Painlevé VI Equations with Algebraic Solutions and Family of Curves

Hossein Movasati and Stefan Reiter

CONTENTS

2000 AMS Subject Classification: 34M55, 35Q53

Keywords: Painlevé sixth equation, Okamoto transformation, monodromy, convolution

In families of Painleve VI differential equations having common algebraic solutions we classify all the members that come from geometry, i.e., the corresponding linear differential equations that are Picard–Fuchs associated to families of algebraic varieties. In our case, we have one family with zero-dimensional fibers and all others are families of curves. We use the classification of families of elliptic curves with four singular fibers carried out by Herfurtner in 1991 and generalize the results of Doran in 2001 and Ben Hamed and Gavrilov in 2005.

1. INTRODUCTION

Along the solutions of the sixth Painlevé differential equation written in the vector field form

$$PVI_\theta : \ \frac{\partial K}{\partial \mu}\frac{\partial}{\partial \lambda} - \frac{\partial K}{\partial \lambda}\frac{\partial}{\partial \mu} + \frac{\partial}{\partial t} \tag{1–1}$$

in $\mathbb{C}^3$ with coordinates (λ, μ, t), where

$$\begin{aligned} t(t-1)K & \\ &= \lambda(\lambda-1)(\lambda-t)\mu^2 \\ &\quad - \big(\theta_2(\lambda-1)(\lambda-t) + \theta_3\lambda(\lambda-t) + (\theta_1-1)\lambda(\lambda-1)\big)\mu \\ &\quad + \kappa(\lambda-t), \quad \kappa = \frac{1}{4}\Big(\Big(\sum_{i=1}^{3}\theta_i - 1\Big)^2 - \theta_4^2\Big), \end{aligned}$$

and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ is a fixed multiparameter, the linear differential equation

$$y'' + p_1(z)y' + p_2(z)y = 0, \tag{1–2}$$

where

$$\begin{aligned} p_1(z) &:= \frac{1-\theta_1}{z-t} + \frac{1-\theta_2}{z} + \frac{1-\theta_3}{z-1} - \frac{1}{z-\lambda}, \\ p_2(z) &:= \frac{\kappa}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \end{aligned}$$

1058-6458/2010 $ 0.50 per page
Experimental Mathematics **19**:2, page 161

is isomonodromic, i.e., its monodromy group representation is constant. In the literature the Painlevé VI equation is usually written in λ and t parameters (see [Iwasaki et al. 91, p. 119])

$$\begin{aligned}\frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 \\ &\quad -\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &\quad +\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2} \\ &\qquad \left(\alpha-\beta\frac{t}{\lambda^2}+\gamma\frac{t-1}{(\lambda-1)^2}+\left(\frac{1}{2}-\delta\right)\frac{t(t-1)}{(\lambda-t)^2}\right),\end{aligned}$$

where

$$\alpha=\frac{1}{2}\theta_4^2,\quad \beta=\frac{1}{2}\theta_2^2,\quad \gamma=\frac{1}{2}\theta_3^2,\quad \delta=\frac{1}{2}\theta_1^2.$$

It is obtained from the ordinary differential equation associated to (1–1) by discarding the μ parameter.

We say that the linear differential equation (1–2) *comes from geometry* if there are a proper family of algebraic varieties $X \to \mathbb{P}^1$ over $\mathbb{C}$ and a differential form $\omega \in H^i_{dR}(X/\mathbb{P}^1)$ such that the periods $\int_{\delta_z}\omega$, where $\delta_z \in H_i(X_z,\mathbb{Z})$ is a continuous family of cycles, spans the solution space of the linear differential equation. Such linear differential equations are also called Picard–Fuchs equations (for further details see [André 89, Section II.1]). In the present text we mainly encounter families of curves, and ω is represented by a meromorphic differential 1-form with no residues around its poles.

If further, the fibration $X \to \mathbb{P}^1$ lies in a family $X_b \to \mathbb{P}^1$, $b \in \mathbb{P}^1$, then one usually gets algebraic solutions of (1–1). An algebraic solution of (1–1) is a curve in (λ,μ,t)-space that is tangent to the vector field (1–1). In this article we prove the following theorem.

Theorem 1.1. *The linear differential equations* (1–2) *with the parameters in columns* 1 *and* 2 *of Table* 1 *come from geometry if and only if the corresponding exponent parameters* θ_1, θ_2, θ_3, *and* θ_4 *are rational numbers. The corresponding families of algebraic varieties and differential forms are listed in column* 3.

It is a well-known fact that if a linear differential equation comes from geometry, then its exponents are rational numbers (see, for instance, [Katz 70]). Therefore, the nontrivial part of Theorem 1.1 is that this condition is also sufficient for linear differential equations of type (1–2) with parameters in the first and second columns of Table 1.

In Table 1, b is an arbitrary parameter. The corresponding algebraic curve in (λ,t) is independent of the θ's. In this way, Table 1, columns 1 and 2, is exactly the classification of families of Painlevé VI equations with common algebraic solution done in [Ben Hamed and Gavrilov 05, Table 2.1]. Note that we have written our table up to Möbius transformations in z of the differential equation (1–2).

In the first row of Table 1, the corresponding linear differential equation (1–2) is the Gauss hypergeometric equation, and the geometric interpretation is classical; it has nice applications in the theory of the special values of Gauss hypergeometric functions (see [Shiga et al. 04]). Note that in this case, the projection of the corresponding algebraic curve in the (λ,μ,t)-space into the (λ,t)-space is just the zero-dimensional variety $\{(0,0),(1,1)\}$. In all other cases it is a curve in the (λ,t)-space. In the literature one finds mainly the equations of such curves.

In Table 1 let us put the parameters a and c in column 2 and set rows $3,4,5,6$ equal to $\frac{1}{2}$. In row 3, we put also $\tilde{a}=\frac{1}{2}$ (in this case we have two apparently different geometric interpretations for the same Painlevé equation). We obtain five families of elliptic curves $y^2=4x^3-g_2(\tilde{z},\tilde{b})x-g_3(\tilde{z},\tilde{b})$ with exactly four singular fibers and with j-invariant depending on an extra parameter b (see Table 2). These are exactly the only families of elliptic curves with the mentioned properties. This follows from the classification of elliptic curves with exactly four singular fibers appearing in [Herfurtner 91].

In [Doran 01], the author took these five families of elliptic curves and obtained algebraic solutions for (1–1) (see Table 1, columns 1 and 2, rows 3–6). In [Ben Hamed and Gavrilov 05], the authors took zero-dimensional families of three points varieties $4x^3-g_2(\tilde{z},\tilde{b})x-g_3(\tilde{z},\tilde{b})=0$, and they obtained the same algebraic solutions in the (t,λ)-space (see Table 1, columns 1 and 2, rows 3–6 for $a=c=0$) but for different parameters θ. In this way they also noticed that in the parameter space of Painlevé equations (1–1), the points obtained by them and Doran lie in families with algebraic solutions for which the projections in the (t,λ)-space is independent of the θ parameter of the family.

Then by a straightforward calculation they showed that up to the Okamoto transformations corresponding to the Möbius transformation of $\mathbb{P}^1$, such families of Painlevé equations are given by the first and second columns of Table 1. For an overview of the symmetries of Painlevé VI and Okamoto transformations, see, for instance, [Boalch 06, Section 2]. We further prove the following theorem.

Algebraic Solution	$(\boldsymbol{\theta_1, \theta_2, \theta_3, \theta_4})$	**Family of Algebraic Varieties**
$\{\lambda = t = 0\} \cup \{\lambda = t = 1\}$	$(0, 1-\alpha_3, \alpha_3-\alpha_1-\alpha_2, \alpha_2-\alpha_1)$	$y = x^{1-\alpha_1}(1-x)^{\alpha_2}(z-x)^{1+\alpha_1-\alpha_3}, \ \frac{dx}{y}$
$\lambda = \left(\frac{-a+1}{a+2c-3}\right)b, \ t = b^2$ $\mu = \frac{-a-2c+3}{2b}$	$\left(\frac{1}{2}, a-1, \frac{1}{2}, -(a+2c-3)\right)$	zero-dimensional varieties
$\lambda = -b, \ t = b^2$ $\mu = \frac{-a-2c+2}{2b}$	$\left(c-\frac{1}{2}, a+c-1, c-\frac{1}{2}, a+c-1\right)$	$y = \left(4x^2 - \tilde{g}_2 x + \tilde{g}_3\right)^c (x + \tilde{g}_2/4)^a, \ \frac{dx}{y}$ $\tilde{g}_2 = 4\left(\tilde{z}^2 + \tilde{z}\right)$ $\tilde{g}_3 = \left(-9b^2\tilde{z}^3 - 8b\tilde{z}^4 + 2b\tilde{z}^3 - 8b\tilde{z}^2 - 9\tilde{z}^3\right)/b$ $\tilde{z} = \frac{-1}{b}z$ If $c = \frac{5}{6}, \ a = \tilde{a} - \frac{1}{3}$ we have also $y = \left(4x^3 - g_2 x - g_3\right)^{\tilde{a}}, \ \frac{dx}{y}$ $g_2 = 3(z-1)\left(z-b^2\right)^3$ $g_3 = (z-1)\left(z-b^2\right)^4(z-b)$
$\lambda = \frac{-2b-1}{b^2}, \ t = \frac{2b+1}{b^4+2b^3}$ $\mu = \frac{(-3a+2)b^2(b+2)}{3(b+1)^2}$	$\left(a-\frac{1}{2}, 3\left(a-\frac{1}{2}\right), a-\frac{1}{2}, a-\frac{1}{2}\right)$	$y = \left(4x^3 - g_2 x - g_3\right)^a, \ \frac{dx}{y}$ $g_2 = 12\tilde{z}^2\left(\tilde{z}^2 + 2\tilde{b}\tilde{z} + 1\right)$ $g_3 = 4\tilde{z}^3\left(2\tilde{z}^3 + 3\left(\tilde{b}^2 + 1\right)\tilde{z}^2 + 6\tilde{b}\tilde{z} + 2\right)$ $\tilde{b} = \frac{2}{3}\left(b + \frac{1}{b}\right) - \frac{1}{3}, \ \tilde{z} = -\frac{b^2+2b}{3}z$
$\lambda = \frac{(b+1)\left(b^2+3\right)}{(b-1)^2(b+3)}, \ t = \frac{(b+1)^3(b-3)}{(b-1)^3(b+3)}$ $\mu = \frac{(3a-2)(b-1)^2(b+3)}{24(b+1)}$	$\left(a-\frac{1}{2}, \frac{1}{2}, a-\frac{1}{2}, a-\frac{1}{2}\right)$	$y = \left(4x^3 - g_2 x - g_3\right)^a, \ \frac{dx}{y}$ $g_2 = 3\tilde{z}^3\left(\tilde{z} + \tilde{b}\right), \ g_3 = \tilde{z}^5(\tilde{z}+1)$ $\tilde{b} = \frac{2}{3}\frac{b^2-3}{b^2+3} + \frac{1}{3}, \tilde{z} = -\frac{b^3-3b^2+3b-1}{b^3-3b^2+3b-9}z$
$\lambda = \frac{-2b^2-4}{b^4-6b^2}, \ t = \frac{-12b^2+8}{b^6-6b^4}$ $\mu = \frac{(-3a+2)b^2\left(b^2+2\right)\left(b^2-6\right)}{12\left(b^2-2\right)^2}$	$\left(a-\frac{1}{2}, \frac{1}{3}, a-\frac{1}{2}, 2a-1\right)$	$y = \left(4x^3 - g_2 x - g_3\right)^a, \ \frac{dx}{y}$ $g_2 = 3\tilde{z}^3\left(\tilde{z} + 2\tilde{b}\right)$ $g_3 = \tilde{z}^4\left(\tilde{z}^2 + 3\tilde{b}\tilde{z} + 1\right)$ $\tilde{b} = \frac{1}{4}\left(b + \frac{2}{b}\right), \ \tilde{z} = -\frac{2b^3}{3b^2-2}z$

TABLE 1. Algebraic solutions of families of the sixth Painlevé equation.

Theorem 1.2. *The Painlevé VI equation with parameters in Table* 1*, row* 2 *(respectively row* 4*) is Okamoto equivalent to the Painlevé VI equation associated to the parameters in Table* 1*, row* 3 *(respectively row* 5*).*

For rational numbers a and c in row 2 we show that the monodromy group of the linear equation (1–2) is a dihedral group and so is finite. Also, the other families are related via the middle convolution to (third-order) differential equations whose monodromy groups are finite imprimitive reflection groups for rational parameters.

In [Boalch 03], the author started with third-order differential equations with finite monodromy to obtain algebraic solutions and the parameters of the corresponding Painlevé VI differential equations. Recently it was shown in [Cantat and Loray 07, Proposition 5.4] that any algebraic solution of a Painlevé VI differential equation having degree 2, 3, or 4 belongs (up to Okamoto transformation) to one of the families in Table 1. (This was done via the classification of finite braid group orbits of length 2, 3, and 4.)

Linear differential equations (1–2) with finite monodromy come automatically from geometry, and this is the origin of many algebraic solutions known until now (see [Boalch 06] and the references therein). One can also obtain equations (1–2) coming from geometry by taking pullbacks of the Gauss hypergeometric equation (see [Kitaev 05] and the references therein).

Our proofs of Theorems 1.1 and 1.2 are heavily based on computer calculations done with Singular (see [Greuel et al. 01]). The details of the calculation are explained only for the example in row 3 of Table 1.

Let us explain the content of each section. In Section 2 we introduce systems of linear differential equations in two and three variables. Pulling these back, we get Fuchsian systems with four singularities. In Section 3 we recall some well-known facts about linear differential equations, and in Section 4 we explain the Schlesinger system associated to (1–2). In Section 5 we recall some basic facts about the middle convolution. Sections 6 and 9 are dedicated to the proof of Theorem 1.1. Finally, in Sections 7 and 8 we prove Theorem 1.2.

2. LINEAR SYSTEMS IN TWO VARIABLES

For $a, b, c \in \mathbb{C}$ fixed, we consider the following family of transcendent curves:

$$E : y = f(x), \tag{2–1}$$

with

$$f(x) := (t_1 - t_3)^{\frac{1}{2}(1-a-c)}(t_1 - t_2)^{\frac{1}{2}(1-a-b)} \times (t_2 - t_3)^{\frac{1}{2}(1-b-c)}(x - t_1)^a(x - t_2)^b(x - t_3)^c.$$

Here $t = (t_1, t_2, t_3)$ is a parameter in

$$T := \{t \in \mathbb{C}^3 \mid (t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \neq 0\}.$$

We distinguish three, not necessarily closed, paths in E. In the x-plane let $\tilde{\delta}_i$, $i = 1, 2, 3$, be the straight path connecting t_{i+1} to t_{i-1}, $i = 1, 2, 3$ (by definition, $t_4 := t_1$ and $t_0 = t_3$). There are many paths in E that are mapped to $\tilde{\delta}_i$ under the projection on x. We choose one of them and call it δ_i. We can make our choices so that $\delta_1+\delta_2+\delta_3$ is a limit of a closed and homotopic-to-zero path in E.

We have the integral

$$\int_\delta \frac{p(x)dx}{y} = \int_{\tilde{\delta}} \frac{p(x)dx}{f(x)}, \quad p \in \mathbb{C}[x], \tag{2–2}$$

where δ is one of the paths explained above. By a linear change in the variable x such integrals can be written in terms of the Gauss hypergeometric function (see [Iwasaki et al. 91]).

Another way to study the integrals (2–2) is with Pochhammer cycles. For simplicity we explain this for the pairs (t_1, t_2). The Pochhammer cycle associated to the points $t_1, t_2 \in \mathbb{C}$ and the path $\tilde{\delta}_3$ is the commutator

$$\tilde{\alpha}_3 = [\gamma_1, \gamma_2] = \gamma_1^{-1} \cdot \gamma_2^{-1} \cdot \gamma_1 \cdot \gamma_2,$$

where γ_1 is a loop along $\tilde{\delta}_3$ starting and ending at some point in the middle of $\tilde{\delta}_1$ that encircles t_1 once counterclockwise, and γ_2 is a similar loop with respect to t_2. It is easy to see that the cycle $\tilde{\alpha}_3$ lifts to a closed path α_3 in E_t, and if $a, b \notin \mathbb{Z}$, then

$$\int_{\alpha_3} \frac{p(x)dx}{y} = (1 - e^{-2\pi i a})(1 - e^{-2\pi i b}) \int_{\tilde{\alpha}_3} \frac{p(x)}{f(x)}\, dx$$

(see [Iwasaki et al. 91, Proposition 3.3.7]).

For a fixed $\mathsf{a} \in T$, the period map is given by

$$\mathsf{pm} : (T, \mathsf{a}) \to \mathrm{GL}(2, \mathbb{C}),\ t \mapsto \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_2} \frac{dx}{y} \\ \int_{\delta_1} \frac{xdx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}, \tag{2–3}$$

where (T, a) means a small neighborhood of a in T.

The map pm can be extended along any path in T with the starting point a. The period map pm is a fundamental system for the linear differential equation $dY = AY$ in $\mathbb{C}^3$, where

$$A = \frac{1}{(t_1 - t_2)(t_1 - t_3)} \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} dt_1 + (\cdots)dt_2 + (\cdots)dt_3, \tag{2–4}$$

where

$$\begin{aligned} N_1 &= \frac{1}{2}(b + c - 2)t_1 + \frac{1}{2}(a + c - 1)t_2 + \frac{1}{2}(a + b - 1)t_3, \\ N_2 &= -a - b - c + 2, \\ N_3 &= at_2t_3 + (b - 1)t_1t_3 + (c - 1)t_1t_2, \\ N_4 &= -\frac{1}{2}(b + c - 2)t_1 - \frac{1}{2}(a + c - 1)t_2 \\ &\quad - \frac{1}{2}(a + b - 1)t_3, \end{aligned}$$

and the matrix coefficient $(\cdots)$ of dt_2 (respectively dt_3) is obtained by permutation of t_1 with t_2 and a with b (respectively t_1 with t_3 and a with c) in the matrix coefficient of dt_1 written above. Now, for the multivalued function

$$y = (27t_3^2 - t_2^3)^{\frac{1}{2}(\frac{1}{2}-a)}(4x^3 - t_2x - t_3)^a$$

we have the system

$$A = \frac{1}{27t_3^2 - t_2^3} \left(\begin{pmatrix} \frac{1}{4}t_2^2 & -27at_3 + 18t_3 \\ -\frac{9}{4}at_2t_3 + \frac{3}{4}t_2t_3 & -\frac{1}{4}t_2^2 \end{pmatrix} dt_2 + \begin{pmatrix} -\frac{9}{2}t_3 & 18at_2 - 12t_2 \\ \frac{3}{2}at_2^2 - \frac{1}{2}t_2^2 & \frac{9}{2}t_3 \end{pmatrix} dt_3 \right), \tag{2–5}$$

and for

$$y = \left(t_2^2 + 2t_3\right)^{\frac{1}{2}(1-a-c)} \left(t_2^2 - 16t_3\right)^{\frac{1}{2}\left(\frac{1}{2}-c\right)} \left(4x^2 - t_2x + t_3\right)^c \times \left(x + \frac{1}{4}t_2\right)^a$$

we have

$$A = \frac{1}{(t_2^2 - 16t_3)(t_2^2 + 2t_3)} \times \left(\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} dt_2 + \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} dt_3 \right), \tag{2–6}$$

where

$$
\begin{aligned}
M_1 &= \left(6at_2t_3 - 6ct_2t_3 - \frac{1}{2}t_2^3 + 5t_2t_3\right),\\
M_2 &= \left(-48at_3 - 96ct_3 + 96t_3\right),\\
M_3 &= \left(12at_3^2 - 3ct_2^2t_3 + t_2^2t_3 - 4t_3^2\right),\\
M_4 &= \left(-6at_2t_3 + 6ct_2t_3 + \frac{1}{2}t_2^3 - 5t_2t_3\right),\\
N_1 &= \left(-3at_2^2 + 3ct_2^2 + t_2^2 + 8t_3\right),\\
N_2 &= \left(24at_2 + 48ct_2 - 48t_2\right),\\
N_3 &= \left(-6at_2t_3 + \frac{3}{2}ct_2^3 - \frac{1}{2}t_2^3 + 2t_2t_3\right),\\
N_4 &= \left(3at_2^2 - 3ct_2^2 - t_2^2 - 8t_3\right).
\end{aligned}
$$

The calculation of the matrix A for the elliptic case $a = b = c = \frac{1}{2}$ is classical and goes back to [Griffiths 66]. In fact, the matrix A in (2–4) can also be calculated from the well-known Fuchsian system for hypergeometric functions, i.e., for the case $t_1 = 0$, $t_2 = 1$, $t_3 = t$. The algorithms for calculating such a matrix are explained in [Movasati 08], and the computer implementation of such algorithms can be found at the first author's homepage.

3. REVIEW OF FUCHSIAN DIFFERENTIAL EQUATIONS

Here we collect some basic facts about Fuchsian differential equations, all of which can be found in [Iwasaki et al. 91]. Let $D = \frac{d}{dz}$ and let

$$D^2y + p_1(z)Dy + p_2(z)y = 0 \tag{3–1}$$

be a Fuchsian differential equation with regular singularities at $t_1, \dots, t_m \in \mathbb{C}$ and $t_{m+1} = \infty$. Then by [Iwasaki et al. 91, Proposition 4.2],

$$
\begin{aligned}
p_1(z) &= \sum_{i=1}^{m} \frac{a_i}{z - t_i},\\
p_2(z) &= \sum_{i=1}^{m} \frac{b_i}{(z - t_i)^2} + \sum_{i=1}^{m} \frac{c_i}{(z - t_i)}, \quad a_i, b_i, c_i \in \mathbb{C},
\end{aligned}
$$

where $p_2(z)\prod_{i=1}^{m}(z-t_i)^2$ is a polynomial in $\mathbb{C}[z]$ of degree at most $2(m-1)$, i.e., $\sum c_i = 0$.

The exponents s_j^i, $j = 1, 2$, at the singularity t_i, $i = 1, \dots, m$, are the roots of $s(s-1) + a_is + b_i = 0$ (see [Iwasaki et al. 91, p. 170]), and at t_{m+1} the exponents satisfy

$$s(s+1) - \Big(\sum_{i=1}^{m} a_i\Big)s + \Big(\sum_{i=1}^{m} b_i + \Big(\sum_{i=1}^{m} c_it_i\Big)\Big) = 0.$$

Furthermore, we have the Fuchs relation

$$\sum_{i=1}^{m+1}\sum_{j=1}^{2} s_j^i = m - 1.$$

All these facts are summarized in the Riemann scheme

$$\begin{pmatrix} t_1 & \cdots & t_{m+1} \\ s_1^1 & \cdots & s_{m+1}^1 \\ s_1^2 & \cdots & s_{m+1}^2 \end{pmatrix}.$$

The second-order Fuchsian differential equation (3–1) can be transformed into SL-form by replacing y by fy, where $0 \neq f$ satisfies $Df = -\frac{1}{2}p_1(z)f$. Then by [Iwasaki et al. 91, p. 166],

$$
\begin{aligned}
&D^2y = p(z)y,\\
&p(z) = -p_2(z) + \frac{1}{4}p_1(z)^2 + \frac{1}{2}Dp_1(z).
\end{aligned} \tag{3–2}
$$

From a two-dimensional system $DY = QY$, $Q = (q_{ij})$ (not necessarily a Fuchsian system), we obtain the second-order differential equation (3–1) for the first coordinate y_1 of $Y = (y_1, y_2)^{\mathrm{tr}}$ (see [Iwasaki et al. 91, Lemma 6.1.1]) with

$$
\begin{aligned}
p_1(z) &= -D\log(q_{12}(z)) - \mathrm{Tr}(Q),\\
p_2(z) &= \det(Q(z)) - Dq_{11} + q_{11}D\log q_{12}.
\end{aligned} \tag{3–3}
$$

If λ is a zero of q_{12} of order r and $\lambda \notin \{t_1, \dots, t_m\}$, then $z = \lambda$ is an apparent singular point with the exponents 0 and $r + 1$ (see [Iwasaki et al. 91, Lemma 6.1.2]).

4. SCHLESINGER SYSTEM

In this section we describe the system

$$
\begin{aligned}
&DY = QY, \quad Q = \sum_{k=1}^{3} \frac{Q_k}{z - t_k},\\
&Q_k = Q_k(t) = (q_{ij}^k) \in \mathrm{Mat}_2(\mathbb{C}), \quad k = 1, 2, 3,
\end{aligned} \tag{4–1}
$$

of Schlesinger type with the four regular singularities at $t_1 = t$, $t_2 = 0$, $t_3 = 1$, $t_4 = \infty$ associated to (1–2) (see [Iwasaki et al. 91]). We can assume that θ_i and 0 are the eigenvalues of Q_i and that (if $\theta_4 \neq 1$)

$$-\sum_{i=1}^{3} Q_i(t) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha + \theta_4 - 1 \end{pmatrix}, \quad 2\alpha + \sum_{i=1}^{4}\theta_i = 1.$$

By [Iwasaki et al. 91, Proposition 6.3.1] the matrices Q_i, $i = 1, 2, 3$, can be expressed as follows:

$$
\begin{aligned}
&Q_i = \begin{pmatrix} M_i(W_i - W) & -M_i \\ -(W_i - W)(M_i(W - W_i) + \theta_i) & \theta_i - M_i(W_i - W) \end{pmatrix},\\
&i = 1, \dots, 3,
\end{aligned} \tag{4–2}
$$

with

$$
\begin{aligned}
M_1 &= \frac{\lambda - t}{t\,(t-1)}, \quad W_1 = \lambda\,(\lambda-1)\left(\mu + \frac{\alpha}{\lambda}\right),\\
M_2 &= \frac{\lambda}{t}, \quad W_2 = (\lambda - t)\,(\lambda-1)\left(\mu + \frac{\alpha}{\lambda}\right) - \frac{t\alpha}{\lambda},\\
M_3 &= \frac{\lambda - 1}{1-t}, \quad W_3 = \lambda\,(\lambda - t)\left(\mu + \frac{\alpha}{\lambda}\right),\\
(\theta_4 - 1)\,W &= \sum_{i=1}^{3} W_i\,(M_i W_i - \theta_i),\\
\mu &= \nu - \frac{1}{2}\sum_{i=1}^{3}\frac{1-\theta_i}{\lambda - t_i},\\
\alpha &= -\frac{1}{2}\Big(\sum_{i=1}^{4}\theta_i - 1\Big).
\end{aligned}
$$

Note that a zero of $Q_{1,2}$ has order $r = 1$, which is the apparent singularity λ for the first coordinate (see [Iwasaki et al. 91, Lemma 6.1.2]).

Remark 4.1. Considering λ, μ, t, and the θ_i's as parameters, one can check with any commutative algebra software ([Greuel et al. 01], for instance) that the first coordinate of a solution of the system (4–1) with $\theta_4 \neq 1$ satisfies (1–2), and this fact does not depend on the nonresonance conditions that appear in [Iwasaki et al. 91, pp. 204, 169]. The same is true for obtaining the SL-form of (1–2). One must only take care about the poles of the entries of the matrices Q_1, Q_2, Q_3 (see Remark 5.2).

5. MIDDLE CONVOLUTION

The middle convolution functor on the category of perverse sheaves was introduced in [Katz 96]. It preserves important properties of local systems (respectively perverse sheaves) such as the index of rigidity and irreducibility but changes in general the rank and the monodromy group. A down-to-earth version of this functor was presented in [Dettweiler and Reiter 00] and [Dettweiler and Reiter 07a].

Thus given a Fuchsian system together with its monodromy group generators, it is possible to write down explicitly the new Fuchsian system and its monodromy group generators under the effect of the convolution functor. This explicit approach reveals that the middle convolution also commutes with the action of the braid group (cf. [Dettweiler and Reiter 07a, Theorem 2.4.iv]).

This has the following consequence in the Painlevé VI (P_{VI}) case: If one has an algebraic solution of a P_{VI} equation, the corresponding Fuchsian system has a finite braid group orbit. Thus if one applies a suitable middle convolution operation (which is always possible) that does not change the rank of the Fuchsian system but does change the local monodromy, one gets again an algebraic solution of a different P_{VI} equation. (The corresponding transformation of the parameters θ_i, $i = 1, \dots 4$, of the P_{VI} equation and the birational transformation of the algebraic solution were found by Okamoto (cf. [Haraoka and Filipuk 07, Section 5]).

A further possibility can be found in [Boalch 03], where the author started with third-order differential equations with finite monodromy to obtain algebraic solutions and the parameters of the corresponding Painlevé VI differential equations. This can also be interpreted as an application of the middle convolution (cf. [Dettweiler and Reiter 07b]).

For the convenience of the reader we give here a short review of the middle convolution for Fuchsian systems (see [Dettweiler and Reiter 07a]). Let $f(z) = (f_1(z), \dots, f_n(z))^{\mathrm{tr}}$ be a solution of the Fuchsian system

$$DY = AY = \sum_{i=1}^{r} \frac{A_i}{z - t_i} Y, \quad A_i \in \mathrm{Mat}_n(\mathbb{C}),$$

and let $[t_i, y] = \gamma_{t_i}^{-1}\gamma_y^{-1}\gamma_{t_i}\gamma_y$ be a Pochhammer cycle around t_i and y. Then the Euler transform

$$\begin{pmatrix} \int_{[t_i,y]} f(z)(y-z)^{\mu}\frac{dz}{z-t_1} \\ \vdots \\ \int_{[t_i,y]} f(z)(y-z)^{\mu}\frac{dz}{z-t_r} \end{pmatrix}, \quad i = 1, \dots, r,$$

of $f(z)$ with respect to $\mu \in \mathbb{C}$ and $[t_i, y]$ is a solution of the (Okubo) system

$$
\begin{aligned}
(yI_{nr} - T)DY = BY &:= \left(\begin{pmatrix} A_1 & \dots & A_r \\ & \vdots & \\ A_1 & \dots & A_r \end{pmatrix} + \mu\, I_{nr}\right) Y,\\
T = \mathrm{diag}(t_1 I_n, \dots, t_r I_n) &\Leftrightarrow DY = \sum_{i=1}^{r}\frac{B_i}{y - t_i}Y,\\
B_i &\in \mathrm{Mat}_{nr}(\mathbb{C}),
\end{aligned}
$$

where I_n is the $n \times n$ identity matrix.

In general, this system is not irreducible and has the following two $\langle B_1, \dots, B_r\rangle$-invariant subspaces:

$$
\begin{aligned}
\mathfrak{k} &= \oplus_{i=1}^{r} \ker(A_i),\\
\mathfrak{l} &= \ker(B) = \Big\langle (v, \dots, v)^{\mathrm{tr}} \mid v \in \ker\Big(\sum_{i=1}^{r} A_i + \mu I_n\Big)\Big\rangle.
\end{aligned}
$$

Factoring out this subspace, we obtain a Fuchsian system in dimension m:

$$m = nr - \sum_{i=1}^{r} \dim(\ker A_i) - \dim\Big(\ker\Big(\sum_{i=1}^{r} A_i + \mu I_n\Big)\Big).$$

Let M_i be the monodromy of the system $DY = AY$ at t_i. Then if the system is irreducible and

$$\operatorname{rk}(A_i) = \operatorname{rk}(M_i - I_n), \quad i = 1, \ldots, r$$
$$\operatorname{rk}\Big(\sum_{i=1}^{r} A_i + \mu I_n\Big) = \operatorname{rk}(M_1 \cdots M_r \lambda - I_n), \quad \lambda = e^{2\pi i \mu},$$

then the factor system is again irreducible, and it is called the *middle convolution* of $DY = AY$ with μ.

We start with a two-dimensional system in Schlesinger form (4–1), where the parameters are $(\theta_1, \ldots, \theta_4)$. Then there are in general two possibilities to obtain again a new two-dimensional system (4–1) via the middle convolution. Here we have $n = 2$, $r = 3$, and $\dim(\ker A_i) = 1$, $i = 1, 2, 3$.

If we choose μ such that $\dim(\ker(\sum_{i=1}^{r} A_i + \mu I_2) = 1)$, we obtain a Fuchsian system in dimension $2 \cdot 3 - (1 + 1 + 1) - 1 = 2$. For this case we can apply the middle convolution either with $\mu = \alpha$ or with $\mu = \alpha + \theta_4 - 1$ and diagonalize the residue matrix at ∞ to

$$\begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\alpha} + \tilde{\theta}_4 - 1 \end{pmatrix}.$$

This changes the parameters as follows:

$$(\tilde{\theta}_1, \ldots, \tilde{\theta}_4) = (\theta_1 + \mu, \ldots, \theta_3 + \mu, \theta_4 - \mu + 2\alpha),$$
$$\tilde{\alpha} = -\mu.$$

Remark 5.1. The described construction of the middle convolution results also in a transformation of the apparent singularity, which is known as an Okamoto transformation. This is worked out in detail in [Haraoka and Filipuk 07, Section 5].

Since we also want to determine the monodromy group of the transformed differential equation, it is more convenient to use the middle convolution than the Okamoto transformation. In the latter case, one still needs one parameter (e.g., K) to determine the differential equation (1–2).

Remark 5.2. For the proof of Theorem 1.2 we use the system (4–1) of Schlesinger type, which has poles at $\theta_4 = 1$. Theorem 1.2 is still true for the case $\theta_4 = 1$. In this case an Okamoto transformation corresponding to a convenient Möbius transformation in z gives us a linear differential equation (1–2) with $\theta_4 \neq 1$.

6. PROOF OF THEOREM 1.1 FOR ROWS 3, 4, 5, 6 OF TABLE 1

In this section we explain how to obtain the algebraic solutions of the Painlevé VI differential equation starting from the families of curves in Table 1, column 3, rows 3, 4, 5, 6, which are constructed directly from the Herfurtner list of families of elliptic curves (see [Herfurtner 91]). For the convenience of the reader we have listed the five families that we need in Table 2.

Let us consider $f(x) = 4x^3 - g_2 x - g_3$, where g_2, g_3 correspond to one of the five families of elliptic curves with parameter $\tilde{b}$ in the Herfurtner list. Since the roots of the discriminant $\Delta = 27g_2^3 - g_3^2$ of f, which will be the singular points of the corresponding differential equation for $\int_\delta \frac{dx}{y}$, are not all rational functions in $\tilde{b}$, we substitute $\tilde{b}$ by a suitable rational function in b such that the transformed roots are rational in b. Such substitutions are effected by the equalities $\tilde{b} = \cdots$ in Table 1, column 3.

In the next step we check whether the polynomial $f(x)$ factors over $\mathbb{Q}(b, z)$. It turns out that this happens only for the second family. In that case we have

$$f(x) = (4x^2 - \tilde{g}_2 x + \tilde{g}_3)\left(x + \frac{\tilde{g}_2}{4}\right). \qquad (6\text{–}1)$$

Substituting $\tilde{g}_2$ and $\tilde{g}_3$ in row 2 (respectively 1, 3, 4, 5) of Table 2 for t_2 and t_3 in (2–6) (respectively (2–5)), we obtain a system $DY = AY$.

We can now compute the second-order differential equation satisfied by $\int_\delta \frac{dx}{y}$, where δ is a Pochhammer cycle, using (3–3) and (3–2). It always turns out that (3–1) has four singularities, at $t_1 = t$, $t_2 = 0$, $t_3 = 1$, and $t_4 = \infty$, and one apparent singularity at λ with exponents 0 and 2.

Hence the SL-form $y'' = p(z)y$ can be written as follows [Iwasaki et al. 91, pp. 173–174]; see also Remark 4.1:

$$\frac{t(t-1)\cdot L}{z(z-t)(z-1)} + \frac{3}{4}\frac{1}{(z-\lambda)^2} - \frac{\lambda(\lambda-1)\cdot\nu}{z(z-1)(z-\lambda)},$$

where

$$a_i = \frac{1}{4}(\theta_i^2 - 1), \quad i = 1, 2, 3,$$

$$a_4 = -\frac{1}{4}\Big(\sum_{i=1}^{3} \theta_i^2 - \theta_4^2 - 1\Big) - \frac{1}{2},$$

and

$$L = K + \frac{(1-\theta_1)}{2}\left(\frac{1-\theta_2}{t} + \frac{1-\theta_3}{t-1} + \frac{1}{\lambda - t}\right).$$

Name	Deformation	
1	$g_2 = 3(\tilde z - 1)(\tilde z - \tilde b^2)^3$	$g_3 = (\tilde z - 1)(\tilde z - \tilde b^2)^4(\tilde z - \tilde b)$
2	$g_2 = 12\tilde z^2(\tilde z^2 + \tilde b\tilde z + 1)$	$g_3 = 4\tilde z^3(2\tilde z^3 + 3\tilde b\tilde z^2 + 3\tilde b\tilde z + 2)$
3	$g_2 = 12\tilde z^2(\tilde z^2 + 2\tilde b\tilde z + 1)$	$g_3 = 4\tilde z^3(2\tilde z^3 + 3(\tilde b^2 + 1)\tilde z^2 + 6\tilde b\tilde z + 2)$
4	$g_2 = 3\tilde z^3(\tilde z + \tilde b)$	$g_3 = \tilde z^5(\tilde z + 1)$
5	$g_2 = 3\tilde z^3(\tilde z + 2\tilde b)$	$g_3 = \tilde z^4(\tilde z^2 + 3\tilde b\tilde z + 1)$

TABLE 2. List of deformable families of elliptic curves $y^2 = 4x^3 - g_2x - g_3$ with four singular fibers and with nonconstant j-invariant along the deformation.

Example 6.1. We demonstrate how the above algorithm yields the results for the third row of Table 1. We start with the second family in Table 2. The roots of the discriminant of f are

$$0, \quad \omega_1, \quad \omega_2, \quad \omega_{1,2} = -\frac{1}{3}\left(2\tilde b - 1 \pm 2\sqrt{\tilde b^2 - \tilde b - 2}\right).$$

Solving the Diophantine equation $\alpha^2 - \alpha - 2 = \beta^2$, we substitute $\tilde b$ by $\frac{3}{4}(b + \frac{1}{b}) + \frac{1}{2}$ in g_2 and g_3. Thus the new roots are

$$t_1 = -b, \quad t_2 = 0, \quad t_3 = -\frac{1}{b}.$$

Since $f(x) = 4x^3 - g_2x - g_3$ factors, we obtain (6–1) with the new coefficients

$$\tilde g_2 := 4(\tilde z^2 + \tilde z),$$
$$\tilde g_3 := \frac{-9b^2\tilde z^3 - 8b\tilde z^4 + 2b\tilde z^3 - 8b\tilde z^2 - 9\tilde z^3}{b}.$$

Substituting $\tilde g_2$ and $\tilde g_3$ for t_2 and t_3 in (2–6), we obtain a system that does not fit in our paper, and we do not write it here. Computing the SL-form (3–2) of the differential equation for the first coordinate of the system, we obtain the following parameters:

$$(\theta_1, \theta_2, \theta_3, \theta_4) = \left(c - \frac{1}{2}, a + c - 1, c - \frac{1}{2}, a + c - 1\right),$$
$$\lambda = -b, \quad t = b^2,$$
$$\mu = \frac{-a - 2c + 2}{2b}, \quad \nu = -\frac{3}{4b},$$

and

$$L = \frac{1}{8b^4 - 8b^2}\big(2a^2b^2 + 4a^2b + 2a^2 + 4acb^2 + 8acb + 4ac - 4ab^2 - 8ab - 4a + 4c^2 - 2cb^2 - 4cb - 6c + 3b^2 + 6b\big).$$

We conclude that for a and c rational numbers, the SL-form of the Picard–Fuchs equation of the family of curves $y = (4x^2 - \tilde g_2x + \tilde g_3)^c(x + \tilde g_2/4)^a$ and the differential form $\frac{dx}{y}$ are of type (1–2) with the above parameters. Thus Theorem 1.1 for row 3 of Table 1 follows.

In a similar way, we have calculated all the data in Table 1, rows 4, 5, 6.

7. PROOF OF THEOREM 1.2, PART I

In this section we show that the algebraic solution in row 2 of Table 1 can be obtained via the middle convolution of the Schlesinger system corresponding to the algebraic solution in row 3 of Table 1.

Proposition 7.1. *The Painlevé VI equation with parameters in Table* 1*, row* 2*, is Okamoto equivalent to the Painlevé VI equation associated to the parameters in Table* 1*, row* 3*.*

Proof. We will show that the middle convolution relates the Schlesinger systems corresponding to the Painlevé VI equation with parameters in Table 1, row 2, and the Painlevé VI equation with the parameters in Table 1, row 3. Then the claim follows from Remark 5.1.

We continue with our example in Section 6.1 by writing down in Table 3 the corresponding Schlesinger system (4–1) using (4–2).

Now we proceed as explained in Section 5: We determine the middle convolution of $DY = QY$ with $\mu = -c$. Hence for the (Okubo) system

$$(zI_6 - T)DY = BY \iff DY = \sum_{i=1}^{3} \frac{B_i}{y - t_i}Y,$$
$$B_i \in \mathrm{Mat}_6(\mathbb{C}),$$

we get

$$B = \begin{pmatrix} Q_1 - c \cdot I_2 & Q_2 & Q_3 \\ Q_1 & Q_2 - c \cdot I_2 & Q_3 \\ Q_1 & Q_2 & Q_3 - c \cdot I_2 \end{pmatrix}$$

and $T = \mathrm{diag}(t, t, 0, 0, 1, 1)$.

$$
\begin{aligned}
q_{11}^1 &= \frac{(ba+a+2bc-3b-1)(a+2c-2)}{4b(a+c-2)},\\
q_{12}^1 &= \frac{1}{b(b-1)},\\
q_{21}^1 &= \frac{(b-1)(ba+a-2b+2c-2)(-ba-a-2bc+3b+1)(a+2c-2)(a-1)}{16b(a+c-2)^2},\\
q_{22}^1 &= -q_{11}^1 + (c-\frac{1}{2})\\
q_{11}^2 &= \frac{(-b^2+2b-1)(a+2c-2)(a-1)}{4b(a+c-2)},\\
q_{12}^2 &= \frac{1}{b},\\
q_{21}^2 &= \frac{(b-1)^2((2-a-2c)(a-1)(b^2+1)+(-2a^2-4ac+6a-4c^2+8c-4)b)(a+2c-2)(a-1)}{16b(a+c-2)^2},\\
q_{22}^2 &= -q_{11}^2 + (a-c-1),\\
q_{11}^3 &= \frac{(ba+a-b+2c-3)(a+2c-2)}{4(a+c-2)},\\
q_{12}^3 &= -\frac{1}{b-1},\\
q_{21}^3 &= \frac{(b-1)(ba+a-b+2c-3)(ba+a+2bc-2b-2)(a+2c-2)(a-1)}{16(a+c-2)^2},\\
q_{22}^3 &= -q_{11}^3 + (c-\frac{1}{2})\\
-(Q_1+Q_2+Q_3) &= \begin{pmatrix} -(a+2c-2) & 0 \\ 0 & -c \end{pmatrix}.
\end{aligned}
$$

TABLE 3. The coefficients of the Schlesinger system of Table 1, row 3.

The $\langle B_1, B_2, B_3\rangle$-invariant subspace has dimension 4, since

$$\ker Q_i = \left\langle \begin{pmatrix} q_{1,2}^i \\ -q_{1,1}^i \end{pmatrix} \right\rangle, \quad \mathfrak{I} = \langle (0,1,0,1,0,1)^{\mathrm{tr}} \rangle.$$

In order to obtain a 2-dimensional factor system, we set

$$S = \begin{pmatrix} q_{1,2}^1 & 0 & 0 & 0 & 0 & 0 \\ -q_{1,1}^1 & 0 & 0 & 1 & 0 & 0 \\ 0 & q_{1,2}^2 & 0 & 0 & 1 & 0 \\ 0 & -q_{1,1}^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & q_{1,2}^3 & 0 & 0 & 1 \\ 0 & 0 & -q_{1,1}^3 & 1 & 0 & 0 \end{pmatrix}.$$

Thus

$$
\begin{aligned}
D(SY) &= \sum_{i=1}^{3} \frac{SB_iS^{-1}}{z-t_i}(SY)\\
&= \begin{pmatrix} * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & \tilde{A} \end{pmatrix}(SY),
\end{aligned}
$$

and we obtain the 2-dimensional factor system $DY = \tilde{A}Y = \sum_{i=1}^{3} \frac{\tilde{A}_i}{z-t_i}Y$. Transforming this system into Schlesinger form via

$$Y \mapsto \tilde{S}Y, \quad \tilde{S} = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix},$$

where

$$
\begin{aligned}
N_1 &= -\Big(\sum_{i=1}^{3} \tilde{A}_i + c \cdot I_2\Big)_{1,2},\\
N_2 &= -\Big(\sum_{i=1}^{3} \tilde{A}_i - c \cdot I_2 - a \cdot I_2 + 2 \cdot I_2\Big)_{1,2},\\
N_3 &= \Big(\sum_{i=1}^{3} \tilde{A}_i + c \cdot I_2\Big)_{1,1},\\
N_4 &= \Big(\sum_{i=1}^{3} \tilde{A}_i - c \cdot I_2 - a \cdot I_2 + 2 \cdot I_2\Big)_{1,1},
\end{aligned}
$$

and the columns of $\tilde{S}$ consist of eigenvectors of $\sum_{i=1}^{3} \tilde{A}_i$ with respect to the eigenvalues $-c$ and $c+a-2$, we obtain finally

$$DY = AY, \quad A = \frac{A_1}{z-b^2} + \frac{A_2}{z} + \frac{A_3}{z-1}, \tag{7–1}$$

$$q^1_{1,1} = \frac{(18a^2b^2 + 18a^2b + 18a^2 - 3ab^3 - 30ab^2 - 48ab - 36a + 2b^3 + 12b^2 + 24b + 16)}{18ab + 18a - 27b - 27}$$
$$q^1_{1,2} = \frac{-b^4 - 2b^3}{b^2 - 1}$$
$$q^1_{2,1} = \frac{q^1_{11} q^1_{22}}{q^1_{12}}$$
$$q^1_{2,2} = (a - \frac{1}{2}) - q^1_{1,1}$$
$$q^2_{1,1} = \frac{(3a-2)(b^2 + (-6a+7)b + 1)(b-1)^2}{18ab^2 - 27b^2}$$
$$q^2_{1,2} = b^2 + 2b$$
$$q^2_{2,1} = \frac{q^2_{11} q^2_{22}}{q^2_{12}}$$
$$q^2_{2,2} = (3a - \frac{3}{2}) - q^2_{1,1}$$
$$q^3_{1,1} = \frac{(18a^2b^3 + 18a^2b^2 + 18a^2b - 36ab^3 - 48ab^2 - 30ab - 3a + 16b^3 + 24b^2 + 12b + 2)}{18ab^3 + 18ab^2 - 27b^3 - 27b^2}$$
$$q^3_{1,2} = \frac{b^2 + 2b}{b^2 - 1}$$
$$q^3_{2,1} = \frac{q^3_{11} q^3_{22}}{q^3_{12}}$$
$$q^3_{2,2} = (a - \frac{1}{2}) - q^3_{1,1}$$
$$-(Q_1 + Q_2 + Q_3) = \begin{pmatrix} -3a+2 & 0 \\ 0 & -2a + \frac{1}{2} \end{pmatrix}.$$

TABLE 4. The coefficients of the Schlesinger system of Table 1, row 4.

where

$$A_1 = \frac{1}{4b} \begin{pmatrix} -ab-a-2bc+b+1 & ab+a+2bc-3b-1 \\ -ab-a-2bc+b+1 & ab+a+2bc-3b-1 \end{pmatrix},$$
$$A_2 = \frac{1}{4b} \begin{pmatrix} ab^2+2ab+a-b^2-2b-1 & ab^2-a-b^2+1 \\ -ab^2+a+b^2-1 & -ab^2+2ab-a+b^2-2b+1 \end{pmatrix},$$
$$A_3 = \frac{1}{4} \begin{pmatrix} -ab-a+b-2c+1 & -ab-a+b-2c+3 \\ ab+a-b+2c-1 & ab+a-b+2c-3 \end{pmatrix},$$

and

$$-(A_1 + A_2 + A_3) = \begin{pmatrix} c & 0 \\ 0 & -(a+c-2) \end{pmatrix}.$$

Computing the entries of A, we get

$$A_{11} = \frac{1}{4z(z-1)(z-b^2)} \times \big(-4cz^2 + ((1-a+2c)(b^2+1) + 2(1-a)b)z + (a-1)(b+1)^2 b \big),$$
$$A_{12} = \frac{1}{4z(z-1)(z-b^2)} \times \big((b^2-1)((a+2c-3)z + b(a-1)) \big),$$
$$A_{21} = \frac{1}{4z(z-1)(z-b^2)} \times \big((b^2-1)((1-a-2c)z + (b-ab)) \big),$$
$$A_{22} = \frac{1}{4z(z-1)(z-b^2)} \times \big(4(a+c-2)z^2 + ((5-3a-2c)(b^2+1) + 2(a-1)b)z + (1-a)(b-1)^2 b \big).$$

Thus the parameters are

$$(\theta_1, \ldots, \theta_4) = \left(-\frac{1}{2},\ a-1,\ -\frac{1}{2},\ -(a+2c-3) \right).$$

The apparent singularity λ_1 for the first coordinate and λ_2 for the second coordinate is the zero of the numerator of A_{12} respectively A_{21} (see Section 4):

$$\lambda_1 = \left(\frac{-a+1}{a+2c-3} \right) b, \quad \lambda_2 = \left(\frac{-a+1}{a+2c-1} \right) b.$$

Since $t = b^2$, we obtain the relation

$$\theta_4^2 \lambda_1^2 = t\theta_2^2, \quad (2-\theta_4)^2 \lambda_2^2 = t\theta_2^2.$$

Computing the SL-form for the first coordinate after transforming $Y \mapsto (z-1)^{1/2}(z-t)^{1/2} Y$, which changes

$$q^1_{1,1} = \frac{(3a-2)(b+(-6a+2))(b+2)^2}{9(4a-1)(b+1)}$$
$$q^1_{1,2} = \frac{b+2}{(72a-18)(b+1)}$$
$$q^1_{2,1} = \frac{(3a-2)((6a-4)b^3+(-36a^2+60a-24)b^2+(24a-21)b-5)(-b+(6a-2))(b+2)}{9(4a-1)(b+1)}$$
$$q^1_{2,2} = (-2a+\frac{3}{2}) - q^1_{1,1}$$
$$q^2_{1,1} = \frac{(-3a+2)(b^2+(-6a+4)b+1)(b^2+b+1)}{9(4a-1)b^2}$$
$$q^2_{1,2} = \frac{-(b^2+b+1)}{18(4a-1)b^2}$$
$$q^2_{2,1} = \frac{((4-6a)b^4+(36a^2-54a+20)b^3+(36a^2-96a+33)b^2+(36a^2-54a+20)b+4-6a)}{9(4a-1)b^2}(3a-2)(-b^2+(6a-4)b-1)$$
$$q^2_{2,2} = \frac{1}{2} - q^2_{1,1}$$
$$q^3_{1,1} = \frac{(3a-2)((-6a+2)b+1)(2b+1)^2}{9(4a-1)b^2(b+1)}$$
$$q^3_{1,2} = \frac{2b+1}{18(4a-1)b^2(b+1)}$$
$$q^3_{2,1} = \frac{(3a-2)(-5b^3+(24a-21)b^2+(-36a^2+60a-24)b+(6a-4))((6a-2)b-1)(2b+1)}{9(4a-1)b^2(b+1)}$$
$$q^3_{2,2} = (-2a+\frac{3}{2}) - q^3_{1,1}$$
$$-(Q_1+Q_2+Q_3) = \begin{pmatrix} 3a-2 & 0 \\ 0 & a-\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 3a-2 & 0 \\ 0 & 3a-2+(-2a+\frac{3}{2})-1 \end{pmatrix}.$$

TABLE 5. The coefficients of the Schlesinger system of Table 4 after convolution.

the parameters $(\theta_1,\dots,\theta_4)$ (but not the apparent singularities) to $(1/2, a-1, 1/2, -(a+2c-3))$, we obtain a simple formula for μ_1:

$$\mu_1 = \frac{-a-2c+3}{2b}. \qquad \square$$

8. PROOF OF THEOREM 1.2, PART II

As in the previous example, we show that the middle convolution relates the results in Table 1, rows 4 and 5.

Proposition 8.1. *The Painlevé VI equation with parameters in Table* 1*, row* 4*, is Okamoto equivalent to the Painlevé VI equation associated to the parameters in Table* 1*, row* 5*.*

Proof. We will show as in Proposition 7.1 that the middle convolution relates the Schlesinger systems corresponding to the Painlevé VI equation with parameters in Table 1, row 4, and the Painlevé VI equation with the parameters in Table 1, row 5. Then the claim follows again from Remark 5.1

The system (4–1) in Schlesinger form corresponding to the differential equation satisfied by $\int_\delta \frac{dx}{y}$, where y is taken from row 4 in Table 1, is given in Table 4.

Applying the middle convolution to $DY = QY$ with $\mu = -(3a-2)$ and transforming the 2-dimensional factor system into Schlesinger form with singularities at t, 0, 1, and ∞, we get the system (4–1) values as shown in Table 5.

We obtain the parameters

$$(\theta_1,\dots,\theta_4) = \left(-2a+\frac{3}{2}, \frac{1}{2}, -2a+\frac{3}{2}, -2a+\frac{3}{2}\right).$$

The apparent singularity for the first coordinate is

$$\lambda = \frac{b^2+b+1}{b^3+2b^2}.$$

Using that

$$t = \frac{2b+1}{b^4+2b^3},$$

we see that λ and t satisfy

$$\lambda^4 - 2t\lambda^3 - 2\lambda^3 + 6t\lambda^2 - 2t^2\lambda - 2t\lambda + t^3 - t^2 + t = 0.$$

Since this is also the relation for

$$t = \frac{b^4 - 6b^2 - 8b - 3}{b^4 - 6b^2 + 8b - 3}, \quad \lambda = \frac{b^3 + b^2 + 3b + 3}{b^3 + b^2 - 5b + 3},$$

the claim follows from Table 1, row 5. □

9. PROOF OF THEOREM 1.1 FOR ROW 2 OF TABLE 1

In this section we show that (7–1) arises also as a pullback of hypergeometric differential equations. By determining the monodromy group, it turns out that this monodromy group is finite, and therefore the claim follows by a well-known result of Klein. Furthermore, we indicate that all the other families of Picard–Fuchs equations are related to those with finite monodromy.

Proposition 9.1. *The monodromy group of* (7–1) *is (up to conjugation) contained in the group* $T \rtimes 2$*, where* T *denotes the group of diagonal matrices in* $\mathrm{GL}_2(\mathbb{C})$*. Moreover, if the parameters* a *and* c *are rational numbers, i.e., if* (7–1) *is a Picard–Fuchs equation, then the monodromy group is even finite, i.e., a dihedral group, and the differential equation is a pullback of a hypergeometric differential equation.*

Proof. To prove this statement, let M_t, M_0, M_1 denote the monodromy at t, 0, and 1 of (7–1). If the parameter b tends to 1, we see that the system becomes reducible and the monodromy group is abelian:

$$A \to \begin{pmatrix} \frac{-a-cz+1}{z^2-z} & 0 \\ 0 & \frac{a+c-2}{z-1} \end{pmatrix}.$$

Hence $M_t M_1$ and M_0 are diagonal matrices and therefore commute. Since M_t and M_1 are reflections, the group generated by them is a dihedral group. Thus M_t and M_1 also normalize M_0, and we get that

$$\langle M_t, M_1, M_0 \rangle \subseteq T \rtimes 2.$$

Hence if the parameters a and c are rational numbers, the monodromy group is finite. And in this case the differential equation is a pullback of a hypergeometric one by a well-known result of Klein; see [Klein 84, I, Chapter 3]. □

Proposition 9.2. *Via the middle convolution of the Schlesinger system corresponding to row* 5 *(respectively row* 6*), column* 2*, in Table* 1*, we obtain a three-dimensional system with imprimitive monodromy group contained in* $T \rtimes S_3$*, where* T *denotes the group of diagonal matrices in* $\mathrm{GL}_3(\mathbb{C})$ *and* S_3 *the symmetric group on three letters. Moreover, these groups are finite imprimitive reflection groups if the parameter* a *is a rational number.*

Proof. We only sketch the proof, since it requires the explicit construction of the 3-dimensional Fuchsian systems to determine the monodromy.

Applying the Möbius transformation $z \mapsto \frac{1}{z}$ that permutes the residue matrices Q_i of the Schlesinger system corresponding to row 5 (respectively row 6), column 2, in Table 1 and scaling the new Q_2, we get the following pairs of eigenvalues for $Q_1, Q_2, Q_3, -(Q_1+Q_2+Q_3)$; see Section 5:

$$\left(a - \frac{1}{2}, 0\right), \quad \left(a - \frac{3}{2}, 0\right),$$
$$\left(a - \frac{1}{2}, 0\right), \quad \left(-\frac{3a-3}{2}, -\frac{3a-2}{2}\right),$$

respectively

$$\left(a - \frac{1}{2}, 0\right), \quad (-2a+2, 0),$$
$$\left(a - \frac{1}{2}, 0\right), \quad \left(-\frac{1}{3}, -\frac{2}{3}\right).$$

The middle convolution with $\mu = -(a-1)$ yields a three-dimensional Fuchsian system with the following triples of eigenvalues of the residue matrices

$$\left(\frac{1}{2}, 0, 0\right), \quad \left(-\frac{1}{2}, 0, 0\right),$$
$$\left(\frac{1}{2}, 0, 0\right), \quad \left(-\frac{a-1}{2}, -\frac{a}{2}, a-1\right),$$

respectively

$$\left(\frac{1}{2}, 0, 0\right), \quad (-3a+3, 0, 0),$$
$$\left(\frac{1}{2}, 0, 0\right), \quad \left(-\frac{1}{3} + a - 1, -\frac{2}{3} + a - 1, a-1\right).$$

Using the explicit construction for the middle convolution and similar arguments as in the above proposition, one easily sees that the monodromy groups of these third-order differential equations are imprimitive reflection groups contained in $T \rtimes S_3$, where T denotes the group of diagonal matrices in $\mathrm{GL}_3(\mathbb{C})$. These become finite if the parameter a is rational. □

Remark 9.3. In the previous section we showed that in Table 1, rows 4 and 5 are also related via the middle convolution. Hence all the families of Picard–Fuchs equations corresponding to Table 1 are related to those with finite monodromy.

Since the corresponding Picard–Fuchs differential equation of row 1 in Table 1 is the hypergeometric one, it is well known that it is obtained via the convolution of a one-dimensional differential equation with finite monodromy. Since a differential equation with finite monodromy has obviously a finite braid group orbit and the middle convolution commutes with braiding (see [Dettweiler and Reiter 07a, Theorem 2.4]), our examples corresponding to Table 1 have a finite braid orbit. But this implies that the solutions of the Painlevé VI equations are algebraic (the ramification data at $0, 1, \infty$ of $\mathbb{Q}(\lambda, t)/\mathbb{Q}(t)$ are given by the cycle decomposition of the braids $\beta_1, \beta_2, \beta_1\beta_2$). This yields the remark following Theorem 1.2 in the introduction.

ACKNOWLEDGMENTS

The second author gratefully acknowledges financial support by the CNPq and thanks IMPA for its hospitality. The authors thank L. Gavrilov for useful discussions in the initial steps of preparing the present article.

REFERENCES

[André 89] Y. André. *G-Functions and Geometry*, Aspects of Mathematics E13. Braunschweig: Friedr. Vieweg & Sohn, 1989.

[Ben Hamed and Gavrilov 05] B. Ben Hamed and L. Gavrilov. "Families of Painlevé VI Equations Having a Common Solution." *Int. Math. Res. Not.* 60 (2005), 3727–52.

[Boalch 03] P. Boalch. "Painlevé Equations and Complex Reflections." *Ann. Inst. Fourier* 53:4 (2003), 1009–22.

[Boalch 06] P. Boalch. "Six Results on Painlevé VI, Théories Asymptotiques et Équations de Painlevé." In *Sémin. Congr. 14, Soc. Math. France*, edited by Éric Delabaere and Michéle Loday-Richaud, pp. 1–20. Paris: Université d'Angers, 2006.

[Cantat and Loray 07] S. Cantat and F. Loray. "Holomorphic Dynamics, Painlevé VI Equation and Character Varieties." arXiv :0711.1579v2, 1–69, 2007.

[Dettweiler and Reiter 00] M. Dettweiler and S. Reiter. "An Algorithm of Katz and Its Application to the Inverse Galois Problem." *J. Symbolic Comput.* 30 (2000), 761–98.

[Dettweiler and Reiter 07a] M. Dettweiler and S. Reiter. "Middle Convolution of Fuchsian Systems and the Construction of Rigid Differential Systems." *J. Algebra* 318:1 (2007), 1–24.

[Dettweiler and Reiter 07b] M. Dettweiler and S. Reiter. "Painlevé Equations and the Middle Convolution." *Adv. Geom.* 7:3 (2007), 317–30.

[Doran 01] C. F. Doran. "Algebraic and Geometric Isomonodromic Deformations." *J. Differential Geom.* 59:1 (2001), 33–85.

[Haraoka and Filipuk 07] Y. Haraoka and G. Filipuk. "Middle Convolution and Deformation for Fuchsian Systems." *J. Lond. Math. Soc. (2)* 76:2 (2007), 438–50.

[Greuel et al. 01] G.-M. Greuel, G. Pfister, and H. Schönemann. "SINGULAR 2.0: A Computer Algebra System for Polynomial Computations." Centre for Computer Algebra, University of Kaiserslautern. Available online (http://www.singular.uni-kl.de), 2001.

[Griffiths 66] P. A. Griffiths. "The Residue Calculus and Some Transcendental Results in Algebraic Geometry, I, II." *Proc. Nat. Acad. Sci. USA.* 55 (1966), 1303–09, 1392–95.

[Herfurtner 91] S. Herfurtner. "Elliptic Surfaces with Four Singular Fibres." *Math. Ann.* 291:2 (1991), 319–42.

[Iwasaki et al. 91] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. *From Gauss to Painlevé*, Aspects of Mathematics E16. Braunschweig: Friedr. Vieweg & Sohn, 1991.

[Katz 70] M. N. Katz. "Nilpotent Connections and the Monodromy Theorem: Applications of a Result of Turrittin." *Inst. Hautes Études Sci. Publ. Math.* 39 (1970), 175–232.

[Katz 96] M. N. Katz. *Rigid Local Systems*, Annals of Mathematical Studies 139. Princeton: Princeton UP, 1996.

[Kitaev 05] A. V. Kitaev. "Grothendieck's Dessins d'Enfants, Their Deformations, and Algebraic Solutions of the Sixth Painlevé and Gauss Hypergeometric Equations." *Algebra i Analiz* 17:1 (2005), 224–75.

[Klein 84] F. Klein. 1884. *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, edited by Peter Slodowy. Basel: Birkhäuser and Stuttgart: B. G. Teubner, 1993.

[Movasati 08] H. Movasati. "On Ramanujan Relations between Eisenstein Series." In preparation, 2008.

[Shiga et al. 04] H. Shiga, T. Tsutsui, and J. Wolfart. "Triangle Fuchsian Differential Equations with Apparent Singularities." *Osaka J. Math.* 41:3 (2004), 625–58.

Hossein Movasati, Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil (hossein@impa.br)

Stefan Reiter, Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil (reiter@impa.br)

Received December 3, 2008; accepted June 22, 2009.