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A Numerical Verification Method for Two-Coupled Elliptic Partial Differential Equations

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A numerical verification method of steady state solutions for a system of reaction-diffusion equations is described. Using a decoupling technique, the system is reduced to a single nonlinear equation and a computer-assisted method for second-order elliptic boundary value problems based on the infinite dimensional fixed-point theorem can be applied. Some numerical examples confirm the effectiveness of the method.

Key words: numerical verification, two-coupled equations, fixed-point theorem

1. Introduction

Consider the following system of two-coupled elliptic partial differential equations:

$$\begin{cases} -\varepsilon^2 \Delta u = f(u) - \delta v & \text{in } \Omega, \\ -\Delta v = u - \gamma v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded convex domain in $\mathbb{R}^n$ $(n = 1, 2, 3)$ with piecewise smooth boundary $\partial\Omega$, the parameters $\varepsilon \neq 0$, γ and δ are real and the map f is assumed to satisfy appropriate conditions described later.

The elliptic system (1) represents a steady state case of reaction-diffusion system of interest in mathematical biology [16]. Here, real functions u and v, which are called the activator and inhibitor respectively, can be interpreted as relative concentrations of two substances known as morphogens, and the function f models autocatalytic and saturation effects [12]. A lot of research has been focused on the reaction-diffusion system (1) from theoretical and numerical sides [2, 4, 11, 12, 16, 20].

The aim of this paper is to propose a numerical method to prove the existence of the solutions of the system (1) near an approximate solution obtained by a usual floating point computation. This method is also called computer-assisted proof. The method is based on the infinite dimensional fixed-point theorem using Newton-like operator and the constructive error estimates. All numerical results discussed take into account of the effects of rounding errors in the floating point computations.

The contents of this paper are as follows. A fixed-point formulation and an existence theorem in certain appropriate function spaces using Newton-like iteration is considered in Section 2. Estimation of a linear boundary value problem in decoupling technique is described in Section 3. Finally, some numerical results which prove the existence of solutions are presented in Section 4.

2. Fixed-point formulation

For some integer m, let $H^m(\Omega)$ denote the L^2-Sobolev space of order m on Ω. We shall find weak solutions of the eq. (1) in

$$H_0^1(\Omega) := \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$$

with the inner product $(\nabla u, \nabla v)_{L^2}$ and the norm $\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)}$ where $(u, v)_{L^2}$ implies L^2-inner product on Ω.

Let S_h be an approximate finite dimensional subspace of $H_0^1(\Omega)$ dependent on the parameter h. For example, S_h is taken to be a finite element subspace with mesh size h or a set of finite Fourier expansion with truncation number N $(= 1/h)$.

We apply decoupling technique which reduces the system (1) to a single nonlinear equation [12] using the particular property that one of the two equations is linear. Although it would be possible to treat problem (1) as a system, the decoupling technique has the advantages from a computational point of view. Assume that when u to be known, the boundary value problem:

$$\begin{cases} -\Delta v + \gamma v = u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

is uniquely solved by

$$v = Bu,$$

where the operator B is bounded invertible linear operator from $L^2(\Omega)$ into $H_0^1(\Omega)$ in the weak sense. If $\gamma < 0$, this assumption is not trivial. We will propose a validation method to check the solvability of problem (2) and an upper bound of $\|Bw\|_{H_0^1(\Omega)}$ in Section 3.

Substituting $v = Bu$ into the first equation of (1), the problem

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2}(f(u) - \delta Bu) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

is arrived. Here, numerical verification methods for nonlinear elliptic boundary value problems [9, 10, 19] can be applied to the single equation (3). Here we adopt computer-assisted approaches which have been developed by Nakao and coauthors [6, 7, 8, 18]. For the self-containedness of the paper, we shall sketch the verification method briefly.

The nonlinear operator $g\colon H_0^1(\Omega) \to L^2(\Omega)$ defined by

$$g(u) := \frac{1}{\varepsilon^2}(f(u) - \delta Bu)$$

is supposed to be continuous, Fréchet differentiable, and also g maps bounded sets in $H_0^1(\Omega)$ into bounded set in $L^2(\Omega)$. For $\xi \in L^2(\Omega)$ let $A\xi$ be the solution of $-\Delta\psi = \xi$, $x \in \Omega$ and $\psi = 0$, $x \in \partial\Omega$, then the operator $A\colon L^2(\Omega) \to H_0^1(\Omega)$ is compact because the compactness of the imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$. Then the nonlinear operator $F := A \circ g$ is also a compact map on $H_0^1(\Omega)$, and the weak form of eq. (3) can be rewritten equivalently as the fixed-point form $u = Fu$ in $H_0^1(\Omega)$.

Next we introduce the Newton-like method. Let $P_h\colon H_0^1(\Omega) \to S_h$ denote the H_0^1-projection defined by

$$(\nabla(w - P_h w), \nabla\phi)_{L^2} = 0, \quad \forall\phi \in S_h, \tag{4}$$

and suppose the following approximation property of P_h:

$$\|w - P_h w\|_{H_0^1(\Omega)} \le Ch|w|_{H^2(\Omega)}, \quad \forall w \in H_0^1(\Omega) \cap H^2(\Omega), \tag{5}$$

where $C > 0$ is a positive constant whose concrete upper bound can be estimated and $|\cdot|_{H^2(\Omega)}$ means the semi-norm on $H^2(\Omega)$ defined by $|w|^2_{H^2(\Omega)} = \sum_{i,j=1}^n \|\partial^2 w/\partial x_i \partial x_j\|^2_{L^2(\Omega)}$. This assumption holds for many finite element subspaces of $H_0^1(\Omega)$ or function spaces of Fourier series with finite truncation.

Since S_h is the closed subspace of $H_0^1(\Omega)$, each element of $H_0^1(\Omega)$ can be uniquely represented as the direct sum of the element of S_h and $S_h^\perp$. Here $S_h^\perp$ stands for the orthogonal complement subspace of S_h in $H_0^1(\Omega)$. Therefore, the fixed-point equation $u = Fu$ in $H_0^1(\Omega)$ can also be uniquely decomposed as the finite dimensional (projection) part and the infinite dimensional (error) part such that

$$\begin{cases} P_h u = P_h F u, \\ (I - P_h)u = (I - P_h)Fu. \end{cases} \tag{6}$$

In order to obtain a solution satisfying eq. (6), we fix an approximate weak solution $u_h \in S_h$ of eq. (3) and define the nonlinear operator $N_h\colon H_0^1(\Omega) \to S_h$ by

$$N_h u := P_h u - L_h^{-1} P_h (u - Fu),$$

where $L_h^{-1}\colon S_h \to S_h$ means the inverse of the operator

$$L_h := P_h\left(I - \frac{1}{\varepsilon^2} A(f'(u_h) - \delta B_h)\right)$$

on S_h and B_h on S_h maps element w_h to the z_h satisfying

$$(\nabla z_h, \nabla x_h)_{L^2} + \gamma(z_h, x_h)_{L^2} = (w_h, x_h)_{L^2}, \quad \forall x_h \in S_h. \tag{7}$$

Note that existence of L_h^{-1} is equivalent to the invertibility of a matrix which is able to be numerically checked in the actual verification process. Because of $P_h u = P_h F u$ and $P_h u = N_h u$ are equivalent, defining the compact operator T on $H_0^1(\Omega)$ by

$$Tu := N_h u + (I - P_h)Fu, \tag{8}$$

two fixed-point problems $u = Tu$ and $u = Fu$ are also equivalent. Therefore Schauder's fixed-point theorem asserts that if for a nonempty, bounded, convex and closed set $U \subset H_0^1(\Omega)$,

$$TU = \{Tu \mid u \in U\} \subset U,$$

holds, then there exists a fixed-point of T in U. We call such a set U expected to be $TU \subset U$ as a *candidate set*.

When a candidate set U is chosen such as

$$U := u_h + U_h + U_h^{\perp}, \quad U_h \subset S_h,\ U_h^{\perp} \subset S_h^{\perp}, \tag{9}$$

a sufficient condition for $TU \subset U$ can be written by

$$\begin{cases} N_h U - u_h \subset U_h, \\ (I - P_h)FU \subset U_h^{\perp} \end{cases} \tag{10}$$

[18]. Note that when the approximate solution $u_h \in S_h$ is sufficiently good, the finite dimensional part will be possibly contractive. On the other hand, the magnitude of the infinite dimensional part of T, i.e., $(I - P_h)Fu$, is expected to be small when the parameter h of S_h are taken to be sufficiently small because of the approximation property (5) of P_h.

3. Validation of linear operator

This section describes the estimation of B for the boundary value problem (2), namely, we consider the existence of a weak solution $v \in H_0^1(\Omega)$ and give an upper bound of $\|v\|_{H_0^1(\Omega)}$ for the problem:

$$\begin{cases} -\Delta v + \gamma v = u_h + \hat{u} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where $\hat{u} \in L^2(\Omega)$. Here, note that the existence of a solution of eq. (11) derives the invertibility of B because B is linear. In actual computation, $\hat{u}$ should be taken as $\hat{u}_h + u_*$, $\hat{u}_h \in U_h$, $u_* \in U_h^{\perp}$, therefore an upper bound of $\|\hat{u}\|_{L^2(\Omega)}$ is known. In the same finite element subspace we compute an approximate solution for $\hat{u} = 0$ in eq. (11) and fix by $v_h \in S_h$. As the similar formulation in the previous section, a fixed-point form $v = \hat{F}v$ in $H_0^1(\Omega)$ is obtained using $\hat{F}v := A(-\gamma v + u_h + \hat{u})$ and also $v = \hat{F}v$ can be decomposed as

$$\begin{cases} P_h v = P_h \hat{F} v, \\ (I - P_h)v = (I - P_h)\hat{F}v. \end{cases}$$

The finite dimensional part $P_h v = P_h \hat{F} v$ is rewritten as Newton-type fixed-point form $P_h v = \hat{N}_h v$ using

$$\hat{N}_h v := [I + \gamma P_h A]_h^{-1} P_h A(u_h + \hat{u} - \gamma(I - P_h)v),$$

where $[I + \gamma P_h A]_h^{-1}$ means the inverse on S_h of the restriction operator $P_h(I + \gamma P_h A)|_{S_h}$. The invertibility of $P_h(I + \gamma P_h A)|_{S_h}$ is equivalent to the existence of G^{-1} described later.

Now we prepare the candidate set $V = v_h + V_h + V_h^{\perp} \subset H_0^1(\Omega)$ as V_h and $V_h^{\perp}$ are balls with radius $\alpha > 0$ and $\beta > 0$ of the form

$$V_h = \{\hat{v}_h \in S_h \mid \|\hat{v}_h\|_{H_0^1(\Omega)} \le \alpha\}, \quad V_h^{\perp} = \{v_* \in S_h^{\perp} \mid \|v_*\|_{H_0^1(\Omega)} \le \beta\}. \tag{12}$$

Then defining the compact operator $\hat{T}$ on $H_0^1(\Omega)$ by $\hat{T}u := \hat{N}_h u + (I - P_h)\hat{F}u$, two fixed-point problems $u = \hat{T}u$ and $u = \hat{F}u$ are equivalent, and the verification condition:

$$\begin{cases} \hat{N}_h V - v_h \subset V_h, \\ (I - P_h)\hat{F}V \subset V_h^{\perp} \end{cases} \tag{13}$$

assures the existence of the solution of the problem (11) in the set $V \subset H_0^1(\Omega)$. From the definition of $[I+\gamma P_h A]_h^{-1}$ and P_h, a sufficient condition of the verification condition (13) is

$$\begin{cases} \rho \sup\limits_{v \in V} \|P_h A(u_h + \hat{u} - \gamma(I - P_h)v - \gamma v_h) - v_h\|_{H_0^1(\Omega)} \le \alpha, \\ Ch \sup\limits_{v \in V} \|-\gamma v + u_h + \hat{u}\|_{L^2(\Omega)} \le \beta, \end{cases} \tag{14}$$

where $\rho := \|L^{\mathrm{T}} G^{-1} L\|_{\mathrm{E}}$, $\|\cdot\|_{\mathrm{E}}$ is the Euclidean norm of $\mathbb{R}^K$, $K := \dim S_h$, G is $K \times K$ matrix defined by $G_{ij} = (\nabla\phi_j, \nabla\phi_i)_{L^2} + \gamma(\phi_j, \phi_i)_{L^2}$, D is $K \times K$ positive definite matrix defined by $D_{ij} = (\nabla\phi_j, \nabla\phi_i)_{L^2}$ and L is $K \times K$ lower triangle matrix by the Cholesky decomposition: $D = LL^{\mathrm{T}}$ [8] and $\{\phi_i\}_{1 \le i \le K}$ is a basis of S_h. Here, a verified enclosure of G^{-1} can be obtained automatically in computer by interval arithmetic (for example see Rump [13]). Also, interval Cholesky decomposition algorithm [1] is usually feasible because of the positive definiteness of D. The estimate of ρ is, generally, singular value problem of a matrix. There are some verification algorithms to estimate rigorous bound for the maximum (or minimum) singular value [14].

The first term of eq. (14) is estimated as

$$
\begin{aligned}
&\|P_hA(u_h+\hat{u}-\gamma(I-P_h)v-\gamma v_h)-v_h\|_{H_0^1(\Omega)}\\
&\le \|P_hA(u_h-\gamma v_h)-v_h\|_{H_0^1(\Omega)}+\|P_hA\hat{u}\|_{H_0^1(\Omega)}+|\gamma|\,\|P_hA(I-P_h)v\|_{H_0^1(\Omega)}. \quad (15)
\end{aligned}
$$

By setting

$$
u_h=\sum_{i=1}^{K}u_i\phi_i,\quad \boldsymbol{u}:=(u_i)\in\mathbb{R}^K,\quad v_h=\sum_{i=1}^{K}v_i\phi_i,\quad \boldsymbol{v}:=(v_i)\in\mathbb{R}^K,
$$

$$
r_h:=P_hA(u_h-\gamma v_h)-v_h=\sum_{i=1}^{K}r_i\phi_i,\quad \boldsymbol{r}:=(r_i)\in\mathbb{R}^K,\quad M_{ij}=(\phi_j,\phi_i)_{L^2},
$$

from the definition of P_h and A, it holds that

$$
(\nabla r_h,\nabla\phi_i)_{L^2}=-(\nabla v_h,\nabla\phi_i)_{L^2}+(u_h-\gamma v_h,\phi_i)_{L^2},\quad 1\le i\le K,
$$

then r_h can be obtained by

$$
\boldsymbol{r}=D^{-1}(-G\boldsymbol{v}+M\boldsymbol{u}).
$$

Therefore we have

$$
r:=\|P_hA(u_h-\gamma v_h)-v_h\|_{H_0^1(\Omega)}=\sqrt{\boldsymbol{r}^{\mathrm{T}}D\boldsymbol{r}}.
$$

Note that r is expected to be small because v_h is the approximate solution of equation (11) with $\hat{u}=0$.

Next, using the Poincaré constant $C_2>0$ such that

$$
\|v\|_{L^2(\Omega)}\le C_2\|v\|_{H_0^1(\Omega)},\quad v\in H_0^1(\Omega), \tag{16}
$$

we get

$$
\begin{aligned}
\|P_hA\hat{u}\|_{H_0^1(\Omega)}^2 &\le \|A\hat{u}\|_{H_0^1(\Omega)}^2\\
&=(\nabla A\hat{u},\nabla A\hat{u})_{L^2}\\
&=(\hat{u},A\hat{u})_{L^2}\\
&\le \|\hat{u}\|_{L^2(\Omega)}\|A\hat{u}\|_{L^2(\Omega)}\\
&\le C_2\|\hat{u}\|_{L^2(\Omega)}\|A\hat{u}\|_{H_0^1(\Omega)},
\end{aligned}
$$

then

$$
\|P_hA\hat{u}\|_{H_0^1(\Omega)}\le C_2\|\hat{u}\|_{L^2(\Omega)}
$$

holds. Similar procedure and Aubin–Nitsche's technique imply

$$
\|P_hA(I-P_h)v\|_{H_0^1(\Omega)}\le C_2Ch\beta.
$$

Then for $v\in V$ we have

$$
\|P_hA(u_h+\hat{u}-\gamma(I-P_h)v-\gamma v_h)-v_h\|_{H_0^1(\Omega)}\le r+C_2(\|\hat{u}\|_{L^2(\Omega)}+|\gamma|Ch\beta).
$$

On the other hand, the second term of inclusions (14) is estimated as

$$\|-\gamma v + u_h + \hat{u}\|_{L^2(\Omega)} \le \|-\gamma v_h + u_h\|_{L^2(\Omega)} + |\gamma|(C_2\alpha + Ch\beta) + \|\hat{u}\|_{L^2(\Omega)},$$

the sufficient conditions for (13) are

$$\begin{cases} \rho(r + C_2(\|\hat{u}\|_{L^2(\Omega)} + |\gamma|Ch\beta)) \le \alpha, \\ Ch(\|-\gamma v_h + u_h\|_{L^2(\Omega)} + |\gamma|(C_2\alpha + Ch\beta) + \|\hat{u}\|_{L^2(\Omega)}) \le \beta, \end{cases}$$

and the following theorem is led immediately.

Theorem 3.1. *If the inequality*

$$1 - C^2h^2|\gamma|\big(1 + |\gamma|C_2^2\rho\big) > 0$$

is satisfied, the equation (11) *has the solution in the set V defined by* (12) *with*

$$\beta = \frac{Ch(\|-\gamma v_h + u_h\|_{L^2(\Omega)} + |\gamma|C_2\rho(r + C_2\|\hat{u}\|_{L^2(\Omega)}) + \|\hat{u}\|_{L^2(\Omega)})}{1 - C^2h^2|\gamma|(1 + |\gamma|C_2^2\rho)},$$
$$\alpha = \rho(r + C_2(\|\hat{u}\|_{L^2(\Omega)} + |\gamma|Ch\beta)).$$

Note that if $-\gamma$ is an eigenvalue of $-\Delta$, the matrix G has the singularity as $h \to 0$ and the estimation ρ can not be obtained. In the case $\gamma = 0$, it is easily checked that $\rho = 1$ and Theorem 3.1 implies

$$\alpha = r + C_2\|\hat{u}\|_{L^2(\Omega)}, \quad \beta = Ch(\|u_h\|_{L^2(\Omega)} + \|\hat{u}\|_{L^2(\Omega)}).$$

When consider the residual type problem [6], the right hand side $\hat{g} := u_h + \hat{u}$ of eq. (11) happen to be very small norm. In this case we can take $u_h = v_h = 0$ and the radius of the candidate set can be taken by

$$\beta = \frac{Ch\|\hat{g}\|_{L^2(\Omega)}(|\gamma|C_2^2\rho + 1)}{1 - C^2h^2|\gamma|(1 + |\gamma|C_2^2\rho)}, \quad \alpha = \rho(C_2(\|\hat{g}\|_{L^2(\Omega)} + |\gamma|Ch\beta)).$$

4. Numerical examples

We now give some verification results which prove the existence of solutions. The candidate set $U \subset H_0^1(\Omega)$ of eq. (9) is defined by

$$U_h = \sum_{i=1}^{K} [\underline{Y}_i, \overline{Y}_i]\phi_i,$$
$$U_h^{\perp} = \{u_* \in S_h^{\perp} \mid \|v_*\|_{H_0^1(\Omega)} \le \eta\},$$

where $\{\phi_1, \ldots, \phi_K\}$ is a basis of S_h, and $[\underline{Y}_i, \overline{Y}_i]$ $(1 \le i \le K)$ means an interval coefficient and $\eta > 0$.

4.1. One dimensional case

Consider FitzHugh–Nagumo type model problem [16]:

$$\begin{cases} -\varepsilon^2 u'' = u(1-u)(u-a) - \delta v & \text{for } -1 \le x \le 1, \\ -v'' = u - \gamma v & \text{for } -1 \le x \le 1, \\ u = v = 0 & \text{for } x \in \{-1, 1\}, \end{cases} \tag{17}$$

where $0 < a \le 1/2$. The interval $(-1, 1)$ is divided into N equal parts and S_h is taken as the set of piecewise linear functions on each subinterval. Then $K = \dim S_h = N - 1$, $h = 2/N$, $C_2 = 2/\pi$ and a priori constant C in (5) can be taken as $1/\pi$ [5]. The interval arithmetic in each verification step was implemented using Sun ONE Studio 7, Compiler Collection Fortran 95 on Fujitsu PRIMEPOWER 850 (CPU: SPARC64-GP 1.3 GHz, OS: Solaris 8). The approximate solutions u_h and v_h were obtained by Newton–Raphson method using usual floating point arithmetic by double precision.

At the first, we try the case $\delta = 0$. Then the system (17) decouples and reduces to the single equation. The boundary value problem satisfied by the u is well studied, and it is known that if $\varepsilon > 0$ is sufficiently small, there are exactly two nontrivial solutions [15]. Fig. 1 shows the shape of these two approximate solutions u_h and corresponding v_h. The left solution in Fig. 1 is of boundary layer type (BL); the right is a so-called peak-solution (PS). Table 1 shows verification results. In the table $\|\cdot\|_\infty$ means the upper bound of L^∞-norm on $(-1, 1)$. We use the estimate $\|u\|_\infty \le (1/\sqrt{2})\|u\|_{H_0^1(\Omega)}$ for the verified candidate set. The last digit in the mantissa for each of the norm values is rounded-up.

Next, we consider the case $\delta \ne 0$. It was proved that if $\delta = 0$, there is a critical value ε_c in the interval $(0, 0.2387)$ and when $\varepsilon > \varepsilon_c$, no positive solution exists [12]. However, setting $\varepsilon = 0.25$, we can assure that non-trivial solutions do exist for a

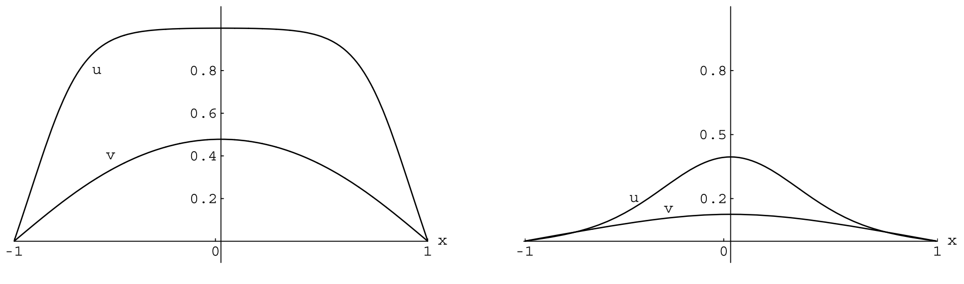

Fig. 1. Approximate solutions for $\delta = 0$, $\varepsilon = 0.1$, $\gamma = -0.05$ and $a = 1/2$.

Table 1. Verification results for $\delta = 0$, $\varepsilon = 0.1$, $\gamma = -0.05$, $a = 1/2$, $N = 1500$.

Type	$\|u_h\|_\infty$	$\|U_h\|_\infty$	$\|U_h^\perp\|_\infty$	$\|v_h\|_\infty$	$\|V_h\|_\infty$	$\|V_h^\perp\|_\infty$
BL	0.99876	1.01499×10^{-4}	1.02739×10^{-2}	0.47804	1.68044×10^{-4}	1.03667×10^{-3}
PS	0.39515	6.07108×10^{-3}	6.27672×10^{-3}	0.12623	7.03572×10^{-3}	2.71132×10^{-4}

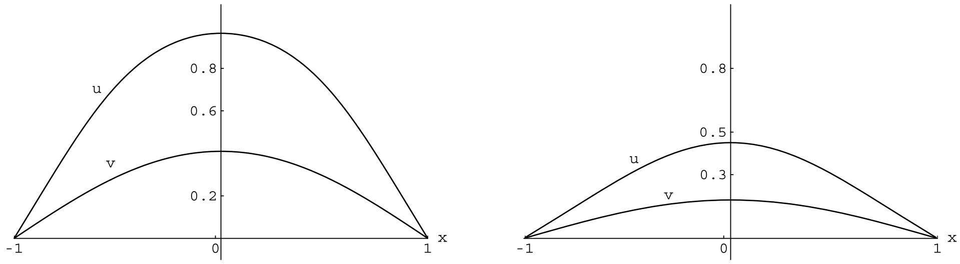

Fig. 2. Approximate solutions for $\delta = -0.2$, $\varepsilon = 0.25$, $\gamma = -0.05$ and $a = 1/2$.

Table 2. Verification results for $\delta = -0.2$, $\varepsilon = 0.25$, $\gamma = -0.05$, $a = 1/2$, $N = 1500$.

Type	$\Vert u_h\Vert_\infty$	$\Vert U_h\Vert_\infty$	$\Vert U_h^\perp\Vert_\infty$	$\Vert v_h\Vert_\infty$	$\Vert V_h\Vert_\infty$	$\Vert V_h^\perp\Vert_\infty$
UP	0.96476	5.30726×10^{-5}	3.12184×10^{-3}	0.40956	8.30218×10^{-5}	8.60449×10^{-4}
LO	0.45028	3.14573×10^{-5}	1.36812×10^{-3}	0.18075	4.19882×10^{-5}	3.77334×10^{-4}

negative value of δ. We call these solution as upper type (UP) and lower type (LO). Fig. 2 shows the shape of approximate solutions u_h and corresponding v_h and Table 2 shows verification results.

When taking lower ε, we have seven approximate solutions: four seem to be symmetric and three seem to be asymmetric. Figs. 3, 4 and 5 show the shape of

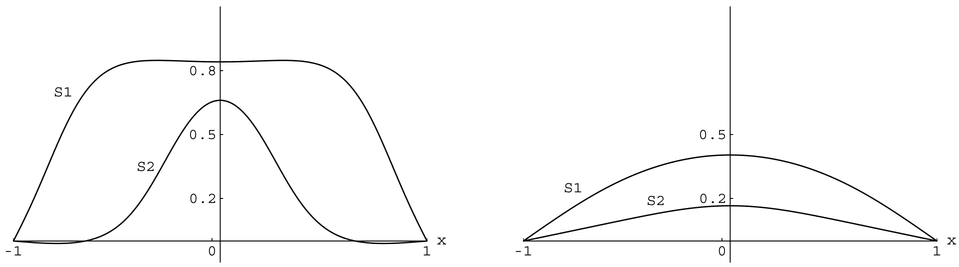

Fig. 3. Approximate "symmetric (S1 and S2)" solutions u_h (left) and v_h (right) for $\delta = 0.2$, $\varepsilon = 0.08$, $\gamma = -0.05$ and $a = 1/2$.

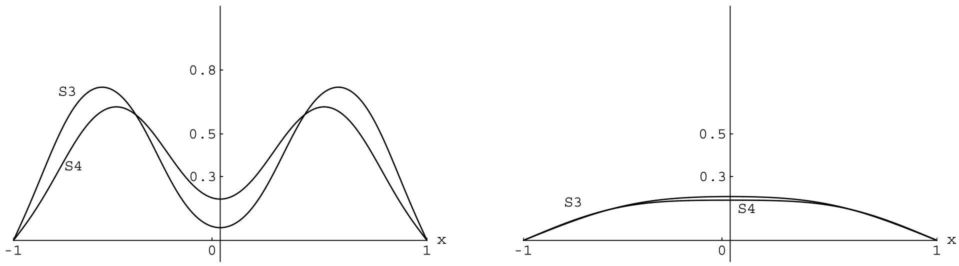

Fig. 4. Approximate "symmetric (S3 and S3)" solutions u_h (left) and v_h (right) for $\delta = 0.2$, $\varepsilon = 0.08$, $\gamma = -0.05$ and $a = 1/2$.

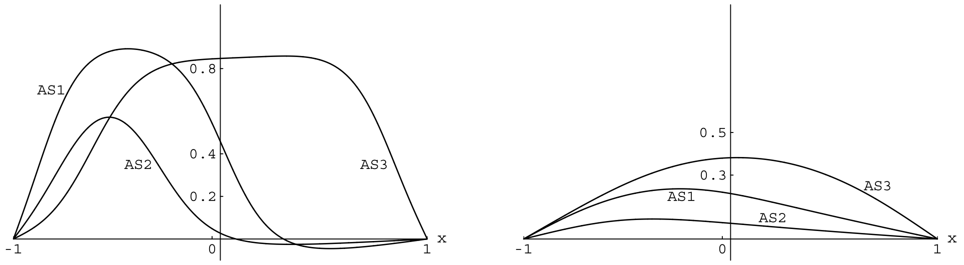

Fig. 5. Approximate "asymmetric (AS1, AS2 and AS3)" solutions u_h (left) and v_h (right) for $\delta = 0.2$, $\varepsilon = 0.08$, $\gamma = -0.05$ and $a = 1/2$.

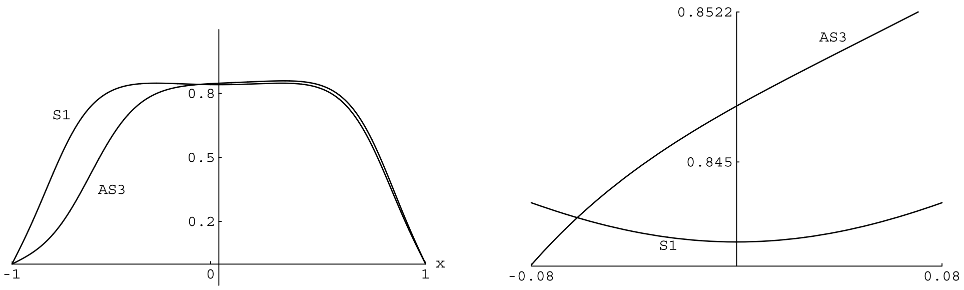

Fig. 6. Shape of "symmetric" approximate solution S1 and "asymmetric" approximate solution AS3 for $\delta = 0.2$, $\varepsilon = 0.08$, $\gamma = -0.05$ and $a = 1/2$.

Table 3. Verification results for $\delta = 0.2$, $\varepsilon = 0.08$, $\gamma = -0.05$, $a = 1/2$, $N = 5000$.

Type	$\|u_h\|_\infty$	$\|U_h\|_\infty$	$\|U_h^\perp\|_\infty$	$\|v_h\|_\infty$	$\|V_h\|_\infty$	$\|V_h^\perp\|_\infty$
S1	0.84803	6.42742×10^{-5}	2.22929×10^{-3}	0.40389	8.38954×10^{-5}	2.61753×10^{-4}
S2	0.66057	5.32970×10^{-4}	2.73014×10^{-3}	0.16555	5.63207×10^{-4}	1.13887×10^{-4}
S3	0.71926	3.03301×10^{-4}	4.20778×10^{-3}	0.18893	4.08560×10^{-4}	1.08497×10^{-4}
S4	0.62685	1.99658×10^{-4}	3.17346×10^{-3}	0.20583	2.77909×10^{-4}	1.55641×10^{-4}
AS1	0.89305	2.03274×10^{-4}	2.79030×10^{-3}	0.23556	1.64205×10^{-4}	1.88588×10^{-4}
AS2	0.57169	7.43865×10^{-5}	2.05645×10^{-3}	0.09379	1.10036×10^{-4}	9.42650×10^{-4}
AS3	0.85895	1.42930×10^{-4}	0.23180×10^{-3}	0.38196	1.31440×10^{-4}	2.45271×10^{-4}

approximate solutions (u_h, v_h). Note that we can assure that more three "reflected" solutions exist corresponding to AS1, AS2 and AS3. Table 3 shows verification results.

4.2. Two dimensional case

Consider the following problem:

$$\begin{cases} -\varepsilon^2 \Delta u = u - u^3 - \delta v & \text{in } \Omega, \\ -\Delta v = u - \gamma v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

where $\Omega = (0,1)\times(0,1)$. The approximate subspace S_h is taken as the set of double finite Fourier series:

$$S_h = \left\{ \sum_{m,n=1}^{N} a_{mn} \sin(\pi m x)\sin(\pi n y) \;\middle|\; a_{mn} \in \mathbb{R} \right\}.$$

Then $\dim S_h = N^2$, $h = 1/N$, $C_2 = 1/(\pi\sqrt{2})$ and a priori constant C can be taken as $N/\big(\pi\sqrt{((N+1)^2+1)}\big)$ [17].

Taking advantage of $\Delta u_h, \Delta v_h \in L^2(\Omega)$ for obtained approximate solutions $u_h, v_h \in S_h$, we rewrite eq. (18) as residual form:

$$\begin{cases} -\varepsilon^2 \Delta\tilde{u} = \varepsilon^2 \Delta u_h + u_h - \delta v_h + \tilde{u} - (\tilde{u}+u_h)^3 - \delta\tilde{v} & \text{in } \Omega, \\ -\Delta\tilde{v} = \Delta v_h + u_h - \gamma v_h + \tilde{u} - \gamma\tilde{v} & \text{in } \Omega, \\ \tilde{u} = \tilde{v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

and find the residual $\tilde{u} = u - u_h$ and $\tilde{v} = v - v_h$ in the candidate set $U_h + U_h^{\perp}$, $V_h + V_h^{\perp} \subset H_0^1(\Omega)$.

We used the Fortran 90 library `INTLIB_90` coded by Kearfott [3] with IBM XL Fortran Enterprise Edition V9.1 Version: 09.01.0000.0007 on IBM eServer p5 model 595 (POWER 5 1.90 GHz Turbo; AIX 5L V5.3) for the verified numerical computations.

Note that if (u,v) is a solution of the system (18), $(-u,-v)$ is also the solution, therefore we try to find "upper" state.

By floating point arithmetic, two types of approximate solutions are obtained for the various ε (see Figs. 7 and 8).

Table 4 shows verification results.

Table 4. Verification results for $\delta = 0.5$, $\gamma = -1.2$, $N = 80$.

Type	ε	$\|u_h\|_{L^2(\Omega)}$	$\|U_h\|_\infty$	$\|U_h^\perp\|_\infty$	$\|v_h\|_{L^2(\Omega)}$	$\|V_h\|_\infty$	$\|V_h^\perp\|_\infty$
I	0.1	0.69266	2.56074×10^{-19}	1.26908×10^{-15}	0.03624	5.03767×10^{-20}	1.95957×10^{-20}
II	0.1	0.05492	1.19191×10^{-22}	1.12352×10^{-15}	0.00057	3.37753×10^{-23}	1.73480×10^{-20}
I	0.08	0.75481	1.38244×10^{-20}	2.37244×10^{-16}	0.03872	2.71964×10^{-21}	3.66325×10^{-21}
II	0.08	0.38917	2.35690×10^{-21}	2.08711×10^{-17}	0.00593	4.63666×10^{-22}	3.22267×10^{-22}
I	0.06	0.81541	1.11203×10^{-16}	1.44528×10^{-11}	0.04071	2.18765×10^{-17}	2.23164×10^{-16}
II	0.06	0.56317	7.98206×10^{-18}	1.20585×10^{-12}	0.01182	1.57029×10^{-18}	1.86193×10^{-17}
I	0.04	0.87478	8.91259×10^{-12}	1.28638×10^{-6}	0.04215	1.75335×10^{-12}	1.98628×10^{-11}
II	0.04	0.71497	5.04207×10^{-12}	8.16969×10^{-7}	0.01627	9.91911×10^{-13}	1.26148×10^{-11}
I	0.02	0.93314	3.85229×10^{-5}	1.408569×10^{-1}	0.04303	7.57850×10^{-6}	2.174954×10^{-6}
II	0.02	0.85819	4.26610×10^{-5}	3.301898×10^{-1}	0.01754	8.39257×10^{-6}	5.098422×10^{-6}

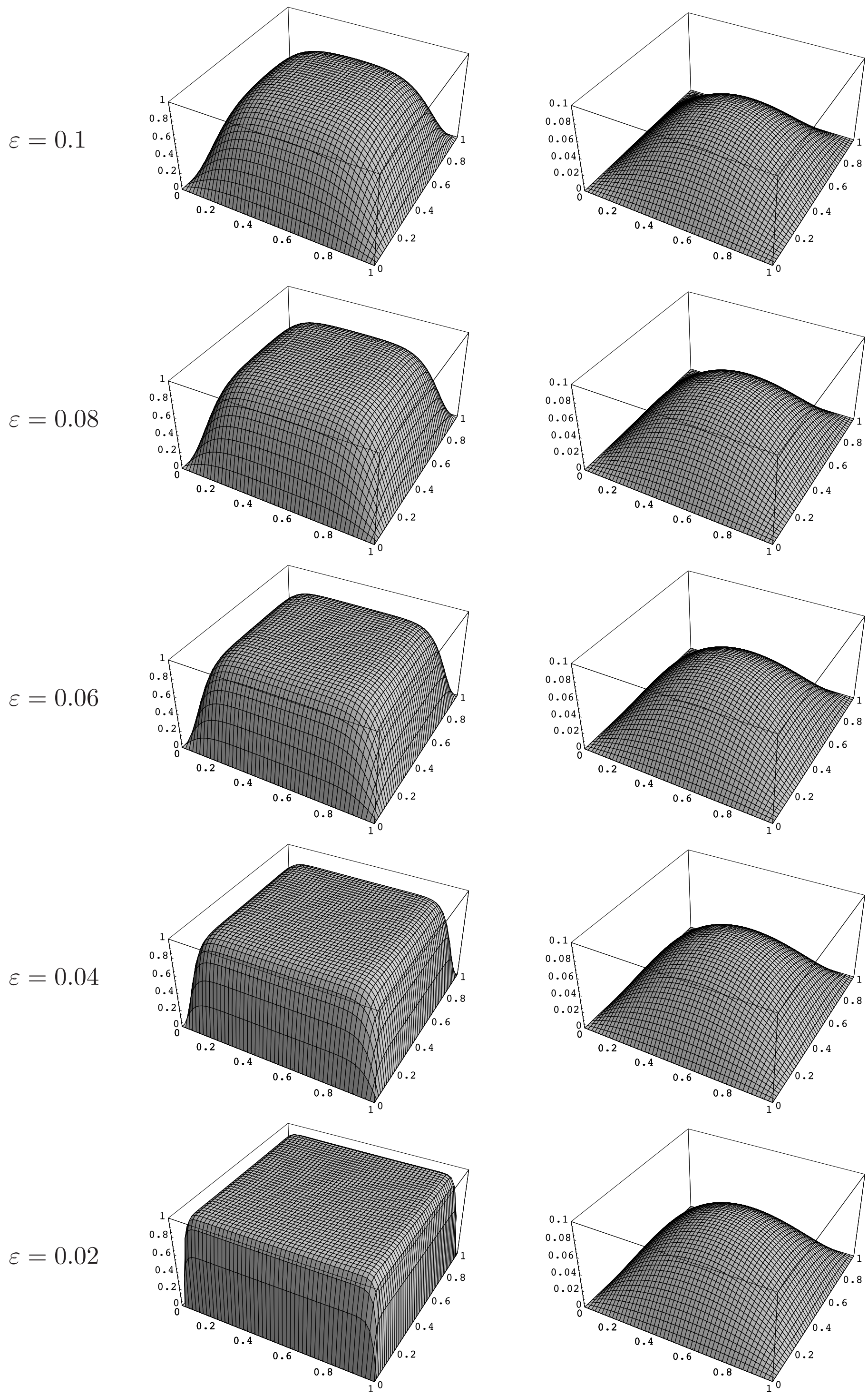

Fig. 7. Approximate solutions u_h (left) and v_h (right) for $\delta = 0.5$, $\gamma = -1.2$ (type I).

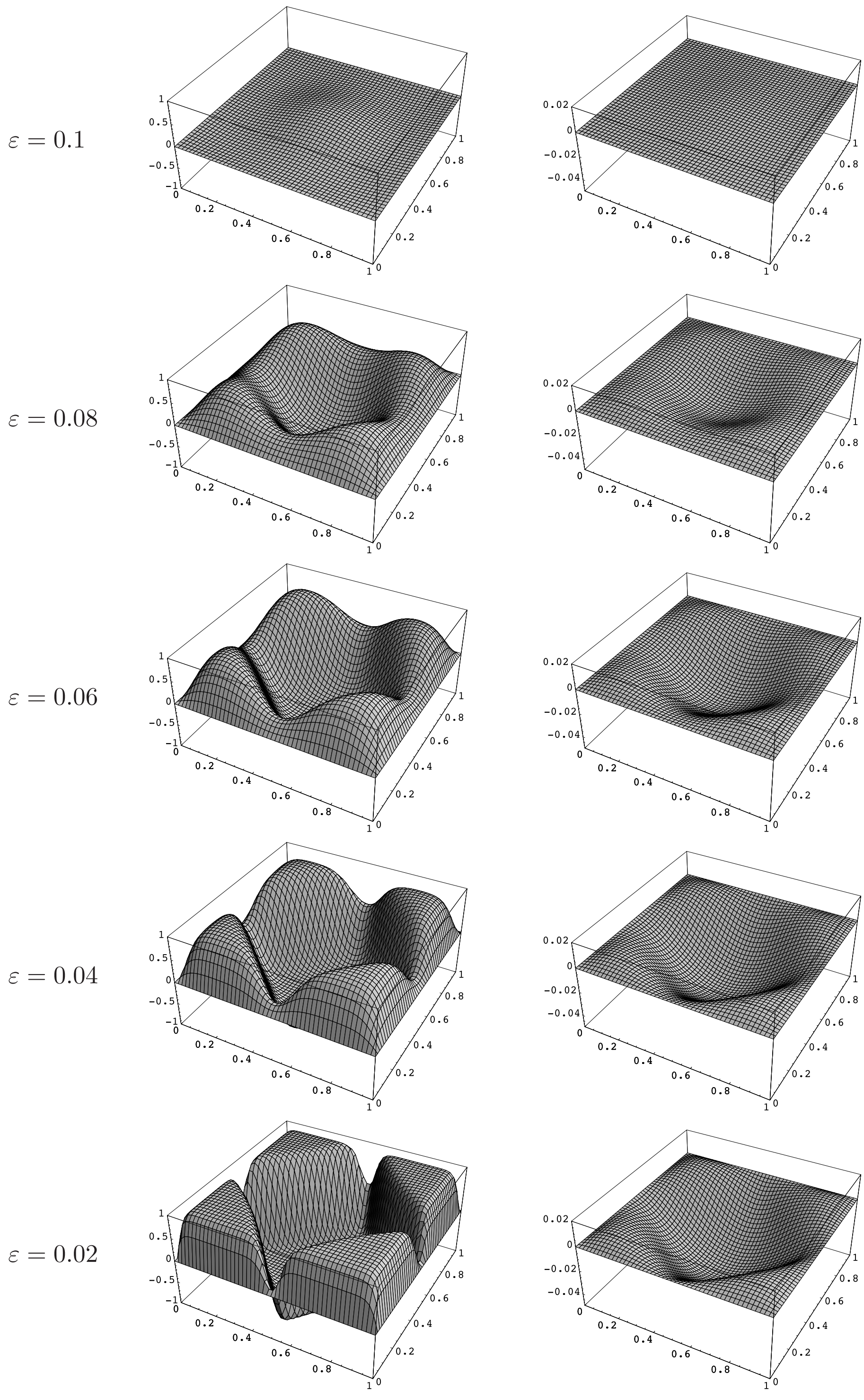

Fig. 8. Approximate solutions u_h (left) and v_h (right) for $\delta = 0.5$, $\gamma = -1.2$ (type II).

5. Conclusion

We propose a numerical method to prove the existence of the solutions of the reaction-diffusion system near an approximate solution obtained by a usual floating point computation. All numerical results discussed are taken into account of the effects of rounding errors in the floating point computations.

The principle of our verification method can be applied to Neumann boundary conditions. This will be given in the forthcoming papers.

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