C*-algebras having the property (T)

By

Tadashi HURUYA

(Received Nov. 30, 1970)

1. Introduction

If A and B are C*-algebras, $A \odot B$ denotes their algebraic tensor product. A norm $\| \ \|_\beta$ in $A \odot B$ is called *compatible* if the completion of $A \odot B$ by $\| \ \|_\beta$ becomes a C*-algebra, and we denote by $A \otimes_\beta B$ the C*-algebra which is the completion of $A \odot B$ with respect to $\| \ \|_\beta$. There are some ways to define compatible norms in $A \odot B$. T. Turumaru [5] introduced the α-norm. As A. Wulfsohn established, the α-norm has the property:

$$\|\sum_{k=1}^{n} x_k \otimes y_k \|_\alpha = \|\sum_{k=1}^{n} \pi_1(x_k) \otimes \pi_2(y_k)\|, \ x_k \in A, \ y_k \in B$$

where π_1 and π_2 are any faithful representations of A and B, respectively. M. Takesaki proved in [4] that the α-norm is not necessarily the unique compatible norm in $A \odot B$ and that it is the least one among the all compatible norms.

On the other hand, A. Guichardet defined the ν-norm and showed that it is the greatest one among the all compatible norms. The ν-norm is defined by the formula

$$\|x\|_\nu = \sup_\pi \ \|\pi(x)\|, \ x \in A \odot B$$

where π runs over the set of all representations of $A \odot B$ which are continuous with respect to any compatible norm in $A \odot B$.

We say that a C*-algebra A has *the property* (T) if, for every C*-algebra B, the α-norm in $A \odot B$ is the unique compatible norm.

This papar is concerned with C*-algebras having the property (T). In § 2, we consider the structure of C*-algebras having the property (T). In §3, we apply the consideration in §2 to tensor products of C*-algebras. Finally in §4 we present that a C*-algebra A has the greatest closed two-sided ideal I having the property (T) and it is the least one such that A/I has no nonzero closed two-sided ideals having the property (T).

2. C*-algebras

We begin with preliminary lemmas.

LEMMA 1. *Let A and B be C*-algebras and let I be a closed two-sided ideal in A. Then there exists a closed two-sided ideal J such that*

$$(A/I)\otimes_{\alpha}B=(A\otimes_{\alpha}B)/J.$$

PROOF. Let π be a representation of A such that the kernel of $\pi=I$ and ι be a faithful representation of B. Then we can consider the canonical homomorphism $\pi\otimes\iota$ of $A\otimes_{\alpha}B$ onto $\pi(A)\otimes_{\alpha}\iota(B)$ and denote its kernel by J. Since $\pi(A)\otimes_{\alpha}\iota(B)$ is isomorphic to $(A/I)\otimes_{\alpha}B$, it follows that $(A/I)\otimes_{\alpha}B=(A\otimes_{\alpha}B)/J$.

LEMMA 2. *If a C*-algebra A has the property (T) and φ is a homomorphism of A, then the image $\varphi(A)$ of A under φ has the property (T).*

PROOF. Let B be a C*-algebra, let π be a representation of $\varphi(A)\odot B$ which is continuous with respect to each compatible norm in $\varphi(A)\odot B$ and let ι be the identity automorphism of B. Then we can consider the canonical homomorphism $\varphi\otimes\iota$ of $A\otimes_{\alpha}B$ onto $\varphi(A)\otimes_{\alpha}B$. The composite $\pi\circ\varphi\otimes\iota$ of $\varphi\otimes\iota$ and π is a representation of $A\odot B$ which is continuous with respect to each compatible norm in $A\odot B$. Since A has the property (T), $\pi\circ\varphi\otimes\iota$ can be extended to the representation of $A\otimes_{\alpha}B$ which is denoted by ν.

On the other hand, since $\varphi(A)\otimes_{\alpha}B=\varphi\otimes\iota(A\otimes_{\alpha}B)$, for each x in $\varphi(A)\otimes_{\alpha}B$ there exists an element y in $A\otimes_{\alpha}B$ such that $\varphi\otimes\iota(y)=x$. Assume that $\varphi\otimes\iota(y)=\varphi\otimes\iota(z)$, where $y\in A\otimes_{\alpha}B$, $z\in A\otimes_{\alpha}B$. Since the kernel of $\varphi\otimes\iota$ is a subset of the kernel of ν, we have $\nu(y)=\nu(z)$. Therefore, we can define the representation $\tilde{\pi}$ of $\varphi(A)\otimes_{\alpha}B$ as follows:

$$\tilde{\pi}(x)=\nu(y),\quad x\in\varphi(A)\otimes_{\alpha}B$$

where y is an element in $A\otimes_{\alpha}B$ such that $\varphi\otimes\iota(y)=x$.
Then $\tilde{\pi}$ is an extention of π. Therefore, the α-norm in $\varphi(A)\odot B$ is greater than the ν-norm in $\varphi(A)\odot B$. This implies the equality of the two norms, and the lemma is proved.

COROLLARY. *Let A be a C*-algebra having the property (T) and let I be a closed two-sided ideal in A. Then A/I has the property (T).*

PROPOSITION 3. *Let I be a closed two-sided ideal in a C*-algebra A. Then A has the property (T) if and only if A/I and I have the property (T).*

PROOF. Let B be a C*-algebra. Assume first that A has the property (T). If π is a non degenerate representation of $I\odot B$ which is continuous with respect to each compatible norm in $I\odot B$. By [2: Proposition 1] there exist representations π_1 of I and π_2 of B such that

$$\pi(x\otimes y)=\pi_1(x)\pi_2(y)=\pi_2(y)\pi_1(x),\ x\in I,\ y\in B.$$

Then π_1 is the non degenerate representation of I and hence π_1 can be extended to the representation of A which is also denoted by π_1. Moreover, we have

$$\pi_1(x)\pi_2(y)=\pi_2(y)\pi_1(x),\ x\in A,\ y\in B.$$

Therefore, we can define the representation $\tilde{\pi}$ of $A\odot B$ as follows:

$$\tilde{\pi}(\sum_{i=1}^{n}x_i\otimes y_i)=\sum_{i=1}^{n}\pi_1(x_i)\ \pi_2(y_i),\quad x_i\in A,\ y_i\in B.$$

Since $\tilde{\pi}$ is continuous with respect to each compatible norm in $A\odot B$, it can be extended to the representation of $A\otimes_\alpha B$ which is also denoted by $\tilde{\pi}$. Since $I\otimes_\alpha B, \subset A\otimes_\alpha B, \tilde{\pi}\mid I\otimes_\alpha B$ is the extention of π. Therefore, I has the property (T).
By Corollary of Lemma 2, A/I has the property (T).

Conversely, assume that A/I and I have the property (T). Let π be a representation of $A\odot B$ on a Hilbert space H_π. If π is continuous with respect to each compatible norm in $A\odot B$, there exist representations π_1 of A and π_2 of B such that

$$\pi(x\otimes y)=\pi_1(x)\pi_2(y)=\pi_2(y)\pi_1(x),\ x\in A,\ y\in B.$$

Then, in case $\pi_1(I)=\{0\}$, π defines the canonical representation $\bar{\pi}$ of $A/I\odot B$ which is continuous with respect to each compatible norm in $A/I\odot B$. Since $A/I\otimes_\nu B=A/I\otimes_\alpha B$, $\bar{\pi}$ can be extended the representation of $A/I\otimes_\alpha B$ which is also denoted by $\bar{\pi}$. By Lemma 1, $\bar{\pi}$ defines the representation ρ of $A\otimes_\alpha B$. Then, $\rho|A\odot B=\pi$.

In case $\pi_1(I)\neq\{0\}$, let H_1 be the closed subspace of H generated by $\{\pi(x)\xi\,|\,x\in I\odot B, \xi\in H\}$. π defines the representation π' of $I\odot B$ on H_1. Then $I\otimes_\alpha B$ is a closed two-sided ideal in $A\otimes_\alpha B$, so π' can be extended to the representation of $A\otimes_\alpha B$ on H_1 which is also denoted by π'. Let (u_λ) and (v_μ) be approximate identities of I and B, respectively. Then we have

$$\underset{\lambda,\mu)}{\text{strong-limit}}\quad \pi(u_\lambda\otimes v_\mu)=I_{H1}$$

where I_{H1} is the identy operator on H_1, and

$$\begin{aligned}\lim_{(\lambda,\mu)}\pi(x\otimes y)\pi(u_\lambda\otimes y_\mu)\xi &=\lim_{(\lambda,\mu)}\pi(xu_\lambda\otimes yv_\mu)\xi\\ &=\lim_{(\lambda,\mu)}\pi'(xu_\lambda\otimes yv_\mu)\xi\\ &=\lim_{(\lambda,\mu)}\pi'(x\otimes y)\pi'(u_\lambda\otimes v_\mu)\xi\end{aligned}$$

for $x\in A, y\in B, \xi\in H_1$.

Hence H_1 is invariant with respect to $\pi(A\odot B)$ and $\pi(x)|H_1=\pi'(x)$ for $x\in A\odot B$.

Let $H^{\perp}{}_1$ be the orthogonal complement of H_1 in H and $\pi|H^{\perp}{}_1$ be the restriction of π on $H^{\perp}{}_1$. Since $\pi|H^{\perp}{}_1(I\odot B)=\{0\}$, $\pi|H^{\perp}{}_1$ can be extended to the representation of $A\otimes_\alpha B$.

Consequently π can be extended to the representation of $A \otimes_\alpha B$, and we obtain the conclusion.

3. Tensor products

Using above results, we consider tensor products of C*-algebras which have the property (T).

PROPOSITION 4. *Let A and B be C*-algebras and let $A \otimes_\alpha B$ be a C*-algebra having the property (T). Then, A and B have the property (T).*

PROOF. Let C be a C*-algebra and let π be a representation of $A \odot C$ on a Hilbert space H_π which is continuous with respect to any compatible norm in $A \odot C$. Then, there exist representations π_1 of A and π_2 of C such that

$$\pi(x \otimes y) = \pi_1(x)\pi_2(y), \quad x \in A,\ y \in C.$$

Now, let ρ be a non degenerate representation of B on a Hilbert space H_ρ. Then we can consider the canonical representation $\rho \otimes \pi_1$ of $B \otimes_\alpha A$.

Here, we define the representation ν of $(B \otimes_\alpha A) \odot C$ as follows:

$$\nu(x \otimes y) = \rho \otimes \pi_1(x)(I \otimes \pi_2(y)), \quad x \in B \otimes_\alpha A,\ y \in C,$$

where I is the identity operator on H_ρ.

$B \otimes_\alpha A$ has the property (T), and $(B \otimes_\alpha A) \otimes_\alpha C = B \otimes_\alpha (A \otimes_\alpha C)$, Therefore, ν can be extended to the representation of $(B \otimes_\alpha A) \otimes_\alpha C$ which is also denoted by ν. Then, there exist representations of ν_1 of $A \otimes_\alpha C$ and ν_2 of B such that

$$\nu(x \otimes y) = \nu_2(x)\nu_1(y), \quad x \in B,\ y \in A \otimes_\alpha C.$$

Let ξ be a unit vector in H_ρ, that is $\|\xi\| = 1$, and let $\nu_1 | \xi \otimes H_\pi$ be the restriction of the representation ν_1 on $\xi \otimes H_\pi$. Then, $\nu_1 | \xi \otimes H_\pi | A \odot C$ is unitarily equivalent to π. Therefore A has the property (T).

COROLLARY. *Let a norm $\|\ \|_\beta$, in $A \odot B$ be compatible and let $A \otimes_\beta B$ be a C*-algebra having the property (T). Then, A and B have the property (T).*

PROOF. We can define a homomorphism π_β of $A \otimes_\beta B$ onto $A \otimes_\alpha B$ as follows:

$$\pi_\beta(x) = \|\ \|_\alpha\text{-}\lim_n x_n, \quad x \in A \otimes_\beta B,$$

where (x_n) is a sequence in $A \odot B$ converging to x with respect to $\|\ \|_\beta$.

Since $\pi_\beta(A \otimes_\beta B) = A \otimes_\alpha B$, it follows from Lemma 2 and Proposition 3 that A and B have the property (T).

PROPOSITION 5. *Let A and B be C*-algebras and I_1 and I_2 be closed two-sided ideals in A and B, respectively. Suppose that A has the property (T), then, we have*

$$A/I_1 \otimes_\alpha B/I_2 = A \otimes_\alpha B/(I_1 \otimes_\alpha B + A \otimes_\alpha I_2).$$

PROOF. By Proposition 3, we have

$$A/I_1 \otimes_\alpha B/I_2 = A/I_1 \otimes_\nu B/I_2.$$

On the other hand, by [3], we have

$$A/I_1 \otimes_\nu B/I_2 = A \otimes_\nu B/(I_1 \otimes B + A \otimes_\nu I_2).$$

Since A has the property (T), we have

$$A \otimes_\nu B/I_1 \otimes_\nu B + A \otimes_\nu I_2 = A \otimes_\alpha B/(I_1 \otimes_\alpha B + A \otimes_\alpha I_2).$$

Consequently we have

$$A/I_1 \otimes B/I_2 = A \otimes_\alpha B/(I_1 \otimes_\alpha B + A \otimes_\alpha I_2).$$

4. Two-sided ideals

In this section, we consider closed two-sided ideals having the property (T).

LEMMA 6. *Let I_1 and I_2 be closed two-sided ideals having the property (T) in a C*-algebra. Then $I_1 + I_2$ has the property (T).*

PROOF. Since $(I_1 + I_2)/I_1$ is isomorphic to $I_2/I_1 \cap I_2$, it follows from Corollary of Lemma 2 that $(I_1 + I_2)/I_1$ has the property (T). Then, by Proposition 3, $I_1 + I_2$ has the property (T).

PROPOSITION 7. *Let A be a C*-algebra. Then there exists the greatest closed two-sided ideal I having the property (T), and it is the least one such that A/I has no nonzero closed two-sided ideals having the property (T).*

PROOF. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be an increasing family of closed two-sided ideals I_λ in A. By [4; Theorem 5], the closure of $\bigcup_{\lambda \in \Lambda} I_\lambda$ has the property (T). Hence there exists a maximal closed two-sided ideal I having the property (T). By the maximality and Proposition 3, A/I has no nonzero closed two-sided ideals having the property (T). By Lemma 6, I is the greatest.

Now, let J be a closed two-sided ideal in A such that A/J has no non-zero ideals having the property (T). Then, $I/J \cap I$ is isomorphic to the closed two-sided ideal in A/J. Hence $I/J \cap I = \{0\}$. Consequently $J \supset I$, this completes the proof.

NIIGATA UNIVERSITY

References

1. J. DIXMIER: Les C*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.
2. A. GUICHARDEFT: *Caractères et représentations de produits tensoriels de C*-algebres*, Ann. Ecole Norm. Sup., 81(1964), 189–206.
3. ———: *Tensor products of C*-algebras*, Soviet Math., 6(1965), 210–213. (Translation of Doklady Akademii Nauk SSSR, 160(1965), 986–989.)
4. M. TAKESAKI: *On the cross-norm of the direct product of C*-algebras*, Tohoku Math. Journ., 16(1964), 111–122.
5. T. TURUMARU: *On the direct product of operator algebras I*, Tohoku Math. Journ., 4(1956), 242–251.
6. A. WULFSOHN: *Produit tensoriel de C*-algèbres*, Bull. Sci. Math., 87(1963), 13–27.
7. ———: *Le produit tensoriel de certaines C*-algèbres*, C. R. Acad. Sc. Paris, 258(1964), 6052–6054.