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C^*-algebras of type R or non type R by K-theory and Fredholm index

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Abstract

We introduce a notion for C^*-algebras to divide them into two classes by using K-theory of C^*-algebras. The index map of the six term exact sequence of K-groups for extensions by C^*-algebras plays a key role. Also, we introduce another notion for C^*-algebras to divide them into two classes by using the Fredholm index of Fredholm operators. We establish some basic properties for these notions and give some illustrative examples such as the group C^*-algebras of (solvable) Lie groups of type R or non type R.

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Introduction

Lie groups have been divided into two classes. One is the class of Lie groups of type R, and the other is the class of Lie groups of non type R. In particular, simply connected solvable Lie groups of type R or non type R have been of some interest (see L. Auslander and C.C. Moore [1]). It is also well known that the unitary representation theory of Lie or locally compact groups corresponds to the representation theory of their group C^*-algebras (see Dixmier [3] or Pedersen [11]). Actually, an irreducible unitary representation π of a locally compact group G corresponds to an irreducible representation Π of its (full) group C^*-algebra $C^*(G)$ as follows:

$$\pi \leftrightarrow \Pi, \quad \Pi(f) = \int_G f(g)\pi_g dg$$

for $g \in G$ and dg the Haar measure on G, and $f \in L^1(G)$ the Banach $*$-algebra of all integrable measurable functions on G with convolution and involution. Also, the group C^*-algebra $C^*(G)$ is defined to be the norm closure of $\Phi(L^1(G))$ in $\mathbb{B}(L^2(H_\Phi))$, where Φ is the universal representation of $L^1(G)$ and $\mathbb{B}(H_\Phi)$ is the C^*-algebra of all bounded operators on the Hilbert space H_Φ of Φ. Furthermore, the unitary dual of G is identified with the spectrum of $C^*(G)$ that consists of unitary equivalence classes of irreducible representaions of $C^*(G)$.

Therefore, it is very natural to study group C^*-algebras of Lie groups to divide them into two classes that should correspond to the classes of Lie groups above. Our first attempt for this study has been made in [16] to show that the unitizations of the group C^*-algebras of all simply connected solvable Lie groups of non type R are not ASH, where ASH means approximately subhomogeneous or inductive limits of subhomogeneous C^*-algebras. For the proof, we use both a result of Auslander and Moore [1] for simply connected solvable Lie groups of non type R and a method by Fredholm index of Fredholm operators for the group C^*-algebras of certain simply connected solvable Lie groups of non type R such as the real or complex $ax+b$ groups (see Rosenberg [13]). Moreover, it is shown in [19] that all CCR C^*-algebras that contain the group C^*-algebras of connected nilpotent or semi-simple Lie groups are ASH.

This paper is organized as follows. In Preliminaries below we review some definitions and facts about Lie groups of type R or non type R and the K-theory of C^*-algebras for the covenience to readers. In Section 1 we introduce a notion for C^*-algebras to divide them into two classes by using the K-theory of C^*-algebras. The index map of the six term exact sequence of K-groups of C^*-algebras plays an important role in our theory. We consider some C^*-algebras of non type R by K-theory in our definition. In particular, we prove that the group C^*-algebras of all simply connected solvable Lie groups of non type R are of non type R by K-theory. In Section 2 we consider some C^*-algebras of type R by K-theory. In Section 3 we consider more examples. From our considerations in these sections we understand that the class of C^*-algebras of type R by K-theory is very close to the class of C^*-algebras with stable rank one, where the stable rank for C^*-algebras was introduced by Rieffel [12]. It also turns out that our type R or non type R by K-theory for C^*-algebras are not fit to type R or non type R of Lie groups. However, our intension to define type R or non type R of C^*-algebras is to define a (topological) notion in terms of C^*-algebras generalizing non type R of Lie groups. Since (topological) K-theory has been a standard tool for C^*-algebras, our attempt here would be some significant and useful in the future research. In Section 4 we introduce another notion for C^*-algebras to divide them into two classes by using Fredholm index of Fredholm operators, and study its some properties. It turns out that this (analytical) notion (since Fredholm index is an analytical notion in the literature) is quite fit to type R or non type R of Lie groups. However, some main parts of this section has been considered in [16] and [19].

Preliminaries

Lie groups of type R or non type R Recall that a Lie group G is said to be of type R if its Lie algebra $\mathfrak{G}$ is type R, that means that for any $X \in \mathfrak{G}$, the adjoint operator $\mathrm{ad}(X)$ on $\mathfrak{G}$ has pure imaginary eigenvalues (that may be zero). If G is connected, then G is of type R if for any $g \in G$, the adjoint operator $\mathrm{Ad}(g)$ on G has eigenvalues the absolute values one (see [1] or Onishchik-Vinberg [10]).

Recall that a locally compact group G is CCR (or liminal) if its group C^*-algebra $C^*(G)$ is CCR (or liminal), i.e., for any irreducible representation π of $C^*(G)$, we have $\pi(C^*(G)) = \mathbb{K}(H_\pi) = \mathbb{K}$ the C^*-algebra of compact operators on the representation Hilbert space H_π of π. Also, G is of type I (or GCR) if $C^*(G)$ is of type I (or GCR), i.e., for any irreducible representation π of $C^*(G)$, we have $\pi(C^*(G)) \supset \mathbb{K}(H_\pi) = \mathbb{K}$. See [3] or [11].

It is shown by [1] that CCR groups are of type R, and connected solvable Lie groups of type R are CCR if and only if they are GCR. It is also known that connected nilpotent Lie groups, connected semi-simple Lie groups and compact groups are all of type R since they are CCR (cf. [3]). Some simply connected solvable Lie groups of non type I such as Mautner group and Dixmier group are known to be of type R. On the other hand, the real (or complex) $ax + b$ group is of non type R. Moreover, a Lie group that has the $ax + b$ group as a quotient is also of non type R ([1]).

K-theory of C^*-algebras Let $\mathfrak{A}$ be a unital C^*-algebra. The K_0-group $K_0(\mathfrak{A})$ of $\mathfrak{A}$ is defined to be the abelian group of stable equivalence classes of projections of the union $\cup_{n=1}^\infty M_n(\mathfrak{A})$ of matrix algebras $M_n(\mathfrak{A})$ over $\mathfrak{A}$, that is,

$$K_0(\mathfrak{A}) = \{[p] - [q] \mid p, q \in \cup_{n=1}^\infty M_n(\mathfrak{A}) \quad \text{projections}\},$$

where $[p], [q]$ are stable equivalence classes of p, q (cf. [23]). For a nonunital C^*-algebra $\mathfrak{A}$, its K_0-group $K_0(\mathfrak{A})$ is defined from the following short exact sequence:

$$0 \to K_0(\mathfrak{A}) \to K_0(\mathfrak{A}^+) \to K_0(\mathbb{C}) \to 0$$

associated with: $0 \to \mathfrak{A} \to \mathfrak{A}^+ \to \mathbb{C} \to 0$ the splitting extension with $\mathfrak{A}^+$ the unitization of $\mathfrak{A}$ by $\mathbb{C}$. Note that $K_0(\mathfrak{A}^+) \cong K_0(\mathfrak{A}) \oplus K_0(\mathbb{C})$ and $K_0(\mathbb{C}) \cong \mathbb{Z}$.

The K_1-group $K_1(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ (or its unitization by $\mathbb{C}$) is defined by

$$K_1(\mathfrak{A}) = \{[u] \mid u \in \cup_{n=1}^\infty M_n(\mathfrak{A}) \quad \text{unitaries (or invertible)}\},$$

where $[u]$ is the homotopy class of u in the union of the unitary groups $U_n(\mathfrak{A})$ (or $GL_n(\mathfrak{A})$) of unitary (or invertible) matrices of $M_n(\mathfrak{A})$, where $U_n(\mathfrak{A})$ is embedded in $U_{n+1}(\mathfrak{A})$ canonically (respectively). Also, $K_1(\mathfrak{A})$ is isomorphic to the inductive limit of the quotient groups $U_n(\mathfrak{A})/U_n(\mathfrak{A})_0$ (or $GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$) by $U_n(\mathfrak{A})_0$ (or $GL_n(\mathfrak{A})_0$) connected components containing the identify matrices. Note that $[u] \cdot [v] = [uv] = [u \oplus v]$, where $\oplus$ means the diagonal sum.

For a short exact sequence: $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{D} \to 0$ of C^*-algebras, its six term exact sequence of K-groups is:

$$\begin{tikzcd}
K_0(\mathfrak{I}) \arrow[r] & K_0(\mathfrak{A}) \arrow[r] & K_0(\mathfrak{A}/\mathfrak{I}) \arrow[d] \\
K_1(\mathfrak{A}/\mathfrak{I}) \arrow[u, "\partial"] & K_1(\mathfrak{A}) \arrow[l] & K_1(\mathfrak{I}) \arrow[l]
\end{tikzcd}$$

where ∂ is the index map defined by

$$\partial([u]) = [v 1_n v^*] - [1_n],$$

where $[u] \in K_1(\mathfrak{A}/\mathfrak{I})$ for $u \in U_n(\mathfrak{A}/\mathfrak{I})$, $v \in U_{2n}(\mathfrak{A})$ a unitary lift of $u \oplus u^*$, and 1_n is the $n \times n$ identity matrix. See Murphy [7] or Wegge-Olsen [23] for more details.

1 C^*-algebras of non type R by K-theory

Definition 1.1 Let $\mathfrak{A}$ be a C^*-algebra. We say that $\mathfrak{A}$ is of non type R by K-theory if there exists a quotient C^*-algebra $\mathfrak{B}$ that is decomposed into an extension:

$$0 \to \mathfrak{I} \to \mathfrak{B} \to \mathfrak{D} \to 0$$

such that the index map from the K_1-group $K_1(\mathfrak{D})$ of the quotient $\mathfrak{D}$ to the K_0-group $K_0(\mathfrak{I})$ of the closed ideal $\mathfrak{I}$ is nonzero, where we allow the case $\mathfrak{B} = \mathfrak{A}$.

If $\mathfrak{A}$ is not of non type R by K-theory, then we say that $\mathfrak{A}$ is of type R by K-theory.

In particular,

Definition 1.2 We say that the extension $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{D} \to 0$ of C^*-algebras (or the extension for $\mathfrak{A}$) is of non type R by K-theory if the index map from $K_1(\mathfrak{D})$ of $\mathfrak{D}$ to $K_0(\mathfrak{I})$ of $\mathfrak{I}$ is nonzero.

If the extension for $\mathfrak{A}$ is not of non type R by K-theory, then we say that the extension for $\mathfrak{A}$ is of type R by K-theory.

Remark. A C^*-algebra extension for a C^*-algebra of non type R by K-theory in our sense is not necessarily of non type R by K-theory. See examples below. We may define a C^*-algebra to be of non type R if it is a C^*-algebra extension of non type R (and it might be better in other situations), but our Definition 1.1 is the main notion in our theory in this paper and we are interested in more than extensions so that we use this for C^*-algebras to be of type R or non type R.

Anyhow, first of all, we consider some examples by extensions.

Example 1.3 Let $\mathbb{K}$ be the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space H and $\mathbb{B}$ the C^*-algebra of bounded operators on H. Then we have the following exact sequence:

$$0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{B}/\mathbb{K} \to 0,$$

where $\mathbb{B}/\mathbb{K}$ is called Calkin algebra, and its six term exact sequence of K-groups implies

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & K_0(\mathbb{B}/\mathbb{K}) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow \\ K_1(\mathbb{B}/\mathbb{K}) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

since $K_0(\mathbb{K}) \cong \mathbb{Z}$, $K_1(\mathbb{K}) \cong 0$, and $K_j(\mathbb{B}) \cong 0$ for $j = 0, 1$. Hence, the index map ∂ is nonzero. Therefore, the extension for $\mathbb{B}$ is of non type R by K-theory and $\mathbb{B}$ is of non type R by K-theory. In fact, the index map ∂ corresponds to the Fredholm index of Fredholm operators contained in $\mathbb{B}$.

Example 1.4 Let $\mathfrak{T}$ be the Toeplitz algebra, which is defined to be the C^*-algebra generated by the shift on the Hilbert space(s). It is known that $\mathfrak{T}$ is decomposed into the following exact sequence:

$$0 \to \mathbb{K} \to \mathfrak{T} \to C(\mathbb{T}) \to 0,$$

where $C(\mathbb{T})$ is the C^*-algebra of continuous functions on the torus $\mathbb{T}$, and its six term exact sequence of K-groups implies

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[u, "\partial"] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

since $K_0(\mathfrak{T}) \cong \mathbb{Z}$, $K_1(\mathfrak{T}) \cong 0$, and $K_j(C(\mathbb{T})) \cong \mathbb{Z}$ for $j = 0, 1$. Hence. the index map ∂ is nonzero. Therefore, the extension for $\mathfrak{T}$ is of non type R by K-theory and $\mathfrak{T}$ is of non type R by K-theory. In fact, the index map ∂ corresponds to the Fredholm index of Fredholm operators contained in $\mathfrak{T}$.

Example 1.5 Let A_2 be the real $ax + b$ group defined by

$$A_2 = \{g = \begin{pmatrix} e^t & s \\ 0 & 1 \end{pmatrix} \mid t, s \in \mathbb{R}\}.$$

Then A_2 is isomorphic to the semi-direct product $\mathbb{R} \rtimes \mathbb{R}$ via the identification: $g = (s, t)$. Let $C^*(A_2)$ be the group C^*-algebra of A_2. Then $C^*(A_2)$ is isomorphic to the crossed product $C_0(\mathbb{R}) \rtimes \mathbb{R}$, where $C_0(\mathbb{R})$ is the C^*-algebra of continuous functions on $\mathbb{R}$ vanishing at infinity. Moreover, we have the following exact sequence:

$$0 \to \mathbb{K} \oplus \mathbb{K} \to C^*(A_2) \to C_0(\mathbb{R}) \to 0,$$

which follows from that the origin of $\mathbb{R}$ is fixed, and $C_0(\mathbb{R}_+) \rtimes \mathbb{R} \cong \mathbb{K}$. Then the six term exact sequence of K-groups from the extension above:

$$\begin{tikzcd}
K_0(\mathbb{K} \oplus \mathbb{K}) \arrow[r] & K_0(C^*(A_2)) \arrow[r] & K_0(C_0(\mathbb{R})) \arrow[d] \\
K_1(C_0(\mathbb{R})) \arrow[u, "\partial"] & K_1(C^*(A_2)) \arrow[l] & K_1(\mathbb{K} \oplus \mathbb{K}) \arrow[l]
\end{tikzcd}$$

implies

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[d] \\
\mathbb{Z} \arrow[u, "\partial"] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $K_j(C_0(\mathbb{R})) \cong K_{j+1}(\mathbb{C})$ and $K_j(C_0(\mathbb{R}) \rtimes \mathbb{R}) \cong K_{j+1}(C_0(\mathbb{R}))$ for $j = 0, 1 \bmod 1$ by Connes' Thom isomorphism for crossed products of C^*-algebras by actions of $\mathbb{R}$. The diagram above shows that the index map ∂ is nonzero. Hence, the extension for $C^*(A_2)$ is of non type R by K-theory and $C^*(A_2)$ is of non type R by K-theory.

Example 1.6 Let $A_2 \times \mathbb{R}$ be the product of the real $ax + b$ group and $\mathbb{R}$, and $C^*(A_2 \times \mathbb{R})$ its group C^*-algebra. Then we have $C^*(A_2 \times \mathbb{R}) \cong C^*(A_2) \otimes C_0(\mathbb{R})$, and it follows from the example above that

$$0 \to (\mathbb{K} \oplus \mathbb{K}) \otimes C_0(\mathbb{R}) \to C^*(A_2 \times \mathbb{R}) \to C_0(\mathbb{R}^2) \to 0.$$

Its six term exact sequence of K-groups gives:

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ {\scriptstyle\partial}\uparrow & & & & \downarrow \\ 0 & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z}^2 \end{array}$$

from which the index map ∂ is zero. Therefore, the above extension for $C^*(A_2 \times \mathbb{R})$ is not of non type R. However, we have the quotient from $C^*(A_2 \times \mathbb{R})$ to $C^*(A_2)$. Since $C^*(A_2)$ is of non type R by K-theory, $C^*(A_2 \times \mathbb{R})$ is of non type R by K-theory.

More generally, we obtain

Theorem 1.7 *Let G be a simply connected solvable Lie group of non type R. Then its group C^*-algebra $C^*(G)$ is of non type R by K-theory.*

Proof. It is known by [1, Proposition 2.2, p. 172] that if G is a simply connected solvable Lie group of non type R, then there exists a quotient that is isomorphic to one of the following: the real $ax+b$ group A_2, the semi-direct product $B_3 = \mathbb{R}^2 \rtimes_{\alpha^c} \mathbb{R}$ and $B_4 = \mathbb{R}^2 \rtimes_\beta \mathbb{R}^2$, where the actions α^c, β are defined by

$$\alpha_t^c = e^{ct}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \beta_{(s,t)} = e^s \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

for $c \in \mathbb{R} \setminus \{0\}$, $s, t \in \mathbb{R}$.

It follows from the fact above that the group C^*-algebra $C^*(G)$ of G non type R has a quotient that is isomorphic to one of the following: $C^*(A_2)$, $C^*(B_3)$ and $C^*(B_4)$. By Example 1.4, $C^*(A_2)$ is of non type R by K-theory.

We now consider the structure of $C^*(B_3)$. Since B_3 is the semi-direct product $\mathbb{R}^2 \rtimes_{\alpha^c} \mathbb{R}$, $C^*(B_3)$ is isomorphic to the crossed product $C_0(\mathbb{R}^2) \rtimes_{\alpha^c} \mathbb{R}$. Furthermore, we have

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes_{\alpha^c} \mathbb{R} \to C^*(B_3) \to C_0(\mathbb{R}) \to 0$$

since the origin of $\mathbb{R}^2$ is fixed under the action α^c. Moreover,

$$C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes_{\alpha^c} \mathbb{R} \cong C(\mathbb{T}) \otimes C_0(\mathbb{R}_+) \rtimes \mathbb{R} \cong C(\mathbb{T}) \otimes \mathbb{K},$$

where we use that the orbit space of $\mathbb{R}^2 \setminus \{0\}$ by $\mathbb{R}$ is homeomorphic to $\mathbb{T}$ and the action α^c on each orbit in $\mathbb{R}^2 \setminus \{0\}$ is identified with the shift action on $\mathbb{R}_+$. Then the six term exact sequence of K-groups of the extension above implies

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\ \mathbb{Z} \arrow[u, "\partial"] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l] \end{tikzcd}$$

from which the index map ∂ is nonzero. Hence, $C^*(B_3)$ is of non type R by K-theory.

We next consider the structure of $C^*(B_4)$. Since B_4 is the semi-direct product $\mathbb{R}^2 \rtimes_\beta \mathbb{R}^2$, we have $C^*(B_4) \cong C_0(\mathbb{R}^2) \rtimes_\beta \mathbb{R}^2$. Furthermore, we have

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes_\beta \mathbb{R}^2 \to C^*(B_4) \to C_0(\mathbb{R}) \to 0$$

since the origin of $\mathbb{R}^2$ is fixed under the action β. By the identification $\mathbb{R}^2 \setminus \{0\} = \mathbb{R}_+ \times \mathbb{T}$ where the action $\beta_{(s,0)}$ is the shift on $\mathbb{R}_+$, and the action $\beta_{(0,t)}$ is the rotation on $\mathbb{T}$, we obtain

$$C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes_\beta \mathbb{R}^2 \cong (C_0(\mathbb{R}_+) \rtimes \mathbb{R}) \otimes (C(\mathbb{T}) \rtimes \mathbb{R}).$$

Furthermore, we have $C_0(\mathbb{R}_+) \rtimes \mathbb{R} \cong \mathbb{K}$ and

$$C(\mathbb{T}) \rtimes \mathbb{R} \cong C(\mathbb{R}/\mathbb{Z}) \rtimes \mathbb{R} \cong C^*(\mathbb{Z}) \otimes \mathbb{K}$$

by the imprimitivity theorem, and $C^*(\mathbb{Z}) \cong C(\mathbb{T})$. Therefore, the structure of $C^*(B_4)$ as an extension is the same as $C^*(B_3)$. Thus, $C^*(B_4)$ is of non type R by K-theory.

It follows from the arguments above that $C^*(G)$ is of non type R by K-theory as desired. □

Remark. In fact, we introduced our definition for C^*-algebras to be of non type R by K-theory to imply the theorem above. However, we see below later that our definition is somewhat weak and loose to divide the group C^*-algebras of solvable Lie groups into two classes of type R or of non type R.

We next consider some basic properties of C^*-algebras of non type R by K-theory.

Proposition 1.8 *Let $\mathfrak{A}$ be a C^*-algebra of non type R by K-theory. Then $\mathfrak{A} \otimes M_n(\mathbb{C})$ and $\mathfrak{A} \otimes \mathbb{K}$ are also non type R by K-theory, where $\mathbb{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. Furthermore, the unitization $\mathfrak{A}^+$ of $\mathfrak{A}$ by $\mathbb{C}$ is also of non type R by K-theory.*

Proof. By definition, there exists a quotient C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ that is decomposed into an extension: $0 \to \mathfrak{J} \to \mathfrak{B} \to \mathfrak{D} \to 0$ such that the index map from $K_1(\mathfrak{D})$ to $K_0(\mathfrak{J})$ is nonzero. Then it is clear that $\mathfrak{B} \otimes M_n(\mathbb{C}) \cong M_n(\mathfrak{B})$ and $\mathfrak{B} \otimes \mathbb{K}$ are quotient C^*-algebras of $\mathfrak{A} \otimes M_n(\mathbb{C}) \cong M_n(\mathfrak{A})$ and $\mathfrak{A} \otimes \mathbb{K}$ respectively, and they are decomposed into the following extensions:

$$0 \to \mathfrak{I} \otimes M_n(\mathbb{C}) \to \mathfrak{B} \otimes M_n(\mathbb{C}) \to \mathfrak{D} \otimes M_n(\mathbb{C}) \to 0,$$
$$0 \to \mathfrak{I} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K} \to \mathfrak{D} \otimes \mathbb{K} \to 0.$$

The conclusion follows from that $K_j(\mathfrak{C}) \cong K_j(\mathfrak{C} \otimes M_n(\mathbb{C})) \cong K_j(\mathfrak{C} \otimes \mathbb{K})$ for $j = 0, 1$ and a C^*-algebra $\mathfrak{C}$ in general.

Furthermore, the unitization $\mathfrak{B}^+$ of $\mathfrak{B}$ by $\mathbb{C}$ is a quotient C^*-algebra of $\mathfrak{A}^+$ that is decomposed into an extension: $0 \to \mathfrak{I} \to \mathfrak{B}^+ \to \mathfrak{D}^+ \to 0$. The conclusion follows from that $K_1(\mathfrak{D}^+) \cong K_1(\mathfrak{D})$ and $K_0(\mathfrak{B}^+) \cong K_0(\mathfrak{B}) \oplus \mathbb{Z}$, which implies that the index map is the same as before taking the unitization by $\mathbb{C}$. □

Proposition 1.9 *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has a quotient C^*-algebra of non type R by K-theory, then $\mathfrak{A}$ is also of non type R by K-theory.*

Proof. It is evident. □

In particular,

Proposition 1.10 *Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t \in X})$ be the C^*-algebra of a continuous field on a locally compact Hausdorff space X with fibers $\mathfrak{A}_t$. If there exists a fiber $\mathfrak{A}_t$ that is of non type R by K-theory, then $\mathfrak{A}$ is also of non type R by K-theory.*

Proposition 1.11 *Let $\mathfrak{A}$ be a C^*-algebra of non type R by K-theory and $\mathfrak{B}$ a C^*-algebra that has a quotient that is isomorphic to either $\mathbb{C}$, $M_n(\mathbb{C})$ or $\mathbb{K}$, that is, an elementary quotient. Then the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ is also of non type R by K-theory.*

Remark. In particular, we can take $\mathfrak{B}$ as a type I C^*-algebra, which has a composition series such that its subquotients are of continuous trace, and a C^*-algebra of continuous trace has an elementary quotient (see [3] or [11]).

2 C^*-algebras of type R by K-theory

Proposition 2.1 *Let $\mathfrak{A}$ be a simple C^*-algebra. Then $\mathfrak{A}$ is of type R by K-theory.*

Proof. It is evident from that $\mathfrak{A}$ has no quotient C^*-algebras. □

Proposition 2.2 *Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras of type R by K-theory. Then their direct sum $\mathfrak{A} \oplus \mathfrak{B}$ is of type R by K-theory.*

Proposition 2.3 *Let $\mathfrak{A}$ be a C^*-algebra of type R by K-theory. Then any quotient C^*-algebra of $\mathfrak{A}$ is also of type R by K-theory.*

Proof. If there exists a quotient C^*-algebra of $\mathfrak{A}$ that is of non type R by K-theory, then we have the contradiction to that $\mathfrak{A}$ is of type R by K-theory. □

Proposition 2.4 *Let $\mathfrak{A}$ be an AF C^*-algebra, that is, an inductive limit of finite dimensional C^*-algebras. Then $\mathfrak{A}$ is of type R by K-theory.*

Proof. Note that any quotient C^*-algebra of $\mathfrak{A}$ is also AF. Since K_1-groups of AF algebras are always zero, the index maps of K-groups associated with AF quotients are always zero. Thus, $\mathfrak{A}$ can not be of non type R by K-theory. □

Example 2.5 Let G be a compact group and $C^*(G)$ its group C^*-algebra. Then $C^*(G)$ can be written as a c_0-direct sum of matrix algebras over $\mathbb{C}$. Actually, note also that since G is compact, its dual group is discrete, and any irreducible representation of G is finite dimensional. Thus, $C^*(G)$ is AF. Hence, $C^*(G)$ is of type R by K-theory.

More generally, we obtain

Theorem 2.6 *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has stable rank one, then it is of type R by K-theory.*

Proof. Recall that a C^*-algebra $\mathfrak{A}$ has stable rank one (denoted by $\mathrm{sr}(\mathfrak{A}) = 1$) if the set of invertible elements of $\mathfrak{A}$ is dense in $\mathfrak{A}$ (see [12]).

Suppose that there exists a quotient C^*-algebra $\mathfrak{B}$ such that

$$0 \to \mathfrak{I} \to \mathfrak{B} \to \mathfrak{D} \to 0.$$

Since $\mathrm{sr}(\mathfrak{A}) = 1$, it follows that $\mathrm{sr}(\mathfrak{B}) = 1$ by [12, Theorem 4.3] that implies $\mathrm{sr}(\mathfrak{B}) \leq \mathrm{sr}(\mathfrak{A})$. Since $\mathrm{sr}(\mathfrak{B}) = 1$, it follows that $\mathrm{sr}(\mathfrak{I}) = 1$ by [12, Theorem 4.4], $\mathrm{sr}(\mathfrak{D}) = 1$, and the index map from $K_1(\mathfrak{D})$ to $K_0(\mathfrak{I})$ must be zero by Nistor [9, Lemma 3] or Nagy [8, Corollary 2]. □

Remark. See Rieffel [12] for the stable rank theory of C^*-algebras. The converse of the statement above is false in general. Indeed, it is shown by Villadsen [22] that there exist simple C^*-algebras that have stable rank (any) $n \geq 2$, and they are of type R by K-theory since they are simple.

We now collect some C^*-algebras with stable rank one.

Example 2.7 All the C^*-algebras below have stable rank one:

- $C(X)$ the C^*-algebra of continuous functions on a compact Hausdorff space X with dimension ≤ 1, and $M_n(C(X))$ for any $n \geq 1$ if $\dim X \leq 1$ (see [12, Theorem 6.1]).
- $M_n(\mathbb{C})$, $\mathbb{K}$ and AF C^*-algebras ([12, Proposition 3.5]).
- The irrational rotaion C^*-algebra, which is the universal C^*-algebra generated by unitaries U, V satisfying $VU = e^{2\pi i\theta}UV$ for θ an irrational number.

• AT algebras, i.e., inductive limits of finite direct sums of matrix algebras over $C(\mathbb{T})$ ([12]) ([12, Theorem 5.1]). It is shown by Elliott-Evans [6] that the irrational rotation C^*-algebra is a simple AT algebra.

• The group C^*-algebras of compact groups, the group C^*-algebras of motion groups ([18]), and the reduced group C^*-algebra of $SL_2(\mathbb{R})$ ([14]).

• The reduced group C^*-algebras of the free groups F_n ($n \geq 2$) (Dykema, Haagerup, Rørdam [4] and Dykema, de la Harpe [5]).

• The class of C^*-algebras with stable rank one is closed under taking quotients, closed ideals, and inductive limits ([12, Theorems 4.3, 4.4 and 5.1]). The class is also closed under tensor products with matrix algebras over $\mathbb{C}$ and stable isomorphism ([12, Theorems 3.3 and 3.6]). Note also that a hereditary C^*-subalgebra $\mathfrak{B}$ of a σ-unital C^*-algebra $\mathfrak{A}$ is stably isomorphic to the closed ideal generated by $\mathfrak{B}$ in $\mathfrak{A}$.

Remark. C^*-algebras of non type R by K-theory may have closed ideals of type R by K-theory. Indeed, by Example 1.3, $\mathbb{B}$ is of non type R by K-theory while $\mathbb{K}$ is of type R by K-theory since $\mathbb{K}$ is AF, and by Example 1.4, the Toeplitz algebra $\mathfrak{T}$ is of non type R by K-theory but $\mathbb{K}$ is of type R by K-theory. See also Example 1.5 for the group C^*-algebra $C^*(A_2)$ of the real $ax + b$ group A_2.

On the other hand, it should be true that the class of C^*-algebras of type R by K-theory is closed under taking their closed ideals. However, the situation seems to be subtle. For example, it is known in the general topology that there exists a locally compact Hausdorff space X with $\dim X = 1$ but $\dim X^+ = 0$, where X^+ is the one point compactification of X. Also, there exists a locally compact Hausdorff space Y with $\dim Y = 1$ but $\dim \beta Y = 0$, where βY is the Stone-Čech compactification of Y. Fortunately, in these cases, $C(X^+)$ and $C(\beta Y)$ are AF since $\dim X^+ = 0$ and $\dim \beta Y = 0$ and hence they are of type R by K-theory, but their closed ideals $C_0(X)$ and $C_0(Y)$ are not AF. Anyway, $C_0(X)^+ \cong C(X^+)$ and $C_0(Y)^+$ (and $C(\beta Y)$) have stable rank one and hence they are all of type R by K-theory.

3 More examples

For checking out how weak our definition for C^*-algebras to be of non type R by K-theory is, we consider the case of commutative C^*-algebras. First of all,

Example 3.1 Let $C_0(\mathbb{R})$ be the commutative C^*-algebra of continuous functions on the real line $\mathbb{R}$ vanishing at infinity. Since an open subset U of $\mathbb{R}$ is a union of disjoint open intervals $\{U_j\}_{j=1}^n$ (n finite or infinite) of $\mathbb{R}$, we have the following exact sequence:

$$0 \to \oplus_{j=1}^n C_0(U_j) \to C_0(\mathbb{R}) \to C_0(K) \to 0,$$

where the closed subset K of $\mathbb{R}$ is the complement of U in $\mathbb{R}$, and its six term exact sequence of K-groups implies

$$\begin{array}{ccccc} \oplus_{j=1}^n K_0(C_0(U_j)) & \longrightarrow & K_0(C_0(\mathbb{R})) & \longrightarrow & K_0(C_0(K)) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow \\ K_1(C_0(K)) & \longleftarrow & K_1(C_0(\mathbb{R})) & \longleftarrow & \oplus_{j=1}^n K_1(C_0(U_j)) \end{array}$$

and $K_0(C_0(\mathbb{R})) \cong K_1(\mathbb{C}) \cong 0$, and $K_0(C_0(U_j)) \cong K_0(C_0(\mathbb{R})) \cong 0$ since U_j is homeomorphic to $\mathbb{R}$. Hence, the index map ∂ is zero. Thus, any extension for $C_0(\mathbb{R})$ is of type R by K-theory.

Moreover, let $C_0(K)$ be any quotient C^*-algebra of $C_0(\mathbb{R})$ (as above) that is decomposed into the exact sequence:

$$0 \to C_0(V \cap K) \to C_0(K) \to C_0(V^c \cap K) \to 0,$$

where V is an open subset of $\mathbb{R}$, and V^c means the complement of V in $\mathbb{R}$. Since $V^c \cap K$ is a disjoint union $\cup_{k=1}^l W_k$ (l finite or infinite) of W_k either closed intervals or half closed intervals, we have

$$K_1(C_0(V^c \cap K)) \cong \oplus_{k=1}^l K_1(C_0(W_k)) \cong \oplus_{k=1}^l K_1(\mathbb{C}) \cong 0$$

since $C_0(W_k) = C(W_k)$ for closed intervals W_k are contractible, and $C_0(W_k)$ for half closed intervals W_k has the following splitting exact sequence:

$$0 \to C_0(W_k) \to C(W_k^+) \to \mathbb{C} \to 0$$

where W_k^+ is the one point compactfication that is also a closed interval, which implies that

$$0 \cong K_1(C(W_k^+)) \cong K_1(C_0(W_k)) \oplus K_1(\mathbb{C}) \cong K_1(C_0(W_k)).$$

Hence, the index map from $K_1(C_0(V^c \cap K))$ to $K_0(C_0(V \cap K))$ is zero. Therefore, $C_0(\mathbb{R})$ is of type R by K-theory.

However,

Example 3.2 Let $C_0(\mathbb{R}^2)$ be the commutative C^*-algebra of continuous functions on the real line $\mathbb{R}^2$ vanishing at infinity. Since an open subset U of $\mathbb{R}^2$ is a union of disjoint open intervals $\{U_{1j} \times U_{2j}\}_{j=1}^n$ (n finite or infinite) for U_{1j}, U_{2j} open intervals of $\mathbb{R}$, we have the following exact sequence:

$$0 \to \oplus_{j=1}^n C_0(U_{1j} \times U_{2j}) \to C_0(\mathbb{R}^2) \to C_0(K) \to 0,$$

where the closed subset K of $\mathbb{R}$ is the complement of U in $\mathbb{R}$, and its six term exact sequence of K-groups implies

$$\begin{array}{ccccc} \oplus_{j=1}^n K_0(C_0(U_{1j} \times U_{2j})) & \longrightarrow & K_0(C_0(\mathbb{R}^2)) & \longrightarrow & K_0(C_0(K)) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow \\ K_1(C_0(K)) & \longleftarrow & K_1(C_0(\mathbb{R}^2)) & \longleftarrow & \oplus_{j=1}^n K_1(C_0(U_{1j} \times U_{2j})) \end{array}$$

and $K_0(C_0(\mathbb{R}^2)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$, and $K_0(C_0(U_{1j} \times U_{2j})) \cong K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$ since $U_{1j} \times U_{2j}$ is homeomorphic to $\mathbb{R}^2$. We now assume $n \geq 2$. Then, the index map ∂ can not be zero. In fact, if $\partial = 0$, then we must have an injection from $\mathbb{Z}^n$ to $\mathbb{Z}$, which is impossible. If $n = 1$, note that $K_1(C_0(K)) \cong K_1(C_0(\mathbb{T} \times [1,\infty)) \cong K_1(C(\mathbb{T})) \cong \mathbb{Z}$ since K is homeomorphic to $\mathbb{T} \times [1,\infty)$ and $C_0([1,\infty))$ (the cone) is contractible. Hence, the index map ∂ can not be zero. Thus, any extension for $C_0(\mathbb{R}^2)$ is of non type R by K-theory and thus $C_0(\mathbb{R}^2)$ is of non type R by K-theory.

More generally,

Theorem 3.3 *Let $C_0(\mathbb{R}^n)$ be the commutative C^*-algebra of continuous functions on $\mathbb{R}^n$ vanishing at infinity. If n is odd, then any extension for $C_0(\mathbb{R}^n)$ is of type R by K-theory, and if n is even, then any extension for $C_0(\mathbb{R}^n)$ is of non type R by K-theory.*

However, $C_0(\mathbb{R})$ is of type R by K-theory, but if $n \geq 2$, then $C_0(\mathbb{R}^n)$ is of non type R by K-theory.

Proof. Since an open subset U of $\mathbb{R}^n$ is a union of disjoint open subsets $\{\Pi_{i=1}^n U_{ij}\}_{j=1}^l$ of $\mathbb{R}^n$ (l finite or infinite) for U_{ij} open intervals of $\mathbb{R}$, we have the following exact sequence:

$$0 \to \oplus_{j=1}^l C_0(\Pi_{i=1}^n U_{ij}) \to C_0(\mathbb{R}^n) \to C_0(K) \to 0,$$

where the closed subset K of $\mathbb{R}^n$ is the complement of U in $\mathbb{R}^n$, and its six term exact sequence of K-groups implies

$$\begin{array}{ccccc} \oplus_{j=1}^l K_0(C_0(\Pi_{i=1}^n U_{ij})) & \longrightarrow & K_0(C_0(\mathbb{R}^n)) & \longrightarrow & K_0(C_0(K)) \\ \partial \uparrow & & & & \downarrow \\ K_1(C_0(K)) & \longleftarrow & K_1(C_0(\mathbb{R}^n)) & \longleftarrow & \oplus_{j=1}^l K_1(C_0(\Pi_{i=1}^n U_{ij})) \end{array}$$

and if n is odd, then

$$K_0(C_0(\mathbb{R}^n)) \cong K_1(\mathbb{C}) \cong 0, \quad K_0(C_0(\Pi_{i=1}^n U_{ij})) \cong K_0(C_0(\mathbb{R}^n)) \cong 0$$

since U_{ij} is homeomorphic to $\mathbb{R}$. Similarly, if n is even, then $K_0(C_0(\mathbb{R}^n)) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_0(C_0(\Pi_{i=1}^n U_{ij})) \cong K_0(C_0(\mathbb{R}^n)) \cong \mathbb{Z}$. Therefore, by the same reasoning given as in Examples above, we have the desired conclusion.

If $n \geq 2$, we have a quotient map from $C_0(\mathbb{R}^n)$ to $C_0(\mathbb{R}^2)$. Since $C_0(\mathbb{R}^2)$ is of non type R by K-theory, $C_0(\mathbb{R}^n)$ is also of non type R by K-theory. □

Similarly, we can obtain

Theorem 3.4 *Let $C(\mathbb{T}^n)$ be the commutative C^*-algebra of continuous functions on the n-torus $\mathbb{T}^n$. If n is odd, then any extension for $C(\mathbb{T}^n)$ is of type R by K-theory, and if n is even, then any extension for $C(\mathbb{T}^n)$ is of non type R by K-theory.*

However, $C(\mathbb{T})$ is of type R by K-theory, but if $n \geq 2$, then $C(\mathbb{T}^n)$ is of non type R by K-theory.

Moreover, we can replace $C(\mathbb{T}^n)$ ($n \geq 1$) with $C([0,1]^n)$ the C^*-algebra of continuous functions on the product space $[0,1]^n$ of the closed interval $[0,1]$.

Proof. We use the following exact sequence:

$$0 \to \oplus_{j=1}^{l} C_0(\Pi_{i=1}^{n} U_{ij}) \to C(\mathbb{T}^n) \to C_0(K) \to 0,$$

where U_{ij} are open intervals of $\mathbb{T}$ such that $\{\Pi_{i=1}^{n} U_{ij}\}_{j=1}^{l}$ are disjoint, and K is the complement of their union in $\mathbb{T}^n$. Note that $K_j(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$ for $j = 0, 1$ (see [23]). Also, we can replace $C(\mathbb{T}^n)$ with $C([0,1]^n)$ in the exact sequence above, where U_{ij} are open intervals of $[0,1]$. If some U_{ij} are half closed at 0 or 1, then we can replace them with the intervals that are not half closed. Since $[0,1]^n$ is contractible, we have $K_0(C([0,1]^n) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ and $K_1(C([0,1]^n) \cong K_1(\mathbb{C}) \cong 0$. □

Corollary 3.5 *The class of C^*-algebras of type R by K-theory is not closed under taking tensor products.*

Theorem 3.6 *Let G be a connected solvable Lie group and $C^*(G)$ its group C^*-algebra. If $\dim(G/[G,G])^\wedge \geq 2$, then $C^*(G)$ is of non type R by K-theory, where $[G,G]$ is the commutator of G and $(G/[G,G])^\wedge$ is the dual group of the quotient group $G/[G,G]$.*

Proof. Since the quotient group $G/[G,G]$ is isomorphic to $\mathbb{R}^n \times \mathbb{T}^s$ for some $n, s \geq 0$, we have a quotient map from $C^*(G)$ to $C^*(\mathbb{R}^n \times \mathbb{T}^s) \cong C_0(\mathbb{R}^n \times \mathbb{Z}^s)$, where $\mathbb{R}^n \times \mathbb{Z}^s$ is the dual group of $\mathbb{R}^n \times \mathbb{T}^s$. Thus, if $n \geq 1$, then we have a quotient map from $C^*(G)$ to $C_0(\mathbb{R}^n)$. Since $C_0(\mathbb{R}^n)$ is of non type R by K-theory for $n \geq 2$, $C^*(G)$ is also of non type R by K-theory in this case. □

Theorem 3.7 *Let G be a simply connected solvable Lie group. The group C^*-algebra $C^*(G \times \mathbb{R})$ is of non type R by K-theory.*

Proof. If G is noncommutative, then the quotient group $G/[G,G]$ is isomorphic to $\mathbb{R}^n$ for some $n \geq 1$. Thus, we have a quotient map from $C^*(G \times \mathbb{R})$ to $C^*(\mathbb{R}^{n+1}) \cong C_0(\mathbb{R}^{n+1})$. □

We now review some facts about the stable rank of C^*-algebras.

Example 3.8 Let $C_0(X)$ be the C^*-algebra of continuous functions on a locally compact Hausdorff space X vanishing at infinity. Then

$$\mathrm{sr}(C_0(X)) = [\dim X^+/2] + 1,$$

where $[x]$ means the maximum integer $\leq x$, and X^+ is the one point compactfication of X ([12, Proposition 1.7]).

The group C^*-algebra $C^*(G)$ of a simply connected solvable Lie group G has stable rank one if and only if $G \cong \mathbb{R}$ (see [21]).

The group C^*-algebras of the real 3-dimensional Heisenberg Lie group H_3 ($\cong \mathbb{R}^2 \rtimes_\alpha \mathbb{R}$ where $\alpha_a(c,b) = (c, b + ac)$ for $a, b, c \in \mathbb{R}$), the real $ax + b$ group A_2, the real 5-dimensional Mautner group M_5 ($\cong \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$ where $\alpha_t(z,w) = (e^{2\pi i t}z, e^{2\pi i\theta t}w)$ for $t \in \mathbb{R}$, $z, w \in \mathbb{C}$ and θ an irrational number), and the real 7-dimensional Dixmier group D_7 ($\cong \mathbb{C}^2 \rtimes_\beta H_3$ where $\beta_g(z,w) = (e^{ia}z, e^{ib}w)$ for $z, w \in \mathbb{C}$, $g = (c,b,a) \in H_3$) all have stable rank two (see [20] for H_3, [15] for M_5 and [17] for D_7 respectively). Furthermore, we have $H_3/[H_3, H_3] \cong \mathbb{R}^2$, $A_2/[A_2, A_2] \cong \mathbb{R}$, $M_5/[M_5, M_5] \cong \mathbb{R}$, and $D_7/[D_7, D_7] \cong \mathbb{R}^2$. Therefore, the group C^*-algebras $C^*(H_3)$ and $C^*(D_7)$ are of non type R by K-theory.

Furthermore,

Theorem 3.9 *The group C^*-algebra $C^*(M_5)$ of Mautner group M_5 is of non type R by K-theory.*

Proof. Since $M_5 \cong \mathbb{C}^2 \rtimes_\alpha \mathbb{R}$, we have $C^*(M_5) \cong C^*(\mathbb{C}^2) \rtimes_\alpha \mathbb{R} \cong C_0(\mathbb{C}^2) \rtimes_{\hat{\alpha}} \mathbb{R}$, where $\hat{\alpha}$ is the dual action of α via the duality $\mathbb{C}^2 \cong (\mathbb{C}^2)^\wedge$ (the dual group of $\mathbb{C}^2$). Since the subspace $\mathbb{C} \times \{0\}$ in $\mathbb{C}^2$ is closed in $\mathbb{C}^2$ and invariant under $\hat{\alpha}$, we obtain a $*$-homomophism from $C^*(M_5)$ onto the crossed product $C_0(\mathbb{C} \times \{0\}) \rtimes_{\hat{\alpha}} \mathbb{R}$. Since the origin $(0,0)$ in $\mathbb{C} \times \{0\}$ is fixed under $\hat{\alpha}$, we have the following exact sequence:

$$(E): \quad 0 \to C_0(\mathbb{C} \setminus \{0\}) \rtimes_{\hat{\alpha}} \mathbb{R} \to C_0(\mathbb{C} \times \{0\}) \rtimes_{\hat{\alpha}} \mathbb{R} \to C^*(\mathbb{R}) \to 0.$$

Since the action $\hat{\alpha}$ on $\mathbb{C} \setminus \{0\}$ is the rotation, we have $C_0(\mathbb{C} \setminus \{0\}) \rtimes_{\hat{\alpha}} \mathbb{R} \cong C_0(\mathbb{R}) \otimes C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{R}$. Furthermore, the imprimitivity theorem implies

$$C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{R} \cong C(\mathbb{R}/\mathbb{Z}) \rtimes_{\hat{\alpha}} \mathbb{R} \cong C^*(\mathbb{Z}) \otimes \mathbb{K}(L^2(\mathbb{T})),$$

and $C^*(\mathbb{Z}) \cong C(\mathbb{T})$, where $\mathbb{K}(L^2(\mathbb{T})) = \mathbb{K}$ is the C^*-algebra of compact operators on the Hilbert space $L^2(\mathbb{T})$. Thus, the six term exact sequence associated with the exact sequence (E) above is:

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\ {\scriptstyle\partial}\uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z} \end{array}$$

since $K_j(C_0(\mathbb{R}) \otimes C(\mathbb{T}) \otimes \mathbb{K}) \cong K_{j+1}(C(\mathbb{T})) \cong \mathbb{Z}$ for $j = 0, 1$ (mod 1), and

$$K_0(C_0(\mathbb{C}) \rtimes \mathbb{R}) \cong K_1(C_0(\mathbb{R}^2)) \cong K_1(\mathbb{C}) \cong 0,$$
$$K_1(C_0(\mathbb{C}) \rtimes \mathbb{R}) \cong K_0(C_0(\mathbb{R}^2)) \cong K_0(\mathbb{C}) \cong \mathbb{Z},$$

where we are using the Connes' Thom isomorphism for crossed products of C^*-algebras by $\mathbb{R}$ and the Bott periodicity. It follows that the index map ∂ is nonzero. Hence $C^*(M_5)$ is of non type R by K-theory. $\square$

Corollary 3.10 *Let G be a locally compact group that has Mautner group M_5 as a quotient group. Then $C^*(G)$ is of non type R by K-theory.*

Proof. Since there exists a quotient map from G to Mautner group M_5, we have a $*$-homomorphism from $C^*(G)$ to $C^*(M_5)$. $\square$

Moreover, we obtain

Theorem 3.11 *Let $G = \mathbb{R}^n \rtimes_\alpha \mathbb{R}^m$ a simply connected solvable Lie group. Then $C^*(G)$ is of non type R by K-theory.*

In particular, the group C^*-algebra of a generalized Mautner group in the sense of Auslander-Moore is of non type R by K-theory.

Proof. We have the following exact sequence:

$$1 \to \mathbb{R}^n \to G = \mathbb{R}^n \rtimes_\alpha \mathbb{R}^m \to \mathbb{R}^m \to 1.$$

Then we have a $*$-homomorphism from $C^*(G)$ onto $C^*(\mathbb{R}^m)$. Thus, if $m \geq 2$, then $C^*(\mathbb{R}^m)$ is of non type R by K-theory. Therefore, $C^*(G)$ is also of non type R by K-theory.

Suppose that $m = 1$. Then $C^*(G) \cong C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{R}$. Since the origin of $\mathbb{R}^n$ is fixed under the action $\hat{\alpha}$, we have the following exact sequence:

$$0 \to C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R} \to C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{R} \to C^*(\mathbb{R}) \to 0,$$

and its six term exact sequence is:

$$\begin{array}{ccccc} K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) & \longrightarrow & K_0(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{R}) & \longrightarrow & K_0(C_0(\mathbb{R})) \cong 0 \\ {\scriptstyle\partial}\uparrow & & & & \downarrow \\ K_1(C_0(\mathbb{R})) \cong \mathbb{Z} & \longleftarrow & K_1(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{R}) & \longleftarrow & K_1(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes_{\hat{\alpha}} \mathbb{R}) \end{array}$$

and by Connes' Thom isomorphism and Bott periodicity,

$$K_1(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{R}) \cong K_0(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ even}, \\ 0 & \text{if } n \text{ odd}. \end{cases}$$

Hence, if n is odd, the index map ∂ is nonzero. Therefore, $C^*(G)$ is of non type R by K-theory.

Now suppose that $n \geq 2$ is even. Then

$$K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) \cong K_1(C_0(\mathbb{R}^n \setminus \{0\})) \cong K_1(C_0(\mathbb{R}_+ \times S^{n-1}))$$

since $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to the product space $\mathbb{R}_+ \times S^{n-1}$, where S^{n-1} is the $(n-1)$-dimensional sphere. Thus,

$$K_1(C_0(\mathbb{R}_+ \times S^{n-1})) \cong K_0(C(S^{n-1})) \cong \mathbb{Z}$$

since $n - 1$ is odd. Also,

$$K_0(C_0(\mathbb{R}^n) \rtimes_{\hat{\alpha}} \mathbb{R}) \cong K_1(C_0(\mathbb{R}^n)) \cong K_1(\mathbb{C}) \cong 0$$

since n is even. It follows that the index map ∂ is nonzero. Hence $C^*(G)$ is of non type R by K-theory.

In particular, a generalized Mautner group in the sense of Auslander-Moore is a simply connected solvable Lie group of the form $\mathbb{R}^n \rtimes_\alpha \mathbb{R}^m$ by an action α by orthogonal matrices acting on $\mathbb{R}^n$ such as Mautner group M_5. $\square$

Theorem 3.12 *Let G be a locally compact group that has a semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}^m$ as a quotient group. Then $C^*(G)$ is non type R by K-theory.*

In particular, for G a simply connected solvable Lie group of type R, if it is not semi-simply regular, then $C^*(G)$ is non type R by K-theory.

Proof. It is shown by [1] that for G a simply connected solvable Lie group of type R, if G is not semi-simply regular, then there exists a homomorphism from G to a generalized Mautner group. $\square$

Remark. Recall that for G a solvable Lie group, there exists a unique nilpotent Lie group N (the nilradical of G) and a unique abelian semi-simple group S of automorphisms of N such that G is contained in NS. If S is regular, then G is said to be semi-simply regular. If G is not semi-simply regular, then N is not regularly embedded in G.

Moreover, in fact we can show the following result:

Theorem 3.13 *Let G be a simply connected solvable Lie group that is not isomorphic to $\mathbb{R}$. Then $C^*(G)$ is of non type R by K-theory.*

Proof. We may assume that G is noncommutative. It is known that G is isomorphic to a semi-direct product $H \rtimes_\alpha \mathbb{R}$ for H a simply connected solvable Lie group with dimension $\dim G - 1$. Then $C^*(G) \cong C^*(H) \rtimes_\alpha \mathbb{R}$. Since H is a simply connected solvable Lie group, it follows that $H/[H, H]$ is isomorphic to $\mathbb{R}^k$ for some $k \geq 1$. Since the space of 1-dimensional representations of H is homeomorphic to $\mathbb{R}^k$ and invariant under the action α, there exists a $*$-homomorphism from $C^*(G)$ onto $C_0(\mathbb{R}^k) \rtimes_\alpha \mathbb{R}$. Hence $C^*(G)$ is of non type R weakly by K-theory because we have shown above that the crossed product $C_0(\mathbb{R}^k) \rtimes_\alpha \mathbb{R}$ is of non type R by K-theory. $\square$

As for noncompact connected solvable or amenable Lie groups,

Example 3.14 If $G = \mathbb{R} \times K$ for a compact group K, then $C^*(G) \cong C_0(\mathbb{R}) \otimes C^*(K)$. Since $C^*(K)$ is AF, it follows that $C^*(G)$ (or its unitization by $\mathbb{C}$) has stable rank one. Therefore, $C^*(G)$ is of type R by K-theory.

Let $G = \mathbb{C} \rtimes_\alpha \mathbb{T}$, where the action α is defined by $\alpha_z x = zx$ for $z \in \mathbb{T}$ and $x \in \mathbb{C}$. Then it is shown by [18] that the group C^*-algebra $C^*(G)$ has stable rank one. Therefore, by Theorem 2.6 $C^*(G)$ is of type R by K-theory.

Moreover, let $G = \mathbb{R}^n \rtimes_\alpha SO(n)$, where the action α is defined by $\alpha_z x = zx$ for $z \in SO(n)$ and $x \in \mathbb{R}^n$. The group G is called the motion group and is non-solvable and amenable if $n \geq 3$. Then it is shown by [18] that the group C^*-algebra $C^*(G)$ has stable rank one. Therefore, by Theorem 2.6 $C^*(G)$ is of type R by K-theory.

Furthermore, constructed by [18] are some connected successive semi-direct products (solvable or amenable) such that their group C^*-algebras have stable rank one. Therefore, they are of type R by K-theory.

As for non-amenable Lie groups,

Theorem 3.15 *Let G be a noncompact, connected semi-simple Lie group. If G has real rank ≥ 2, then its reduced group C^*-algebra $C_r^*(G)$ is of non type R by K-theory.*

Also, if G has real rank one, then $C_r^*(G)$ is of type R by K-theory.

Proof. Recall that G has the Iwasawa decomposition KAN. The real rank of G is defined to be the real dimension of A a simply connected commutative Lie group. It is known that a quotient space $A^\wedge/W$ of the dual group $A^\wedge$ of A by the Weyl group W (a finite group) is identified with a closed and open subset of the spectrum of $C_r^*(G)$ (or the reduced dual space of G) (see [14]). Thus $C_r^*(G)$ has a direct summand $\mathfrak{D}$ that has $A^\wedge/W$ as a spectrum. Since $A^\wedge/W$ is a locally compact Hausdorff space and G is CCR, $\mathfrak{D}$ has continuous trace so that $\mathfrak{D}$ is isomorphic to the C^*-algebra of a continuous field on $A^\wedge/W$ with fibers $\mathbb{K}$. Since G has real rank ≥ 2, i.e. $\dim A^\wedge \geq 2$, we have $\dim A^\wedge/W \geq 2$. Thus, there exists a quotient C^*-algebra $\mathfrak{B}$ of $\mathfrak{D}$ such that $\mathfrak{B}$ is isomorphic to $C_0([0,1]^s) \otimes \mathbb{K}$ for $s = \dim A^\wedge$. Hence it follows that $C_r^*(G)$ is of non type R by K-theory since $\mathfrak{B}$ is so.

It is shown that if G has real rank one, then $C_r^*(G)$ has stable rank one ([14]). Therefore, $C_r^*(G)$ is of type R by K-theory. □

Corollary 3.16 *Let G be a noncompact, locally compact group that has a connected semi-simple Lie group S of real rank ≥ 2 as a quotient group and as a semi-simple part. If the kernel for the quotient map from G to S is amenable, then $C^*(G)$ and $C_r^*(G)$ are of non type R by K-theory.*

In particular, if G is a noncompact, connected reductive Lie group whose semi-simple part is $SL_n(\mathbb{R})$ $(n \geq 3)$, then $C^*(G)$ and $C_r^*(G)$ are of non type R by K-theory.

Proof. If G has such S as a quotient, then G is nonamenable. Then we have the $*$-homomorphism from $C^*(G)$ onto $C_r(G)$ (see [11, Sections 7.2 and 7.3]). Since there exists a quotient map from G to S whose kernel is amenable, the reduced dual of S is contained in that of G (this is standard in the unitary representation of locally compact groups) so that we have a $*$-homomorphism from $C_r(G)$ onto $C_r^*(S)$. □

Example 3.17 Let $GL_n(\mathbb{R})_0$ be the connected component of $GL_n(\mathbb{R})$ cotaining the identity matrix. Since $GL_n(\mathbb{R})_0$ is a reductive Lie group and has a qutient map onto $SL_n(\mathbb{R})$, it follows that there exists a $*$-homomorphism from $C_r^*(GL_n(\mathbb{R})_0)$ onto $C_r^*(SL_n(\mathbb{R}))$. Thus, if $n \geq 3$, then $C_r^*(GL_n(\mathbb{R})_0)$ is of non type R by K-theory. Hence $C^*(GL_n(\mathbb{R})_0)$ is also of non type R by K-theory.

We consider the case $n = 2$. Since $GL_n(\mathbb{R})_0 \cong \mathbb{R}_+ \times SL_2(\mathbb{R})$ via the determinant, we have $C^*(GL_2(\mathbb{R})_0) \cong C^*(\mathbb{R}) \otimes C^*(SL_2(\mathbb{R}))$. It follows that $C_r^*(GL_2(\mathbb{R})_0) \cong$

$C^*(\mathbb{R}) \otimes C_r^*(SL_2(\mathbb{R}))$. Since $SL_2(\mathbb{R})$ has real rank one, it follows that there exists a quotient C^*-algebra of $C_r^*(SL_2(\mathbb{R}))$ that is isomorphic to $C([0,1]) \otimes \mathbb{K}$. Therefore, $C_r^*(GL_2(\mathbb{R})_0)$ has a quotient C^*-algebra that is isomorphic to $C_0(\mathbb{R}\times[0,1])\otimes\mathbb{K}$. Since this quotient C^*-algebra is of non type R by K-theory, it follows that $C_r^*(GL_2(\mathbb{R})_0)$ is of non type R by K-theory.

It is shown by [14] that $C_r^*(GL_n(\mathbb{R})_0)$ $(n \geq 2)$ have stable rank two.

On the other hand, let $G = \mathbb{R}^2 \rtimes_\alpha SL_2(\mathbb{R})$, where $\alpha_g x = gx$ for $g \in SL_2(\mathbb{R})$ and $x \in \mathbb{R}^2$. Then it shown by [18] that the reduced group C^*-algebra $C_r^*(G)$ of G has stable rank one. Thus, $C_r^*(G)$ is of type R by K-theory by Theorem 2.6.

However, let $G = \mathbb{R}^n \rtimes_\alpha SL_n(\mathbb{R})$ $(n \geq 3)$, where $\alpha_g x = gx$ for $g \in SL_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. Then $C_r^*(SL_n(\mathbb{R}))$ is a quotient of $C_r^*(G)$. By the corollary above, $C^*(G)$ and $C_r^*(G)$ are of non type R by K-theory.

4 Another notion

Definition 4.1 Let $\mathfrak{A}$ be a C^*-algebra. We say that $\mathfrak{A}$ (or its unitization) is of non type R by Fredholm index if there exists a quotient C^*-algebra $\mathfrak{B}$ that has a faithful representation on a Hilbert space under which the image of $\mathfrak{B}$ contains a Fredholm operator with its Fredholm index nonzero.

If $\mathfrak{A}$ is not of non type R by Fredholm index, then we say that $\mathfrak{A}$ is of type R by Fredholm index.

We have already shown that

Theorem 4.2 [16] *Let G be a simply connected solvable Lie group of non type R. Then the unitization of $C^*(G)$ of G is of non type R by Fredholm index.*

Moreover, we showed

Theorem 4.3 [16] *Let $\mathfrak{A}$ be a C^*-algebra of non type R by Fredholm index. Then $\mathfrak{A}$ is not ASH, where a C^*-algebra is ASH if it is isomorphic to an inductive limit of subhomogeneous C^*-algebras.*

On the other hand, we showed

Theorem 4.4 [16] *Let $\mathfrak{A}$ be a CCR C^*-algebra. Then $\mathfrak{A}$ is of type R by Fredholm index.*

Remark. It is shown in [19] that a CCR C^*-algebra is ASH.

Furthermore, we can show

Theorem 4.5 *Any ASH C^*-algebra is of type R by Fredholm index. Also, inductive limits of CCR C^*-algebras or ASH C^*-algebras are of type R by Fredholm index.*

Proof. Let $\mathfrak{A}$ be an inductive limit of ASH C^*-algebras $\mathfrak{A}_j$. We may assume that $\mathfrak{A}$ and $\mathfrak{A}_j$ are unital by the common unit. Suppose that $\mathfrak{A}$ is of non type R by Fredholm index. Then $\mathfrak{A}_j$ for j large enough must be of non type R by Fredholm index. This contradicts to Theorem 4.3. □

Corollary 4.6 *Let G be a simply connected solvable Lie group of non type R. Then the unitization of $C^*(G)$ of G is not an inductive limit of ASH C^*-algebras. Also, $C^*(G)$ is not.*

Proposition 4.7 *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has a quotient C^*-algebra $\mathfrak{B}$ that acts irreducibly on a Hilbert space H and contains a Fredoholm operator on H with index nonzero, then $\mathfrak{A}$ is of non type R by K-theory.*

Proof. Note that we may assume that $\mathfrak{B}$ is unital. By assumption, let T be a Fredholm operator of $\mathfrak{B}$ with index nonzero. Then the operator $1-TT^*$ is a compact operator on H. Since $\mathfrak{B}$ acts irreducibly on H, $\mathfrak{B}$ must contain $\mathbb{K}(H)$ the C^*-algebra of all compact operators on H. Therefore, $\mathfrak{B}$ is decomposed into the exact sequence:

$$0 \to \mathbb{K}(H) \to \mathfrak{B} \to \mathfrak{B}/\mathbb{K}(H) \to 0,$$

and it is easy to see that the index map of its six term exact sequence must be nonzero. In fact, the index map is just the Fredholm index for Fredholm operators in such a situation. □

Proposition 4.8 *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has an irreducible representation π of type I but not of CCR and the image $\pi(\mathfrak{A})$ of $\mathfrak{A}$ contains a Fredoholm operator on the representation space H_π of π with index nonzero, then $\mathfrak{A}$ is of non type R by K-theory.*

Proof. Let π be the irreducible representation of $\mathfrak{A}$ in the statement. Since π is of type I, the image $\pi(\mathfrak{A})$ of $\mathfrak{A}$ contains $\mathbb{K}(H_\pi)$ the C^*-algebra of all compact operators on the representation space H_π of π. Since π is not of CCR, $\mathbb{K}(H_\pi)$ is strictly contained in $\pi(\mathfrak{A})$. Thus, we have the following exact sequence:

$$0 \to \mathbb{K}(H_\pi) \to \pi(\mathfrak{A}) \to \pi(\mathfrak{A})/\mathbb{K}(H_\pi) \to 0.$$

Since $\pi(\mathfrak{A})$ contains a Fredoholm operator on H_π with index nonzero, we can deduce the conclusion as proved in the proposition above. □

Theorem 4.9 *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{A}$ has stable rank one, then it is of type R by Fredholm index.*

Proof. Note that a C^*-algebra $\mathfrak{A}$ has stable rank one if and only if the set of invertible elements of $\mathfrak{A}$ is dense in $\mathfrak{A}$. Therefore, $\mathfrak{A}$ with stable rank one does not contain a Fredholm operator with nonzero index. In fact, suppose that $\mathfrak{A}$ has a Fredholm operator T with nonzero index n on a Hilbert space H. Since the set $F_n(H)$ of all Fredholm operators with index n is open in $\mathbb{B}(H)$, the intersection $F_n(H) \cap \mathfrak{A}$ must be open in $\mathfrak{A}$. This contradicts to that the set of invertible elements of $\mathfrak{A}$ is dense in $\mathfrak{A}$ since invertible Fredholm operators have index zero.

Furthermore, since $\mathrm{sr}(\mathfrak{A}) = 1$, it follows that any quotient C^*-algebra of $\mathfrak{A}$ has stable rank one by [12, Theorem 4.3]. □

Furthermore, we give

Definition 4.10 *Let $\mathfrak{A}$ be a C^*-algebra. We say that $\mathfrak{A}$ is of type R if it is ASH.*

Remark. This notion seems to be quite suitable by the following reason. It is known that CCR groups are of type R, and connected solvable Lie groups of type R are of type I if and only if they are CCR (see [1]). On the other hand, we obtain that CCR C^*-algebras are ASH, and ASH C^*-algebras are of type I if and only if they are CCR (see [16] and [19]). Therefore, the notion for groups to be of type R just corresponds to the notion for C^*-algebras to be of type R, i.e., ASH. More reasons might be waiting to be discovered.

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