_t(z) | w \rangle = \langle z | \hat{\alpha}_t(w) \rangle$ for $z, w \in \mathbb{C}^n$, $t \in \mathbb{Z}$. Since the origin 0_n of $\mathbb{C}^n$ is $\hat{\alpha}$-invariant, we have the following exact sequence:

$$0 \to C_0(\mathbb{C}^n \setminus \{0_n\}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C_0(\mathbb{C}^n) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C^*(\mathbb{Z}) \to 0$$

because $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by the Fourier transform.

For the sake of convenience, we consider the following example:

Example 1.1. If $G = \mathbb{C} \rtimes_\alpha \mathbb{Z}$, then for some $w \in \mathbb{C} \setminus \{0\}$, $\hat{\alpha}_t(z) = w^t z$ for $z \in \mathbb{C}, t \in \mathbb{Z}$. If $w = 1$, then $C^*(G) \cong C_0(\mathbb{C} \times \mathbb{T})$. When $w \notin \mathbb{T}$, by Green's result [Gr1, Corollary 15],

$$C_0(\mathbb{C} \setminus \{0\}) \rtimes \mathbb{Z} \cong C((\mathbb{C} \setminus \{0\})/\mathbb{Z}) \otimes \mathbb{K} \cong C(\mathbb{T}^2) \otimes \mathbb{K}.$$

If $w = e^{2\pi i\theta} \in \mathbb{T}\setminus\{1\}$, then $C_0(\mathbb{C}\setminus\{0\}) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong C_0(\mathbb{R}) \otimes (C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z})$, where $C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}$ is the rotation algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ by the angle $2\pi\theta$ (cf.[AP], [EE]).

We now investigate general cases in the following. Taking a suitable basis of $\mathbb{C}^n$ for the Jordan decomposition of α_1, and assuming it as a canonical basis of $\mathbb{C}^n$, we may assume that α_1 is equal to the diagonal sum as follows: for $\beta_j \in \mathbb{C}$ $(1 \le j \le l)$,

$$\alpha_1 = \left(\oplus_{j=1}^m \begin{pmatrix} \beta_j & & 0 \\ & \ddots & \\ 0 & & \beta_j \end{pmatrix}\right) \oplus \left(\oplus_{k=m+1}^l \begin{pmatrix} \beta_k & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \beta_k \end{pmatrix}\right)$$

on the direct sum decomposition $\mathbb{C}^n = (\oplus_{j=1}^m \mathbb{C}^{n_j}) \oplus (\oplus_{k=m+1}^l \mathbb{C}^{n_k})$. Then for $t \in \mathbb{Z}$, we have that

$$\hat{\alpha}_t = \left(\oplus_{j=1}^m \begin{pmatrix} \bar{\beta}_j^t & & 0 \\ & \ddots & \\ 0 & & \bar{\beta}_j^t \end{pmatrix}\right) \oplus \left(\oplus_{k=m+1}^l \begin{pmatrix} \bar{\beta}_k^t & t\bar{\beta}_k^{t-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & t\bar{\beta}_k^{t-1} \\ 0 & & & \bar{\beta}_k^t \end{pmatrix}\right)$$

Note that there exists a quotient map from $C^*(G)$ to $C_0(\mathbb{C}^g \times \mathbb{T})$ for some $0 \le g \le n$, where $\mathbb{C}^g \times \mathbb{T}$ is homeomorphic to $\hat{G}_1$, and $\mathbb{C}^g$ is homeomorphic to the subspace of $\mathbb{C}^n$ fixed under $\hat{\alpha}$. If some β_j or β_k are 1, then $g \ge 1$. By (F1) and (F2), we obtain that

$$\begin{cases} \mathrm{sr}(C^*(G)) \ge \mathrm{sr}(C_0(\hat{G}_1) = \dim_{\mathbb{C}} \hat{G}_1, \\ \mathrm{csr}(C_0(\hat{G}_1)) \le [(\dim \hat{G}_1 + 1)/2] + 1 = \dim_{\mathbb{C}} \hat{G}_1 + 1. \end{cases}$$

We consider the restrictions of $\hat{\alpha}$ to the $\hat{\alpha}$-invariant subspaces

$$\mathbb{C}^{g_0} \oplus \left(\oplus_{j=1}^{m'} (\mathbb{C} \setminus \{0\})^{n'_j}\right) \oplus \left(\oplus_{k=m+1}^{l'} (\mathbb{C}^{n'_k} \setminus \{0_{n'_k}\})\right)$$

for $0 \le m' \le m$, $0 \le n'_j \le n_j$, $m+1 \le l' \le l$ and $0 \le n'_k \le n_k$, where $\mathbb{C}^{g_0}$ means the direct sum of $\mathbb{C}^{n_j}$ for $1 \le j \le m$ such that $\beta_j = 1$. Moreover, we need to consider the following decomposition: for $m+1 \le k \le l'$,

$$\mathbb{C}^{n'_k} \setminus \{0_{n'_k}\} = ((\mathbb{C} \setminus \{0\}) \times \{0_{n'_k - 1}\}) \cup (\mathbb{C} \times (\mathbb{C}^{n'_k - 1} \setminus \{0_{n'_k - 1}\}))$$

In addition, we decompose $\mathbb{C}^{n'_k - 1} \setminus \{0_{n'_k - 1}\}$ into the disjoint union of the $\hat{\alpha}$-invariant subspaces $\mathbb{C}^{j'_k - 1} \times (\mathbb{C} \setminus \{0\}) \times \{0_{n'_k - 1 - j'_k}\}$ $(1 \le j'_k \le n'_k)$. We let

$$X_s = \mathbb{C}^{g_0} \oplus \left(\oplus_{j=1}^{m'} (\mathbb{C} \setminus \{0\})^{n'_j}\right) \oplus \left(\oplus_{k=m+1}^{l'} Y_k\right)$$

an $\hat{\alpha}$ invariant subspace obtained as above, where

$$Y_k = \begin{cases} (\mathbb{C}\setminus\{0\}) \times \{0_{n'_k-1}\} & \text{or} \\ \mathbb{C}^{j'_k-1} \times (\mathbb{C}\setminus\{0\}) \times \{0_{n'_k-1-j'_k}\}. \end{cases}$$

If $\beta_k = 1$ for some $m + l \leq k \leq l$, the subspace $(\mathbb{C}\setminus\{0\}) \times \{0_{n_k-1}\}$ is fixed under $\hat{\alpha}$. Thus in this case we assume that $Y_k = \mathbb{C}^{j'_k-1} \times (\mathbb{C}\setminus\{0\}) \times \{0_{n'_k-1-j'_k}\}$ for some j'_k in what follows.

We now note that

$$\begin{pmatrix} \bar{\beta}_j^t & & 0 \\ & \ddots & \\ 0 & & \bar{\beta}_j^t \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_{n'_j} \end{pmatrix} = \begin{pmatrix} \bar{\beta}_j^t z_1 \\ \vdots \\ \bar{\beta}_j^t z_{n'_j} \end{pmatrix}$$

for $(z_1, \cdots, z_{n'_j}) \in (\mathbb{C}\setminus\{0\})^{n'_j}$, and

$$\begin{pmatrix} \bar{\beta}_k^t & t\bar{\beta}_k^{t-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & t\bar{\beta}_k^{t-1} \\ 0 & & & \bar{\beta}_k^t \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_{j'_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ \vdots \\ \bar{\beta}_k^t w_{j'_k-1} + t\bar{\beta}_k^{k-1} w_{j'_k} \\ \bar{\beta}_k^t w_{j'_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $(w_1, \cdots, w_{j'_k}, 0, \cdots, 0) \in \mathbb{C}^{j'_k-1} \times (\mathbb{C}\setminus\{0\}) \times \{0_{n'_k-1-j'_k}\}$. By direct calculation, the action $\hat{\alpha}$ on X_s is one of the following three cases (cf.[Sd4]):

$$\begin{cases} \text{Free and wandering case} \\ \text{Free and nonwandering case} \\ \text{Nonfree case} \end{cases}$$

where the first case is that β_j or $\beta_k \notin \mathbb{T}$ for some j, k, or $j' \geq 2$ for some j', the second one is that all $\beta_j, \beta_k \in \mathbb{T}$ and one of them is an irrational number in $\mathbb{R}$ (mod 2π) identified with $\mathbb{T}$, and the third one is that all β_j, β_k are rational numbers. We consider the crossed product $C_0(X_s) \rtimes \mathbb{Z}$ in each case.

If the action of $\mathbb{Z}$ on X_s is free and wandering, we have by [Gr1, Corollary 15] that

$$C_0(X_s) \rtimes \mathbb{Z} \cong C_0(X_s/\mathbb{Z}) \otimes \mathbb{K}.$$

We note that X_s contains an $\hat{\alpha}$-invariant closed subspace which is a copy of $\mathbb{C}\setminus\{0\}$, and its orbit space by $\hat{\alpha}$ is homeomorphic to $\mathbb{T}^2$. Hence we have $\mathrm{sr}(C_0(X_s/\mathbb{Z})) \geq 2$. Therefore, $\mathrm{sr}(C_0(X_s/\mathbb{Z}) \otimes \mathbb{K}) = 2$.

We next consider the free and nonwandering case. Then

$$X_s = \mathbb{C}^{g_0} \oplus \left(\oplus_{j=1}^{m'}(\mathbb{C}\setminus\{0\})^{n'_j}\right) \oplus \left(\oplus_{k=m'+1}^{l'}((\mathbb{C}\setminus\{0\}) \times \{0_{n'_k-1}\})\right)$$

where the restriction of $\hat{\alpha}$ to each direct factor $\mathbb{C}\setminus\{0\}$ of X_s is a rotation, and one of these restrictions is an irrational rotation. Thus we have that for some $u_s \geq 1$,

$$C_0(X_s) \rtimes \mathbb{Z} \cong C_0(\mathbb{C}^{g_0} \times \mathbb{R}^{u_s}) \otimes (C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z}).$$

Moreover, by [EL2] (cf.[EE]), $C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z}$ is an inductive limit of finite direct sums of matrix algebras over $C(\mathbb{T})$ with their matrix sizes going to infinity. Therefore, by (F3) and [Rfl, Theorem 5.1], we obtain that $\mathrm{sr}(C_0(X_s) \rtimes \mathbb{Z}) \leq 2$ and $\mathrm{csr}(C_0(X_s) \rtimes \mathbb{Z}) \leq 2$.

If $u_s \geq 2$, then we have a quotient as follows:

$$C_0(X_s) \rtimes \mathbb{Z} \to C([0,1]^2) \otimes (C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z}) \to 0.$$

By [NOP, Proposition 5.3], we obtain that $\mathrm{sr}(C_0(X_s) \rtimes \mathbb{Z}) \geq \mathrm{sr}(C([0,1]^2) \otimes (C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z})) \geq 2$.

If $u_s = 1$, we suppose that $\mathrm{sr}(C_0(X_s) \rtimes \mathbb{Z}) = 1$. Then $\mathrm{sr}(C([0,1]) \otimes (C(\mathbb{T}) \rtimes \mathbb{Z})) = 1$. Then the K_1-group of $C([0,1]) \otimes (C(\mathbb{T}) \rtimes \mathbb{Z})$ must be trivial by [NOP, Proposition 5.2]. However, this is impossible since the K-groups of $C(\mathbb{T}) \rtimes \mathbb{Z}$ are $\mathbb{Z}^2$ so that the K_1-group of $C([0,1]) \otimes (C(\mathbb{T}) \rtimes \mathbb{Z})$ is also $\mathbb{Z}^2$ by Künneth formula (cf.[Wo, 9.3.3]). Therefore, $\mathrm{sr}(C_0(X_s) \rtimes \mathbb{Z}) \geq 2$.

Finally, we consider the nonfree case. Then

$$X_s = \mathbb{C}^{g_0} \oplus \left(\oplus_{j=1}^{m'}(\mathbb{C}\setminus\{0\})^{n'_j}\right) \oplus \left(\oplus_{k=m'+1}^{l'}((\mathbb{C}\setminus\{0\}) \times \{0_{n'_k-1}\})\right)$$

where the restriction of $\hat{\alpha}$ to each direct factor $\mathbb{C}\setminus\{0\}$ of X_s is a rational rotation. Then

$$C_0(X_s) \rtimes \mathbb{Z} \cong C_0(\mathbb{R}^{2g_0+u_s}) \otimes (C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z})$$

for some $u_s \geq 1$. Moreover, we have that for a $p \geq 2$,

$$0 \to C_0(\mathbb{R}) \otimes (C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z}_p) \to C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z} \to C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z}_p \to 0$$

with $C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z}_p$ a homogeneous C^*-algebra (cf.[EL1], [Dv, VIII.9] for some cases with $C(\mathbb{T}^{u_s}) \rtimes \mathbb{Z}_p \cong M_p(C(\mathbb{T}^{u_s}))$). By (F1), (F2) and (F3),

$$\begin{aligned}
2 \leq \mathrm{sr}(M_p(C_0(\mathbb{R}^{2g_0+u_s+1} \times \mathbb{T}^{u_s}))) &= \{[(2(g_0+u_s)+1)/2]/p\} + 1 \leq \\
\mathrm{sr}(C_0(X_s) \rtimes \mathbb{Z}) \leq \mathrm{sr}(M_p(C_0(\mathbb{R}^{2g_0+u_s+1} &\times \mathbb{T}^{u_s}))) \vee \mathrm{csr}(M_p(C_0(\mathbb{R}^{2g_0+u_s} \times \mathbb{T}^{u_s}))) \\
&\leq \{[(2(g_0+u_s)+1)/2]/p\} + 1 = \{(g_0+u_s)/p\} + 1, \\
\mathrm{csr}(C_0(X_s) \rtimes \mathbb{Z}) \leq \mathrm{csr}(M_p(C_0(\mathbb{R}^{2g_0+u_s+1} &\times \mathbb{T}^{u_s}))) \vee \mathrm{csr}(M_p(C_0(\mathbb{R}^{2g_0+u_s} \times \mathbb{T}^{u_s}))) \\
&\leq \{[(2(g_0+u_s)+2)/2]/p\} + 1 = \{(g_0+u_s+1)/p\} + 1.
\end{aligned}$$

Summing up the above argument we obtain that

Theorem 1.2. *Let $G = \mathbb{C}^n \rtimes_\alpha \mathbb{Z}$ be a semi-direct product of $\mathbb{C}^n$ by $\mathbb{Z}$. Then there exists a finite composition series $\{\mathfrak{I}_s\}_{s=1}^r$ of $C^*(G)$ such that*

$$\mathfrak{I}_s/\mathfrak{I}_{s-1} \cong \begin{cases} C_0(\mathbb{C}^g \times \mathbb{T}) = C_0(\hat{G}_1),\ g \geq 0 & s = r, \\ \begin{cases} C_0(X_s/\mathbb{Z}) \otimes \mathbb{K} \quad or \\ C_0(\mathbb{R}^{2g_0+u_s}) \otimes (C(\mathbb{T}^{u_s}) \rtimes_{\Theta_s} \mathbb{Z}) \end{cases} & 1 \leq s < r \end{cases}$$

where $u_{s-1} \geq u_s$, $\dim X_{s-1} \geq \dim X_s$ and the action Θ_s of $\mathbb{Z}$ is a multi-rotation.

Moreover, applying (F1) to the above composition series inductively we obtain that

Theorem 1.3. *In the situation of Theorem 1.2, we have that*

$$2 \vee \dim_{\mathbb{C}} \hat{G}_1 \vee \max(\{(g_0 + u_s)/p_s\} + 1) \leq$$
$$\mathrm{sr}(C^*(G)) \leq (1 + \dim_{\mathbb{C}} \hat{G}_1) \vee \max(\{(g_0 + u_s + 1)/p_s\} + 1),$$
$$\mathrm{csr}(C^*(G)) \leq (1 + \dim_{\mathbb{C}} \hat{G}_1) \vee \max(\{(g_0 + u_s + 1)/p_s\} + 1)$$

where p_s means the period of Θ_s when it is a rational rotation.

Remark. By [Eh, Theorem 2.2], we have that $\mathrm{csr}(C^*(G)) \geq 2$. Hence if all the periods p_s of the rational rotations Θ_s are large enough, we can obtain that

$$\begin{cases} \mathrm{sr}(C^*(G)) = 2 \vee \dim_{\mathbb{C}} \hat{G}_1, & \text{if } \dim \hat{G}_1 \text{ even}, \\ 2 \vee \dim_{\mathbb{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq 1 + \dim_{\mathbb{C}} \hat{G}_1, & \text{if } \dim \hat{G}_1 \text{ odd}, \end{cases}$$

$$\begin{cases} \mathrm{csr}(C^*(G)) = 2, & \text{if } \dim_{\mathbb{C}} \hat{G}_1 = 1 \text{ or } 2, \\ 2 \leq \mathrm{csr}(C^*(G)) \leq (1 + \dim_{\mathbb{C}} \hat{G}_1), & \text{otherwise}. \end{cases}$$

Compare Theorem 1.2 and 1.3 with [Sd2], [Sd4] and [ST2].

In particular, we have the following:

Corollary 1.4. *Let $G = \mathbb{C}^n \rtimes_\alpha \mathbb{Z}$ be a semi-direct product of $\mathbb{C}^n$ by $\mathbb{Z}$. We suppose that $C^*(G)$ has no finite dimensional irreducible representations except 1-dimensional ones, that is, any restriction of α to the α-invariant subspaces as above is not a rational rotation. Then we have the rank formulas as in the above remark.*

Remark. By Lie's theorem (cf.[OV, Theorem 5 in §4]), any connected solvable (real or complex) Lie group has either one or infinite dimensional irreducible representations.

Example 1.5. The discrete Mautner group M is defined by $\mathbb{C} \rtimes_\alpha \mathbb{Z}$ with $\alpha_t(z) = e^{it}z$ for $z \in \mathbb{C}$, $t \in \mathbb{Z}$. Note $e^{2\pi i t} = 1$ for $t \in \mathbb{Z}$. Then $C^*(M)$ has the following structure from Example 1.1:

$$0 \to C_0(\mathbb{R}) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z}) \to C^*(M) \to C(\mathbb{T}) \to 0$$

where $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is as in Example 1.1 with $\theta = 1/2\pi$. Then we have $\mathrm{sr}(C^*(M)) = 2 > 1 = \dim_{\mathbb{C}} \hat{M}_1$, and $\mathrm{csr}(C^*(M)) = 2$.

Next let $G = \mathbb{C}^2 \rtimes_\alpha \mathbb{Z}$ with $\alpha_t(z_1, z_2) = (e^{i\pi t}z_1, e^{i\pi t}z_2)$. Then by the same calculation as before Theorem 1.2, we have $\mathrm{sr}(C^*(G)) = 3$, $\mathrm{csr}(C^*(G)) \leq 4$ and $\dim_{\mathbb{C}} \hat{G}_1 = 1$.

If $G = \mathbb{C}^3 \rtimes_\alpha \mathbb{Z}$ with $\alpha_t(z_1, z_2, z_3) = (e^t z_1, e^{i\pi t}z_2, e^{i\pi t}z_3)$, then we have $\mathrm{sr}(C^*(G)) = 3$ or 4, $\mathrm{csr}(C^*(G)) \leq 4$ and $\dim_{\mathbb{C}} \hat{G}_1 = 1$.

Example 1.6. Let $G^\lambda = \mathbb{C}^2 \rtimes_{\alpha^\lambda} \mathbb{Z}$ with $\alpha_t^\lambda(z_1, z_2) = (e^{it}z_1, e^{i\lambda t}z_2)$ for $t \in \mathbb{Z}$, $z_1, z_2 \in \mathbb{C}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Then $C^*(G^\lambda)$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^4$ such that

$$\begin{cases} \mathfrak{I}_4/\mathfrak{I}_3 = C^*(G^\lambda)/\mathfrak{I}_3 \cong C(\mathbb{T}), & \mathfrak{I}_3/\mathfrak{I}_2 \cong C_0(\mathbb{R}) \otimes (C(\mathbb{T}) \rtimes_\theta \mathbb{Z}), \\ \mathfrak{I}_2/\mathfrak{I}_1 \cong C_0(\mathbb{R}) \otimes (C(\mathbb{T}) \rtimes_{\lambda\theta} \mathbb{Z}), & \mathfrak{I}_1 \cong C_0(\mathbb{R}^2) \otimes (C(\mathbb{T}^2) \rtimes_\Theta \mathbb{Z}) \end{cases}$$

where $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ and $C(\mathbb{T}) \rtimes_{\lambda\theta} \mathbb{Z}$ are defined as in Example 1.1 with $\theta = 1/2\pi$, and Θ means the multi-rotation by the multi-angle $(\theta, \lambda\theta)$. Then we have that

$$\operatorname{sr}(C^*(G^\lambda)) = 2 = \operatorname{csr}(C^*(G^\lambda)) > 1 = \dim_{\mathbb{C}} \hat{G}_1^\lambda.$$

Remark. From Theorem 1.2 and [Sd4] we see that the tensor products $C^*(G) \otimes \mathbb{K}$, $C^*(G') \otimes \mathbb{K}$ for $G = \mathbb{C}^n \rtimes_\alpha \mathbb{Z}$, $G' = \mathbb{C}^{n'} \rtimes_{\alpha'} \mathbb{R}$ have the almost same structure. But it is not true that $C^*(G)$ is stably isomorphic to $C^*(G')$, since $\hat{G}_1$ has $\mathbb{T}$ as a direct product subspace while $\hat{G}'_1$ is homeomorphic to $\mathbb{R}^k$ for some $k \geq 1$. However, some subquotients of these group C^*-algebras are stably isomorphic.

§2. The case of semi-direct products of $\mathbb{R}^n$ by $\mathbb{Z}$

In this last section, we apply Theorem 1.2 to the cases of semi-direct products $H = \mathbb{R}^n \rtimes_\beta \mathbb{Z}$. By the same way as in [Sd4], we put $G = \mathbb{C}^n \rtimes_\alpha \mathbb{Z}$ with $\alpha_t(x + iy) = \beta_t(x) + i\beta_t(y)$ for $x, y \in \mathbb{R}^n$ $t \in \mathbb{Z}$. Then $C^*(H)$ is a quotient C^*-algebra of $C^*(G)$. Keeping the notation of Theorem 1.2, we have the following:

Theorem 2.1. *Let $H = \mathbb{R}^n \rtimes_\beta \mathbb{Z}$ be a semi-direct product of $\mathbb{R}^n$ by $\mathbb{Z}$. Then there exists a finite composition series $\{\mathfrak{L}_s\}_{s=1}^r$ of $C^*(H)$ such that*

$$\mathfrak{L}_s/\mathfrak{L}_{s-1} \cong \begin{cases} C_0(\hat{H}_1) = C_0(\mathbb{R}^h \times \mathbb{T}),\ h \geq 0 & s = r, \\ \begin{cases} C_0(Y_s) \otimes \mathbb{K} & \text{or} \\ C_0(V_s) \otimes \mathfrak{B}_s \end{cases} & 1 \leq s < r \end{cases}$$

where Y_s is a closed subset of $X_s/\mathbb{Z}$, and V_s is a closed subset of $\mathbb{R}^{2g_0+u_s}$ and $\mathfrak{B}_s$ is equal to $(C(\mathbb{T}^{u_s}) \rtimes_{\Theta_s} \mathbb{Z})$ or its quotient C^*-algebra.

Remark. The above remark is true in the case of $\mathbb{R}^n \rtimes_\beta \mathbb{Z}$ and $\mathbb{R}^{n'} \rtimes_{\beta'} \mathbb{R}$.

Moreover, we obtain that

Theorem 2.2. *In the situation of Theorem 2.1, we have that*

$$\begin{aligned} \dim_{\mathbb{C}} \hat{H}_1 \vee \max(\{[(v_s + u_s + 1)/2]/p_s\} + 1) &\leq \\ \operatorname{sr}(C^*(H)) \leq (1 + \dim_{\mathbb{C}} \hat{H}_1) &\vee \max(\{[(v_s + u_s + 2)/2]/p_s\} + 1), \\ 2 \leq \operatorname{csr}(C^*(H)) \leq (1 + \dim_{\mathbb{C}} \hat{H}_1) &\vee \max(\{[(v_s + u_s + 2)/2]/p_s\} + 1) \end{aligned}$$

where $v_s = \dim V_s$, and p_s means the period of Θ_s when Θ_s is a rational rotation.

Example 2.3. Let $K = \mathbb{R} \rtimes_\beta \mathbb{Z}$ with $\beta_t(x) = e^t x$ for $x \in \mathbb{R}$, $t \in \mathbb{Z}$, which is regarded as a closed normal subgroup of the proper $ax+b$ group. Then we have that

$$0 \longrightarrow \oplus^2(C(\mathbb{T}) \otimes \mathbb{K}) \xrightarrow{\ i\ } C^*(K) \xrightarrow{\ q\ } C(\mathbb{T}) \longrightarrow 0.$$

Then we obtain that $\mathrm{sr}(C^*(K)) = 1$ or 2, and $\mathrm{csr}(C^*(K)) = 2 > 1 = \dim_{\mathbb{C}} \hat{K}_1$. On the other hand, since $C^*(K) \cong C_0(\mathbb{R}) \rtimes_{\hat{\beta}} \mathbb{Z}$, we have $\mathrm{sr}(C_0(\mathbb{R}) \rtimes_{\hat{\beta}} \mathbb{Z}) \le \mathrm{sr}(C_0(\mathbb{R})) + 1 = 2$ by [Rf1, Theorem 7.1]. Moreover, we have the 6-term exact sequence (cf.[Wo]) of K-groups for the above sequence:

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "i_*"] & K_0(C^*(K)) \arrow[r, "q_*"] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[u, "\partial"] & K_1(C^*(K)) \arrow[l, "q_*"'] & \mathbb{Z}^2 \arrow[l, "i_*"']
\end{tikzcd}$$

On the other hand, the Pimsner-Voiculescu sequence (cf.[Bl]) for $C^*(K)$ is given by

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & K_0(C^*(K)) \arrow[d] \\
K_1(C^*(K)) \arrow[u] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

since $K_0(C_0(\mathbb{R})) \cong 0$ and $K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$. It follows that $K_0(C^*(K))$ is assumed to be a subgroup of $\mathbb{Z}$. Now, if the index map ∂ is zero, i_* must be injective so that $K_0(C^*(K))$ contains $\mathbb{Z}^2$ as a subgroup, which is the contradiction. Therefore, ∂ is nonzero. Then Nagy or Nistor's result ([Ny], [Ns2]) implies that $\mathrm{sr}(C^*(K)) \ge 2$.

Example 2.4. Let $H = \mathbb{R}^2 \rtimes_\beta \mathbb{Z}$ with $\beta_t(x,y) = (x+ty, y)$ for $x, y \in \mathbb{R}$, $t \in \mathbb{Z}$, which is regarded as a closed normal subgroup of the Heisenberg Lie group. Then we have that

$$0 \to C_0((\mathbb{R}\setminus\{0\}) \times \mathbb{R}) \rtimes_{\hat{\beta}} \mathbb{Z} \to C^*(H) \to C_0(\mathbb{R} \times \mathbb{T}) \to 0$$

with $C_0((\mathbb{R}\setminus\{0\}) \times \mathbb{R}) \rtimes \mathbb{Z} \cong C_0((\mathbb{R}\setminus\{0\}) \times \mathbb{T}) \otimes \mathbb{K}$, where $\hat{\beta}_t(x', y') = (x', tx' + y')$ for $x', y' \in \mathbb{R}$. Then we obtain that $\mathrm{sr}(C^*(H)) = 2 = \dim_{\mathbb{C}} \hat{H}_1$, and $\mathrm{csr}(C^*(H)) = 2$.

Acknowledgment. The author would like to thank Professor S. Kawakami for stimulating comments and bringing my attention to the paper [Bg], and thank the referee for some corrections and reading the manuscript carefully.

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DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, UNIVERSITY OF THE RYUKYUS, NISHIHARA-CHO, OKINAWA 903-0213, JAPAN.
E-mail address: sudo@math.u-ryukyu.ac.jp

Received March 13, 2001 Revised June 29, 2001