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Spectral Resonances which Become Eigenvalues

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Abstract: The stationary Schrödinger equation is $-\partial_x^2\phi + \lambda V(x)\,\phi = z\phi$ for $\phi \in \mathscr{L}^2(\mathbf{R}^+, dx)$. If the potential is bounded below, singular only at $x = 0$, negative on some compact interval and behaves like $V(x) \sim 1/x^\mu$ as $x \to \infty$ with $2 \geqq \mu > 0$, then the system admits shape resonances which continuously become eigenvalues as λ increases. Here $\lambda > 0$ and for $\mu = 2$ a sufficiently large λ is required. Exponential bounds are obtained on $\mathrm{Im}(z)$ as λ approaches a threshold. The group velocity near threshold is also estimated.

1. Introduction

We study the transition of a *spectral resonance* (s.r.) *value* to an eigenvalue which occurs at thresholds of the coupling parameter λ. A typical system is,

$$H^\lambda = -\frac{d^2}{dx^2} + \lambda V(x) \quad \text{on } \mathscr{L}^2(\mathbf{R}^+, dx), \quad \text{for } \mathbf{R}^+ \equiv (0, \infty)\ , \tag{1.1a}$$

with Dirichlet B.C. at $x = 0$ and having a shape resonance potential of the form,

$$V(x) = \begin{cases} -V_{\min}, & 0 < x < b \\ V_M, & b < x < c\ , \\ V_M(c/x)^\mu, & x > c \end{cases} \tag{1.1b}$$

where $2 \geqq \mu > 0$ and $V_{\min}, V_{\mathrm{M}}$ are positive constants. The physically interesting $\mu = 2$ case requires λ sufficiently large. For $\mu > 2$ our methods break down. One serious problem is that the Agmon length of V at 0 energy is finite if $\mu > 2$. We refer the reader to [6] for a discussion which does not use shape-resonance theory.

The shape resonance problem has been studied by many authors (see [1] for an extensive list) but mostly in the non-threshold cases $-V_{\min} > V(\infty) = 0$ (see [8] for a consideration of the threshold case). Here we continue the work of [5] by studying the past-threshold case (i.e. $-V_{\min} < 0$). It is demonstrated that the appearance of eigenvalues from the bottom of the essential spectrum, in the $\mu \leqq 2$ cases, is due to the smooth transition of an s.r.value to an eigenvalue. We use an improved *basic resolvent estimate* which allows one to avoid the introduction of an exterior Hamiltonian. The problem studied here has application to the multichannel scattering theory of fiber optics [4].

In the remainder of the paper the word *threshold* will correspond to the following,

Definition 1.1. *Let A be a positive self-adjoint operator with $\sigma(A) = [0, \infty)$. Let B be a symmetric A-compact operator which is bounded from below and let $\mu_n(\lambda)$ be the n^{th}-eigenvalue of $H^\lambda \equiv A + \lambda B$, $\lambda > 0$ Then $\tau_n \equiv \liminf\{\lambda \mid \mu_n(\lambda) < 0\}$ is called the **n^{th}-threshold** of H^λ. We say that τ_n is **non-degenerate** if $\mu_n(\lambda)$ has multiplicity 1 for $\lambda > \tau_n$ and $\tau_{n-1} < \tau_n < \tau_{n+1}$.*

The following is a result of Simon:

Theorem 1.2. *If* $\ker(A) = \{0\}$ *in the above definition and τ_n is non-degenerate, then*

$$\lim_{\lambda \downarrow \tau_n} \frac{\mu_n(\lambda)}{\lambda - \tau_n} = \mu_n'(\tau_n) , \tag{1.2a}$$

exists and is non-zero (negative) iff H^{τ_n} has a zero eigenvalue with normalized eigenfunction ϕ_n. In this case,

$$\mu_n'(\tau_n) = \langle \phi_n, B\phi_n \rangle \equiv -\mathscr{V}_n . \tag{1.2b}$$

Two types of threshold behaviors occur: Either $\mu_n'(\tau_n) = 0$, in which case $\mu_n(\lambda) \sim o(\lambda - \tau_n)$ (i.e. super-linear), or $\mu_n'(\tau_n) \neq 0$, so that $\mu_n(\lambda) = \mu_n'(\tau_n)(\lambda - \tau_n) + o(\lambda - \tau_n)$ (i.e. linear; see Fig. 1). Here we study the linear case and show that there is a differentiable continuation of the eigenvalue into the 2$^{\text{nd}}$ Riemann sheet of $\sigma(H^\lambda)$. We will not use the explicit techniques of [3], although the work of Klaus and Simon reveals a close connection between conditions (H1) and (H2) below.

We consider a class of potentials which satisfy the conditions (see Fig. 2);

(H1) a) $\exists V_{\min}$, $V_M > 0$ and $c > b > a \geqq 0$: essinf $V = -V_{\min}, \inf_{(b,c)} V(x) \geqq V_M$, $\inf_{(0,a)} V(x) \geqq V_M$, and if $a > 0$ then $V(x) \to \infty$ as $x \to 0^+$ is allowed;
b) $\forall x \geqq c$ **i)** $\exists \mu_i \in (0,2)$ and $V_\infty > 0 : xV'/V$, $\ln(V_\infty/V)/\ln(x/c) \in [\mu_1, \mu_2]$,
OR ii) $\mu_2 = 2$ and $\exists V_i > 0 : x^{2-\mu_1} V_1 \leqq x^2 V(x) \leqq V_2$, $\mu_1 \leqq (-xV'/V) \leqq 2$.

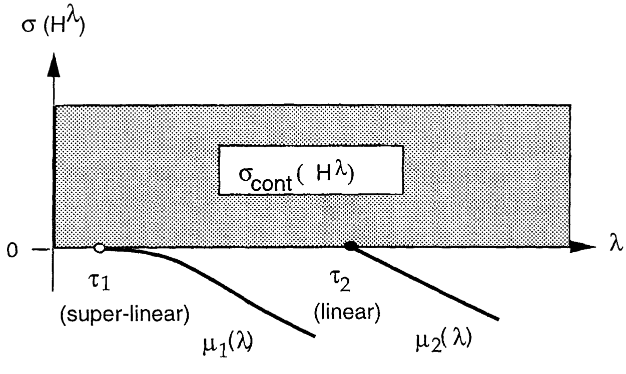

Fig. 1. Spectral dependence on coupling parameter

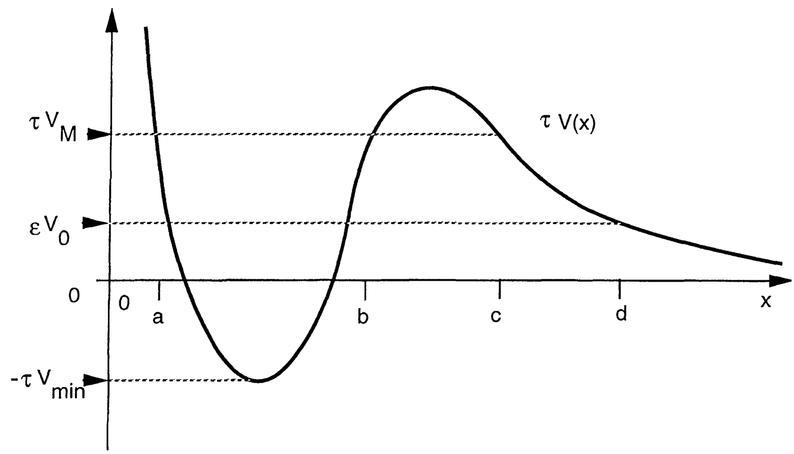

Fig. 2. Past-threshold shape resonance potential

(H2) $\exists \tau > 0$ a non-degenerate threshold of H^λ with eigenvalue E^λ so that,
a) $E^\tau = 0 \in \sigma_{pp}(H^\tau)$;
b) $\exists \delta_0 > 1$ so that for $\mathscr{V}$ in (1.2b) and $\lambda > \tau$: $|E^\lambda + (\lambda - \tau)\mathscr{V}| = \mathscr{O}(\lambda - \tau)^{\delta_0}$.

(H3) $\exists \beta_0,\ \varepsilon_0 > 0$ and an analytic function $\hat{V}(\kappa)$ defined on the truncated cone,

$$\mathscr{C}_{\beta_0}(c) = \{c + \varepsilon_0 + \rho e^{i\beta} | \rho > \varepsilon_0, |\beta| \leqq \beta_0\},$$

where

a) $\hat{V}(x) = V(x)$ for $x \in (c + 2\varepsilon_0, \infty)$;
b) $\forall d > c,\ \kappa \in \mathscr{C}_{\beta_0}(d), |(\kappa - d)^j \hat{V}^{(j)}(\kappa)/(\hat{V}(\kappa) - V(d))| < \infty$ uniformly, $j \leqq 3$;
c) $\exists C_i > 0$ so that $\forall \kappa \in \mathscr{C}_{\beta_0}(c)$, $|\mathrm{Im}\, \hat{V}(\kappa)| \leqq C_1 |\mathrm{Re}\, \hat{V}(\kappa)| + C_2$.

Main Theorem. *Suppose that conditions* (H1)–(H3) *hold* i) *If* $\mu_2 < 2$ *in* (H1)b) *then* H^λ *has a unique s.r. value* $z(\lambda)$ *for* $\lambda < \tau$ *sufficiently large, so that* $z(\tau^-) = 0$ *and* $\exists \delta_2 > 1$ *and* $C_i > 0$ *so that,*

$$|(\lambda - \tau)\mathscr{V} - z(\lambda)| \leqq C_1(\tau - \lambda)^{\delta_2}, \tag{1.3a}$$

$$0 < \mathrm{Im}(z(\lambda)) < C_2 e^{-\rho_\mu(\tau,\lambda)}, \tag{1.3b}$$

$$|\mathscr{V} + z'(\lambda)| \leqq C_3(\tau - \lambda)^{\delta_2 - 1}, \tag{1.3c}$$

where $\rho_\mu = C\tau^{(2-\mu_1-\mu_2)/2\mu_1}(\tau - \lambda)^{-(2-\mu_2)/2\mu_1} - C_0\sqrt{\tau}$. ii) *If* $\mu_2 = 2$ *then* $\delta_2 = 1$, *however* C_1 *and* C_3 *can be decreased if* τ *is increased. Furthermore* $\rho_2 = -C\sqrt{\tau}(\ln(1 - \lambda/\tau) + C_0)$.

Remark. Conditions (H1) and (H2) are satisfied by many single-bottom potentials of interest including those found in higher-dimensional problems [3]. The requirement $\mu_1 > 0$ in (H1)b) is not essential but is best handled on a case-by-case basis. Much of our techniques still work if $\mu_n(\lambda)$ approaches 0 super-linearly, but we do not consider the details here. The non-degeneracy condition can be dropped if extra conditions on the splitting of eigenvalues are imposed [8].

The paper is organized as follows. In Sect. 2 some notation is defined. In Sect. 3 we consider the details for handling the $2 > \mu > 0$ cases. Modifications required to handle the $\mu = 2$ case are discussed in Sect. 4. We also complete the proof of the Main Theorem. In particular, details are given for estimation of the group velocity.

2. Definitions and Notations

In this paper we work on the Hilbert space $\mathscr{H} \equiv \mathscr{L}^2(\mathbf{R}^+, dx)$ with norm denoted $\|\cdot\|$ and inner product $\langle\cdot\,,\,\cdot\rangle$. The essential-sup norm is $\|\cdot\|_\infty$. If $S \subset \mathbf{R}^+$ then $\chi_S(x)$ is the characteristic function of S. The domain of an operator A is denoted $\mathscr{D}(A)$ and its adjoint is A^*. The commutator of operators A and B is $[A,B] \equiv AB - BA$. A full derivative d/dx is simply written as ∂_x, whereas the momentum operator is $p \equiv -\mathrm{i}d/dx$. Constants C are bounded, positive and independent of relevant parameters (unless indicated).

Let H be a self-adjoint operator on a Hilbert space $\mathscr{H}$. A complex number $z \notin \sigma(H)$, is called an *s.r.value* if $\exists$ a group of operators $\mathscr{U}_\theta$ (which are unitary for $\theta \in \mathbf{R}$) so that $z \in \sigma(H(\theta))$, where $H(\theta) \equiv \mathscr{U}_\theta H \mathscr{U}_{-\theta}$ is an analytic family of operators with $\mathscr{D}(H(\theta)) = \mathscr{D}(H)$. See Fig. 3. In most applications $\mathrm{Im}(\theta)$ is restricted, hence not all resonance values can be studied in this way.

When H is a Schrödinger operator the *Agmon length* of the potential $V(x)$ at energy E is defined to be

$$\rho = \int_{b_-}^{\infty} \sqrt{\max\{0, V(x) - E\}}\, dx \;, \tag{2.1}$$

where $b_- \equiv \sup\{x \in (a,b) \,|\, V(x) < E\}$ is the *interior turning point*. This provides an important measure of the exponential decay of eigenfunctions in the Classically Forbidden region of H at E. For a detailed discussion see [2, Sect. 7], or [8, Sect. 11].

3. The $2 > \mu_i > 0$ Cases

We assume (H1)–(H3) hold throughout our discussion. This ensures that eigenvalues have finite multiplicity and that $\sigma(H^\lambda)$ has no positive eigenvalues or singular

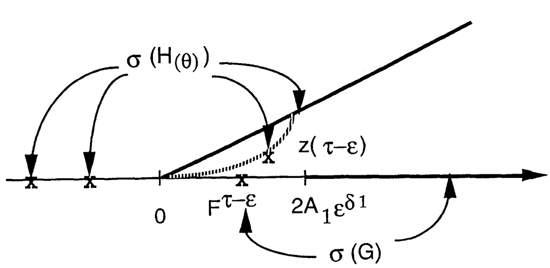

Fig. 3. Comparison of spectra

continuous spectrum. For a non-degenerate threshold $\tau > 0$ we define a comparison operator to H^λ for $\lambda < \tau$, as originally used in [8],

$$G^\lambda = -\partial_x^2 + \lambda V(x)\chi_{I_c} + 2(\tau - \lambda)^{\delta_1} A_1 \chi_{J_c} \quad \text{on } \mathscr{H} , \tag{3.1}$$

with Dirichlet B.C. at $x = 0$. Here $\tau V(c_1) \equiv A_1(\tau - \lambda)^{\delta_1}$ for some fixed value of $\delta_1 \in (\max\{2 - \delta_0, \mu_2/2\}, 1)$, $A_1 > 0$ and $I_c \equiv (0, c_1)$, $J_c \equiv (c_1, \infty)$. To simplify notation define $\varepsilon = \tau - \lambda$ and let $H \equiv H^{\tau-\varepsilon}$, $G \equiv G^{\tau-\varepsilon}$. The proof of the following is given in the appendix:

Lemma 3.1. *With the above conditions* $\exists C, \mathscr{V}, \varepsilon_1 > 0$ *and a unique eigenvalue* $F^{\tau-\varepsilon} \in \sigma_{pp}(G) \cap (0, 2A_1\varepsilon^{\delta_1})$ *so that,*

$$|F^{\tau-\varepsilon} - \varepsilon\mathscr{V}| < C\,\varepsilon^{2-\delta_1}/A_1, \quad \textit{for } 0 \leqq \varepsilon \leqq \varepsilon_1 .$$

We now construct an analytic family of operator associated with H. The group of operators $\mathscr{U}_\theta^\varepsilon$ are obtained from the vector field $v_\varepsilon(x)$ defined on $\mathbf{R}^+$ by,

$$v_\varepsilon(x) = (x - d_1)(1 - \exp[-\sigma(B_1 - (\tau/\varepsilon)V(x))])\chi_{(d_1,\infty)}(x) . \tag{3.2}$$

Here $B_1 > \mathscr{V}$ and $\tau V(d_1) = B_1\varepsilon$. This generates a flow φ_θ on $\mathbf{R}^+$, which is the solution of the differential equation,

$$\frac{d}{d\theta}\varphi_\theta(x) = v_\varepsilon[\varphi_\theta(x)], \quad \varphi_0(x) = x . \tag{3.3a}$$

For $\theta = i\beta$ the analysis of [8] gives,

$$|\varphi_{i\beta}(x) - x| \leqq \beta v_\varepsilon(x) , \tag{3.3b}$$

which implies that Image$(\varphi_\beta) \subset \mathscr{C}_{\beta_0}(c)$ for $|\beta| \leqq \beta_0$. Hence, from condition (H3),

$$V_{i\beta}(x) \equiv \hat{V}(\varphi_{i\beta}(x)) = V(x) + i\beta v_\varepsilon(x)V'(x) + R , \tag{3.3c}$$

with $|R| \leqq C\beta^2 V(x)$. Furthermore, the transformations,

$$\mathscr{U}_\theta^\varepsilon \psi(x) = \mathscr{J}_\theta^{1/2}\psi(\varphi_\theta(x)) , \tag{3.4}$$

are well defined for $|\mathrm{Im}(\theta)| \leqq \beta_0$ and are unitary for $\theta \in \mathbf{R}$. The Jacobian is defined as $\mathscr{J}_\theta = |\partial_x\varphi_\theta|$. The analytic family of Hamiltonians become,

$$H(\theta) \equiv \mathscr{U}_\theta^\varepsilon H \mathscr{U}_{-\theta}^\varepsilon = p_\theta^2 + (\tau - \varepsilon)V_\theta(x) , \tag{3.5a}$$

where the kinetic term is,

$$p_\theta^2 = p\mathscr{J}_\theta^{-2}p + \mathscr{F}_\theta , \tag{3.5b}$$

and the remainder term is,

$$\mathscr{F}_\theta = (1/2)\mathscr{J}_\theta^{-3}\mathscr{J}_\theta'' - (5/4)\mathscr{J}_\theta^{-4}(\mathscr{J}_\theta')^2 . \tag{3.5c}$$

A rather tedious calculation using (3.2) and (H3)b) gives the following bounds:

$$\|1 - \mathscr{J}_{i\beta}\|_\infty \leqq C\beta\sigma, \qquad \|\mathscr{J}_{i\beta}^{(k)}\|_\infty \leqq C\beta\sigma\,\varepsilon^{k/\mu_2} \quad (\text{for } k = 1, 2) , \tag{3.6a}$$

$$\|\mathscr{F}_{i\beta}\|_\infty \leqq C\beta\sigma(1 + \beta\sigma)\varepsilon^{2/\mu_2}, \qquad \|\mathrm{Re}(\mathscr{F}_{i\beta})\|_\infty \leqq C\beta^2\sigma^2\varepsilon^{2/\mu_2} . \tag{3.6b}$$

Next we choose $B_1 > \mathscr{V}$ and define $B_2 \in (\mathscr{V}, B_1)$ so that

$$B_2 < \mathscr{V} \frac{2}{2 - \mu_1 + \mu_1 d_1/d_2} , \tag{3.7}$$

where $\tau V(d_i) = B_i \varepsilon$. This estimate can be made to hold due to (H1)b), since $d_2 \to \infty$ as $B_2 \to \mathscr{V}^+$. It is now clear that on $[d_2, \infty)$, $\exists K_\mu > 0$ so that, ignoring lower order terms,

$$\begin{aligned} 2\varepsilon\mathscr{V} - \tau(2V + vV') &\geqq 2\varepsilon\mathscr{V} - \tau V[2 - (1 - d_1/x)(-xV'/V)] \\ &\geqq \varepsilon(2\mathscr{V} - (2 - \mu_1)B_2 - (d_1/d_2)\mu_1 B_2) \geqq K_\mu \varepsilon , \end{aligned} \tag{3.8}$$

which is the key Exterior estimate. We also fix $B_3 \in (\mathscr{V}, B_2)$. For $A_0 > A_1$ choose the minimal distances c_i which satisfy the condition $\tau V(c_i) = A_i \varepsilon^{\delta_1}$. Now for some smooth step function $0 \leqq q(x) \leqq 1$, define the cutoff functions $g_k(x)$,

$$1 \equiv g_1^2 + g_2^2 + g_3^2, \qquad g_0^2 \equiv g_1^2 + g_2^2, \qquad g_4^2 \equiv g_2^2 + g_3^2, \qquad g_2 = g_0 g_4 ,$$

$$g_1(x) = q\left(\frac{x - c_0}{c_1 - c_0}\right), \qquad g_3(x) = \left[1 - q^2\left(\frac{x - d_2}{d_3 - d_2}\right)\right]^{1/2} ,$$

where $q(x)$ is chosen so that $q(x) = 1$ for $x \leqq 0$ and $q(x) = 0$ for $x \geqq 1$. From the Chain Rule we obtain the estimates, for $j = 0, 1, 2$,

$$\begin{aligned} \max\{\|g_k^{(j)}\|_\infty; k = 1, 2, 4\} &\leqq C_j\, \varepsilon^{j\delta_1/\mu_2}, \\ \max\{\|g_k^{(j)}\|_\infty; k = 0, 3\} &\leqq C_j\, \varepsilon^{j/\mu_2} \end{aligned} \tag{3.9}$$

Throughout we use the notation $\phi_k = g_k \phi$ and consider $\mathscr{C}_2$ to be the positively oriented contour defined by $|\varepsilon\mathscr{V} - \zeta| = A_1 \varepsilon^{\delta_2}$, where $\delta_2 \in (1, \min\{\delta_0, 2\delta_1/\mu_2\})$ is fixed. Then for any $\phi \in \mathscr{D}(H)$,

$$\|(H(\theta) - \zeta)\phi_1\| = \|(G - \zeta)\phi_1\| \geqq A_1 \varepsilon^{\delta_2} C \|\phi_1\| , \tag{3.10}$$

which is the Interior estimate. In the Classically Forbidden region $[c_0, d_3]$ we find,

$$\begin{aligned} \|(H(\theta) - \zeta)\phi_2\| &\geqq \frac{1}{\|\phi_2\|} \mathrm{Re}\langle \phi_2, (H(\theta) - \zeta)\phi_2 \rangle \\ &\geqq \frac{\langle \phi_2, -\partial_x \mathscr{J}_\theta^{-2} \partial_x \phi_2 \rangle}{\|\phi_2\|} + \inf_{[c_0,\, d_3]} (\mathrm{Re}(\mathscr{F}_\theta + (\tau - \varepsilon)V_\theta - \zeta))\|\phi_2\| \\ &\geqq \sqrt{\varepsilon(B_3 - \mathscr{V})/2}\, \|\mathscr{J}_\theta^{-1} \partial_x \phi_2\| + (B_3 - \mathscr{V})(\varepsilon - \varepsilon/2 - C\varepsilon^{\delta_2})\|\phi_2\| . \end{aligned} \tag{3.11a}$$

This gives two estimates involving $g_2(x)$. For the comparison operator,

$$\|(G - \zeta)\phi_4\| \geqq \sqrt{\varepsilon^{\delta_1}(A_1 - \varepsilon^{1-\delta_1}\mathscr{V})/2}\ \|\partial_x \phi_4\| + A_1(\varepsilon^{\delta_1}/2 + \mathscr{O}(\varepsilon))\ \|\phi_4\| . \tag{3.11b}$$

Finally, from (3.8) we obtain,

$$\|(H(\theta) - \zeta)\phi_3\| \geqq \frac{1}{\|\phi_3\|} \mathrm{Im}\{-e^{2\imath\beta\gamma_3} \langle \phi_3, (H(\theta) - \zeta)\phi_3 \rangle\} \geqq \varepsilon\beta K_\mu C \|\phi_3\| , \tag{3.12}$$

with the definition

$$\gamma_3 \equiv \langle p\phi_3, v' p\phi_3\rangle / \|p\phi_3\|^2 \in (1 - C\sigma e^{-\sigma}, 1) .$$

Once σ is chosen sufficiently large and fixed, $\beta > 0$ can be chosen sufficiently small.

We now present,

The Basic Estimate Lemma I 3.2. *Under conditions* (H1)–(H3) *and for* $\theta = i\beta$ *with* $|\beta|$ *sufficiently small, the following holds for all* $\phi \in \mathscr{D}(H)$:

$$\|(H(\theta) - \zeta)\phi\|^2 \geqq A_1^2 \varepsilon^{2\delta_2} C_1 \|\phi\|^2 + \varepsilon^2 C_2 \sum_{k=2}^{4} \|g_k \phi\|^2 + \varepsilon C_3 \|\partial_x g_2 \phi\|^2 + \varepsilon C_4 \sum_{k=0}^{4} \|g_k' \phi\|^2 . \tag{3.13}$$

The same estimate holds for G with $\phi \in \mathscr{D}(G)$ *and* g_2 *replaced with* g_4

Proof. We start with the IMS formula [8, Lemma 6.1],

$$\begin{aligned} &\|(H(\theta) - \zeta)\phi\|^2 \\ &\geqq (1/6) \sum_{k=1}^{3} 2\|(H(\theta) - \zeta)\phi_k\|^2 + (1/6)\|(H(\theta) - \zeta)\phi_2\|^2 - R(\phi) \\ &\geqq A_1^2 \varepsilon^{2\delta_2} D_1 \|\phi\|^2 + \varepsilon^2 D_2 \|g_4 \phi\|^2 + \varepsilon^2 D_3 \|g_2 \phi\|^2 + \varepsilon D_4 \|\partial_x g_2 \phi\|^2 - R(\phi) , \end{aligned} \tag{3.14}$$

so the result is completed by establishing,

Claim 3.3. *For* $R(\phi) \equiv \sum_{k=1}^{3} \|[H(\theta), g_k]\phi\|^2$, $\exists C_i > 0$ *so that,*

$$R(\phi) \leqq \varepsilon^{4\delta_1/\mu_2} C_1 \|\phi\|^2 + C_2 \|(H(\theta) - \zeta)\phi\|^2 - \varepsilon(B_3 - \mathscr{V}) C_3 \sum_{k=0}^{4} \|g_k' \phi\|^2 . \tag{3.15}$$

Proof. First note that,

$$R(\phi) \leqq C_\beta \sum_{k=1}^{3} (\|\mathscr{J}_\theta^{-2} g_k' \partial_x \phi\|^2 + \|g_k'' \phi\|^2 + \|\mathscr{J}_\theta'\|_\infty^2 \|g_k' \phi\|^2) . \tag{3.16}$$

To study this inequality set $h \equiv g_k'$ and $J \equiv \mathscr{J}_\theta^{-2}$. Now consider the term

$$\begin{aligned} \|hJ\partial_x \phi\|^2 &\leqq (1 + \mathscr{O}(\beta)) \mathrm{Re}\langle \phi, -\partial_x h^2 J \partial_x \phi\rangle \\ &\leqq C_1 \|h'\phi\| \|hJ\partial_x \phi\| + C_2 \mathrm{Re}\langle \phi, h^2 (H(\theta) - \zeta)\phi\rangle \\ &\quad - C_2 \mathrm{Re}\langle \phi, h^2 (\mathscr{F}_\theta + \tau V_\theta - \zeta)\phi\rangle \\ &\leqq 2C_1^2 \|h'\phi\|^2 + (1/2)\|hJ\partial_x \phi\|^2 + C_2^2 \|h^2 \phi\|^2 \\ &\quad + \|(H(\theta) - \zeta)\phi\|^2 - C_2 \inf_{[c_0, d_3]} \mathrm{Re}\,(\mathscr{F}_\theta + \tau V_\theta - \zeta)\, \|h\phi\|^2 . \end{aligned}$$

Hence, combining the known estimates gives

$$\|hJ\partial_x\phi\|^2 \leqq \varepsilon^{4\delta_1/\mu_2}D_1\|\phi\|^2 + 2\|(H(\theta)-\zeta)\phi\|^2 - \varepsilon(B_3-\mathscr{V})D_2\|h\phi\|^2\,. \tag{3.17}$$

Now, using (3.6) and (3.9) in (3.16) and employing (3.17) gives the claim. □

A similar estimate to (3.15) holds for G. Combining (3.14) and the claim for $\varepsilon>0$ sufficiently small completes the lemma. □

Define the projection operators,

$$P_\varepsilon = \frac{i}{2\pi}\oint_{\mathscr{C}_2}\frac{d\zeta}{G-\zeta}\,, \qquad Q_\varepsilon(\theta) = \frac{i}{2\pi}\oint_{\mathscr{C}_2}\frac{d\zeta}{H(\theta)-\zeta}\,. \tag{3.18}$$

We now show

Theorem 3.4. *There is a constant $C>0$ so that for $\varepsilon>0$ sufficiently small,*

$$\|P_\varepsilon - Q_\varepsilon(\theta)\| \leqq A_1\varepsilon^{\delta_2-1}C<1, \qquad |F^{\tau-\varepsilon}-z(\tau-\varepsilon)| \leqq 2A_1\varepsilon^{\delta_2}\,. \tag{3.19}$$

Furthermore, $z(\lambda)$ is differentiable for $\lambda<\tau$.

Proof Define the resolvents $R(G)=(G-\zeta)^{-1}$ and $R(H)=(H(\theta)-\zeta)^{-1}$. Then

$$\begin{aligned}\|P_\varepsilon - Q_\varepsilon(\theta)\| &\leqq A_1\varepsilon^{\delta_2}\sup_{\zeta\in\mathscr{C}_2}\{2\|R(G)g_4'\|\|g_4'R(H)\| + \|R(G)g_4'\|\|\partial_x g_2R(H)\| \\ &\quad + \|R(G)g_2\partial_x\|\|g_4'R(H)\| + \|R(G)g_4\| + \|g_4R(H)\|\} \\ &\leqq A_1\varepsilon^{\delta_2}(C\varepsilon^{-1})\,.\end{aligned}$$

Here it is useful to note that $g_1g_1' = -g_4g_4' = -g_2g_4'$. In this way all terms involving g_1 can be replaced with g_2 and g_4 terms. From (3.19) we know that the $\dim(Q_\varepsilon(\theta)) = \dim(P_\varepsilon) = 1$ for ε sufficiently small and this obtains the second statement in (3.19).

For continuity let us define $z=z(\tau-\varepsilon)$, $z_0=z(\tau-\varepsilon_0)$, etc. Note that the contours $\mathscr{C}_*$ can be deformed slightly without changing (3.13). Now we consider,

$$H'(\theta) \equiv \mathscr{U}_\theta^{\varepsilon_0}(p^2+(\tau-\varepsilon)V)\mathscr{U}_{-\theta}^{\varepsilon_0}\,.$$

Due to [8, Theorem 3.2] the pure point spectrum is independent of the scaling $\mathscr{U}_\theta^\varepsilon$ so we have $z\in\sigma(H'(\theta))$ and is non-degenerate. Hence using a common contour for the projections, we obtain,

$$Q'H' - Q_0H_0 \equiv zQ' - z_0Q_0 = (z-z_0)Q' + z_0(Q'-Q_0)\,. \tag{3.20}$$

Rearranging terms and taking the norm gives,

$$\begin{aligned}|z-z_0| &\leqq A_1\varepsilon^{\delta_2}\sup_{\zeta\in\mathscr{C}_2}\{|\zeta-z_0|\,\|R(H')(\varepsilon-\varepsilon_0)V_0R(H_0)\|\} \\ &\leqq |\varepsilon-\varepsilon_0|\varepsilon^{2\delta_2}(A_1^2\|V_\theta\|_\infty/C_1\varepsilon^{2\delta_2})\,.\end{aligned}$$

If $V(x)$ is not bounded at $x=0$ then define the cutoff functions $h_k(x)$,

$$1 = h_1^2 + h_2^2, \qquad h_1(x) = q\left((2x-a)/a\right)\,. \tag{3.21}$$

The IMS formula gives, for $\phi \in \mathscr{D}(H_0)$,

$$\|(H_0 - \zeta)\phi\|^2 \geqq (1/2)\tau^2 \|h_1 V\phi\|^2 - R(\phi)\,,$$

where the remainder term satisfies (see the proof of Claim 3.3),

$$R(\phi) \leqq C\|\phi\|^2 + \|(H_0 - \zeta)\phi\|^2\,.$$

Hence we obtain, for any $u \in \mathscr{H}$,

$$2\|u\|^2 + C\|R(H_0)u\|^2 \geqq (\tau^2/2)\|h_1 VR(H_0)u\|^2\,,$$

which implies the resolvent bound

$$\|h_1 VR(H_0)\| \leqq (2 + A_1 C/\varepsilon^{\delta_2})/\tau\,.$$

We can now return to (3.20) and insert $1 = h_1^2 + h_2^2$ between the resolvent operators on the right-hand side. This implies differentiability. The improved estimate of (1.3c) is obtained below. □

4. The $\mu_2 = 2$ Case

The theory of the previous section requires $\delta_1 = \delta_2 = 1$. The consequence of this is that the threshold τ must be sufficiently large to obtain the convergence of projections.

To obtain the basic estimate we require $A_0 > A_1$ to be sufficiently large. Then $B_1 > B_2 > B_3 > \mathscr{V}$ are chosen as required. We show the following in the Appendix,

Lemma 4.1. *Under conditions* (H1)–(H3) *Lemma* 3.1 *holds with* $\delta_1 = 1$.

The Chain Rule and condition (H2) combine to give,

$$\|g_k^{(j)}\|_\infty \leqq (\varepsilon A_0/\tau)^j C\,. \tag{4.1}$$

Next choosing $\mathscr{C}_0$ to be the circle defined by $|\varepsilon - \zeta| = \varepsilon\beta r_0$ for some $r_0 < 1$, ensures that the Interior estimate (3.10) holds with $\delta_2 = 1$ and ε replaced by $\varepsilon\beta r_0$. The Exterior estimates (3.8) and (3.12) will hold if σ and τ are chosen sufficiently large. For the Classically Forbidden region a careful analysis gives, for β sufficiently small,

$$\|(H(\theta) - \zeta)\phi_2\| \geqq \sqrt{\varepsilon B_3}C_1\|\partial_x\phi_2\| + \varepsilon B_3 C_2\|\phi_2\|\,. \tag{4.2}$$

We now present,

The Basic Estimate Lemma II 4.2. *Under conditions* (H1)–(H3) *and for* $\theta = i\beta$ *with* $|\beta|$ *sufficiently small, the following holds for all* $\phi \in \mathscr{D}(H)$:

$$\begin{aligned}\|(H(\theta) - \zeta)\phi\|^2 \geqq\ & \varepsilon^2\beta^2 r_0^2 C_1\|\phi\|^2 + \varepsilon^2 C_2(B_3^2\|g_2\phi\|^2 + \beta^2\|g_4\phi\|^2) \\ & + \varepsilon B_3 C_3\|\partial_x g_2\phi\|^2 + \varepsilon(B_3 - \mathscr{V})C_4\sum_{k=0}^{4}\|g_k'\phi\|^2\,.\end{aligned} \tag{4.3}$$

The same estimate holds for G with g_2 replaced by g_4.

Proof. Using a relation similar to (3.14), we need only obtain the bound,

$$R(\phi) \leqq \varepsilon^2 (A_0/\tau)^2 C_1 \|\phi\|^2 + C_2 \|(H(\theta) - \zeta)\phi\|^2 - \varepsilon(B_3 - \mathscr{V})C_3 \sum_{k=0}^{4} \|g_k' \phi\|^2 . \qquad (4.4)$$

The details are clear from Sect. 3 however note that $r_0 \to 0^+$ requires $\tau \to \infty$. □

We can now show,

Theorem 4.3. *For τ sufficiently large and $\varepsilon > 0$ sufficiently small, $\exists C > 0$ so that,*

$$\|P_\varepsilon - Q_\varepsilon(\theta)\| \leqq C/\tau < 1, \qquad |F^{\tau-\varepsilon} - z(\tau - \varepsilon)| \leqq 2\varepsilon\beta r_0 . \qquad (4.5)$$

Furthermore, $z(\lambda)$ is differentiable for $\lambda < \tau$.

Proof. Unlike in Sect. 3, uniform convergence of the projection operators requires τ to be sufficiently large. Consider

$$\begin{aligned}\|(P_\varepsilon - Q_\varepsilon(\theta))\| \leqq\ & \varepsilon\beta r_0 \sup_{\zeta \in \mathscr{C}_0} \{2\|R(G)g_4'\|\|g_4' R(H)\| + \|R(G)g_4'\|\|\partial_x g_2 R(H)\| \\ & + \|R(G)g_2\partial_x\|\|g_4' R(H)\| + \|R(G)g_4^2\| + \|g_4^2 R(H)\phi\|^2\} \\ \leqq\ & \varepsilon\beta r_0 (C_1/\varepsilon(B_3 - \mathscr{V}) + C_2/\varepsilon\beta) \leqq C r_0 .\end{aligned}$$

Hence for $r_0 > 0$ sufficiently small we obtain (4.5). Note that this imposes a lower bound on τ from Lemma 4.2.

For the differentiability of the s.r.value, we combine estimates in (4.3) with (3.20) to obtain a bound on $|dz/d\varepsilon|$ independent of ε. A sharper estimate is obtained below. □

Proof of the Main Theorem. To improve the estimate on $|F^\lambda - z(\lambda)|$ in (4.5) to exponential order we refer the reader to [2, Sect. 7], [8, Sect. 14] or [5]. Details for estimating $\mathrm{Im}(z)$ can also be found in these references.

Finally, to obtain bounds on the derivative of $z(\tau - \varepsilon)$ we recall the notation used in the proof of Theorem 3.4. Then

$$(z - z_0)Q_0 = (H' - z)Q' - (H_0 - z)Q_0 . \qquad (4.6)$$

Now let ψ be the resonance which solves $(H_0 - z_0)\psi = 0$. Then we obtain the formula,

$$\frac{z - z_0}{\varepsilon - \varepsilon_0}\langle\psi, \psi\rangle = \langle\psi, Q' V\psi\rangle + i\,\frac{z - z_0}{2\pi}\oint_{\mathscr{C}_0} \langle\psi, R(H')V\psi\rangle\,\frac{d\zeta}{z_0 - \zeta} .$$

From the bounds on the resolvent found in (4.3) and (3.19), we can take the limit as $\varepsilon \to \varepsilon_0$ and find $dz/d\varepsilon = \langle\psi, Q_0 V\psi\rangle/\|\psi\|^2$. Here we need the fact that $\|Q' - Q_0\| = \mathscr{O}(\varepsilon - \varepsilon_0)$. To simplify the derivative let us consider the difference,

$$\begin{aligned}\langle\psi, (Q' - Q_0^*)V\psi\rangle &= \frac{i}{2\pi}\oint_{\mathscr{C}_0} \langle(H' - H_0^*)\psi, R(H)V\psi\rangle\,\frac{d\zeta}{z_0^* - \zeta} \\ &\leqq C(\varepsilon - \varepsilon_0)\frac{\|V\|_\infty^2}{\varepsilon\beta}\|\psi\|^2 + \frac{\|V\|_\infty}{\varepsilon\beta}\|\psi\|\sum_{k=0}^{2} C_k \|\chi_{(d_1,\infty)}\partial_x^k \psi\| \\ &\leqq (C_1(1 - \varepsilon_0/\varepsilon) + C_2 \varepsilon^{-\alpha} e^{-\rho_*})\|\psi\|^2 ,\end{aligned}$$

where α is finite and ρ_μ is easily calculated from Definition (2.1). In particular,

$$\rho \geqq \int_c^d \sqrt{\tau V(x) - \varepsilon \mathscr{V}}\, dx \geqq \sqrt{\tau V_\infty}(1 - \mathscr{V}/B_1)^{1/2} c^{\mu_2/2} \int_c^{d_1} x^{-\mu_2/2} dx \geqq \rho_* > \rho_\mu . \quad (4.7)$$

Exponential bounds on resonance states were obtained by Sigal in [8, Sect. 13]. Combining these estimates give $dz/d\varepsilon = \langle \psi, V\psi \rangle / \|\psi\|^2 + \mathcal{O}(e^{-\rho_\mu})$.

To complete the estimate in (1.3) let ϕ be the normalized zero state of H^τ and define,

$$\psi = g_1\phi - (1 - Q_0)g_1\phi .$$

Calculating as in [8] gives,

$$\frac{dz}{d\varepsilon} = \mathscr{V} + \mathcal{O}_1(\|\psi - \phi\|) + \mathcal{O}_2(\|g_4^{(k)}\partial_x^k \phi\|; k = 0,1,2) . \quad (4.8)$$

It can be shown that $\|(1 - g_1)\phi\|$ and $\mathcal{O}_2$ are $\mathcal{O}(e^{-\rho_\mu})$. Finally we observe, for P_{ε_0} as defined in (3.18),

$$\|(1 - Q_0)g_1\phi\| = \|g_1(1 - P_{\varepsilon_0})\phi\| + \|(g_1 P_{\varepsilon_0} - Q_0 g_1)\phi\| . \quad (4.9)$$

The first term on the right-hand side is handled in (A.3) of the Appendix. As for the last term in (4.9) a bound was obtained in (4.5) (or (3.18) for Sect. 3) with remainder $\mathcal{O}(e^{-\rho_\mu})$. □

Appendix

For τ as in (H2) define $c_1^* \equiv \min\{x | \tau V(x) = 2\varepsilon V_{\min}\}$, $I^* \equiv (a, c_1^*)$ and $J^* \equiv (c_1^*, \infty)$. Then we introduce the operator,

$$G_*^{\tau-\varepsilon} = -\partial_x^2 + (\tau - \varepsilon)V(x)\chi_{I^*} + 2\varepsilon V_{\min}\chi_{J^*} \quad \text{on } \mathscr{H} , \quad (A.1)$$

and present,

Lemma A.1. *For $\varepsilon > 0$ sufficiently small $\exists! F_*^{\tau-\varepsilon} \in \sigma(G_*^{\tau-\varepsilon}) \cap [0, \varepsilon(2V_{\min} - \mathscr{V}) + \mathcal{O}(\varepsilon^{\delta_0})]$*

Proof. By hypothesis $\exists E^\varepsilon \in \sigma(H^{\tau+\varepsilon} + 2\varepsilon V_{\min})$, where $E^\varepsilon = \varepsilon(2V_{\min} - \mathscr{V}) + \mathcal{O}(\varepsilon^{\delta_0})$ for some $\delta_0 > 1$ and $0 < \mathscr{V} < V_{\min}$. Since

$$(H^{\tau+\varepsilon} + 2\varepsilon V_{\min}) \geqq G_*^{\tau-\varepsilon} ,$$

we conclude $E^\varepsilon \geqq F_*^{\tau-\varepsilon}$. Conversely, since

$$G_*^{\tau-\varepsilon} \geqq H^{\tau-\varepsilon} \equiv H^\tau - \varepsilon V ,$$

we have by min-max theory [6] that $F_*^{\tau-\varepsilon} \geqq 0$ and that it is a unique eigenvalue. □

Proof of Lemma 3.1. Let ψ be the normalized eigenfunction corresponding to the eigenvalue F of $G^{\tau-\varepsilon}$. Also let P_* be the projection of $G_*^{\tau-\varepsilon}$ onto F_*. Then defining,

$$\psi_* = g_1\psi - (1-P_*)g_1\psi ,$$

gives $(G_*^{\tau-\varepsilon} - F_*)\psi_* = 0$. Hence,

$$(F-F_*)\langle\psi,\psi_*\rangle = \langle\psi,[(\tau-\varepsilon)V(x)\chi_{(c_1,c_1^*)} + 2\varepsilon(V_{\min}\chi_{J^*} - \varepsilon^{\delta_0-1}A_1\chi_{J_c})]\psi_*\rangle . \quad \text{(A.2)}$$

For the left-hand side note that,

$$\langle\psi,\psi_*\rangle = 1 - \|\psi-\psi_*\|^2 = 1 - \mathcal{O}(e^{-\rho_*}) .$$

The right-hand side of (A.2) is clearly $\mathcal{O}(e^{-\rho_*})$ from [8]. As in (4.7) we calculate,

$$\rho \geqq \int_c^{c_1} \sqrt{\tau V(x) - \varepsilon^{\delta_1}A_1}\, dx \geqq \sqrt{\tau V_\infty}(1 - A_1/A_0)^{1/2} c^{\mu_2/2} \int_c^{c_2} x^{-\mu_2/2} dx > \rho_* .$$

This demonstrates that $F = F_* + \mathcal{O}(e^{-\rho_*})$.

Now let ϕ be the normalized zero state which solves $H^\tau\phi = 0$. Also let P_ε be the projection of $G^{\tau-\varepsilon}$ onto $F^{\tau-\varepsilon}$. Then

$$0 = \langle\phi, P_\varepsilon H^\tau\phi\rangle = F^{\tau-\varepsilon}\langle\phi,P_\varepsilon\phi\rangle + \langle\phi, P_\varepsilon(\varepsilon V\chi_I - 2\varepsilon^{\delta_1}A_1\chi_J)\phi\rangle .$$

Thus rearranging terms gives,

$$F^{\tau-\varepsilon} = -\varepsilon\mathscr{V} + \varepsilon\mathcal{O}_1(\|(P_\varepsilon - 1)\phi\|) + \varepsilon^{\delta_1}\mathcal{O}_2(\|\chi_J\phi\|) .$$

As a contour take $\mathscr{C}_1 = \{\zeta \,|\, |\zeta| = \varepsilon^{\delta_1}A_1\}$ and observe that,

$$\|(P_\varepsilon - 1)\phi\| = \frac{\varepsilon}{2\pi}\left\| \oint_{\mathscr{C}_1} \frac{1}{G^{\tau-\varepsilon}-\zeta} V\phi \frac{d\zeta}{\zeta} \right\| \leqq \varepsilon \frac{1}{\varepsilon^{\delta_1}A_1 - \varepsilon(2V_{\min} - \mathscr{V})} \|V\phi\| . \quad \text{(A.3)}$$

If V is singular at 0 then we can introduce the cutoff functions h_k defined in (3.20) and proceed as in the proof of Theorem 3.4. □

Proof of Lemma 4.1. The details above clearly apply if $\delta_1 = 1$. Note however that (A.3) requires A_1 to be large if $\|P_\varepsilon - 1\|$ is to be small. □

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