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Long-time Asymptotics for Integrable Systems. Higher Order Theory

P.A. Deift[1], **X. Zhou**[2]

[1] Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA
[2] Yale University, New Haven, CT 06520, USA

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Abstract: The authors show how to obtain the full asymptotic expansion for solutions of integrable wave equations to all orders, as $t \to \infty$. The method is rigorous and systematic and does not rely on an a priori ansatz for the form of the solution.

1. Introduction

In [DZ1], the authors introduced a new nonlinear steepest descent-type method for analyzing the asymptotics of oscillatory Riemann–Hilbert (RH) problems. This method has since been used to study rigorously the long-time asymptotics of a wide variety of integrable systems such as the modified Korteweg de Vries (MKdV) equation [DZ1], the nonlinear Schrödinger (NLS) equation [DIZ], the doubly infinite Toda Lattice [K], the autocorrelation function for the transverse Ising chain at critical magnetic field [DZ2], the collisionless shock region for the Korteweg de Vries (KdV) equation [DVZ], and also the Painlevè II equation [DZ3]. In these papers only the leading asymptotics is considered. The purpose of this paper is to show how to obtain the full asymptotic expansion for the solutions in a rigorous and systematic way.

Full asymptotic expansions have been written down in the form of an ansatz for a variety of equations. For example, for NLS

$$iu_t + u_{xx} - 2|u|^2 u = 0\ , \quad u(x,0) = u_0(x) \in S(\mathbb{R})\ , \tag{1.1}$$

Segur and Ablowitz [SA1] introduced the expansion

$$u(x,t) \sim t^{-1/2}\left(\alpha + \sum_{n=1}^{\infty}\sum_{k=0}^{2n}\frac{(\log t)^k}{t^n}\alpha_{nk}\right)e^{ix^2/4t - i\nu\log t}\ , \quad t \to \infty\ , \tag{1.2}$$

where α, α_{nk} and ν are functions of the "slow" variable x/t. The coefficients α_{nk} and the parameter ν can be found explicitly in terms of α via the substitution of (1.2) into

(1.1). For example

$$v=2|\alpha|^2 . \tag{1.3}$$

In [ZM], Zakharov and Manakov derived a formula for α in terms of the reflection coefficient $r(z)$ associated with the initial condition u_0 through the inverse scattering method:

$$|\alpha(z_0)|^2=\frac{v(z_0)}{2}=-\frac{1}{4\pi}\log(1-|r(z_0)|^2) , \tag{1.4}$$

$$\begin{cases} \arg\alpha(z_0)= -3v\log 2-\frac{\pi}{4}+\arg\Gamma(iv)-\arg r(z_0) \\ \qquad\qquad +\frac{1}{\pi}\int_{-\infty}^{z_0}\log|z-z_0|d\log(1-|r(z)|^2) \\ z_0=-x/4t , \quad \Gamma=\text{gamma function} . \end{cases}$$

The expansion (1.2) was also considered by Novokshenov [N]. For the KdV equation,

$$u_t+u_{xxx}-6uu_x=0 , \quad u(x,0)=u_0(x)\in S(\mathbb{R}) , \tag{1.5}$$

Ablowitz and Segur [SA2] and later Buslaev and Sukhanov [BSa], [BSb] considered expansions $t\to\infty$ of the form

$$u(x,t)\sim\sum_{m=-\infty}^{\infty} e^{im\Phi(z_0,t)}t^{imB(z_0)}\sum_{q+|m|\leqq p}\frac{u_{pqm}(z_0)(\log t)^q}{t^{p/2}} , \quad z_0=\sqrt{\frac{-x}{12t}} , \tag{1.6}$$

for suitable functions Φ, B and u_{pqm}, and in [BSa, BSb] it is shown that under certain (nongeneric) assumptions on u_0, the solution $u(x,t)$ does indeed have such an expansion.

As in [DZ1, DIZ] we will consider specific examples to illustrate our method. It will be clear that our approach is general and systematic and applies to all integrable systems solvable through a RH problem. We consider, in particular, the NLS equation (1.1) and the MKdV equation

$$u_t+u_{xxx}-6u^2u_x=0, \quad u(x,0)=u_0(x)\in S(\mathbb{R}) , \tag{1.7}$$

in the so-called similarity region

$$|z_0|=\left|-\frac{r}{4t}\right|\leqq M \quad \text{for NLS} , \tag{1.8}$$

$$\frac{1}{M}\leqq z_0=\sqrt{-x/12t}\leqq M , \quad x<0 , \quad \text{for MKdV} , \tag{1.9}$$

for some $M>1$.

Theorem 1.10. (NLS). *Let $u(x,t)$ be the solution of* (1.1) *with $u_0\in S(\mathbb{R})$. Then*

(a) *For (x, t) in the similarity region* (1.8), *$u(x, t)$ has an asymptotic expansion of the form*

$$u(x, t)\sim e^{\frac{ix^2}{4t}-iv\log t}\sum_{p=1}^{\infty}\frac{u_p(z_0,t)}{t^{p/2}} \quad as\ t\to\infty , \tag{1.11}$$

where ν is given by (1.4) *and*

$$u_p(z_0,t)=\sum_{q=0}^{p-1} u_{pq}(z_0)(\log t)^q\ , \tag{1.12}$$

in the sense that

$$u(x,t)=e^{ix^2/4t-i\nu\log t}\left(\sum_{p=1}^{N}\frac{u_p(z_0,t)}{t^{p/2}}+O\left(\frac{(\log t)^N}{t^{(N+1)/2}}\right)\right),\quad \textit{for any } N\ ,$$

and all $|z_0|=|-x/4t|\leqq M$.

(b) *The asymptotics in* (1.11) *can be differentiated term by term with respect to* x *and* t.

(c) $u_p=0$ *for* p *even and* u_p *can be determined recursively for* p *odd from* $u_0(z_0,t)=u_{10}(z_0)=\alpha(z_0)$, *as follows: for* $p>1$,

$$u_{pq}=\frac{4}{(p-1)^2}\left[\left(i\left(\frac{p-1}{2}\right)-\nu\right)(f_{pq}-i(q+1)u_{p,q+1})+2\omega_0^2\overline{(f_{pq}-i(q+1)u_{p,q+1})}\right], \tag{1.13}$$

where

$$f_{pq}=2\sum_{\substack{p_1+p_2+p_3=p+2,\ q_1+q_2+q_3=q\\ 0\leqq q_i<p_i<p,\ p_i\,\mathrm{odd}}} u_{p_1q_1}\,u_{p_2q_2}\,u_{p_3q_3}$$

$$-\frac{1}{16}\left[u''_{p-2,q}-(\nu')^2u_{p-2,q-2}-i\nu''u_{p-2,q-1}\right] \tag{1.14}$$

and

$$u_{pq}=0\quad \textit{for } q\geqq p\ . \tag{1.15}$$

Here u_{pq} *is determined recursively in decreasing order of* q *starting from* $q=p-1$. □

Definition 1.16. *We define an order* $\prec$ *on* $\mathbb{N}\times\mathbb{N}$: $(k',p')\prec(k,p)$, *if either* $p'-p<k'-k$ *or* $p'-p=k'-k<0$. *For a function* $F=F(\xi,\eta)$, *set* $F'=F_\xi$, $\dot F=F_\eta$.

Theorem 1.17. (MKdV). *Let* $u(x,t)$ *be the solution of* (1.7) *with* $u_0\in S(\mathbb{R})$. *Then*:

(a) *for* x,t *in the similarity region* (1.9), $u(x,t)$ *has an asymptotic expansion of the form*

$$u(x,t)\sim\sum_{k\,\mathrm{odd}}\frac{e^{k\psi}}{t^{ik\nu}}\left(\sum_{p\geqq|k|}\frac{u_{kp}(z_0,t)}{t^{p/2}}\right), \tag{1.18}$$

where

$$u_{kp}(z_0,t)=\sum_{0\leqq q\leqq p-|k|} u_{kpq}(z_0)(\log t)^q,\ u_{kpq}(z_0)=\overline{u_{-kpq}(z_0)}\ , \tag{1.19}$$

and

$$\psi=16itz_0^3,\quad \nu=\nu(z_0)=12z_0|u_{11}(z_0)|^2\ , \tag{1.20}$$

in the sense that

$$u(x,t)=\sum_{k\,\mathrm{odd},\ |k|\leqq p\leqq N,\ 0\leqq q\leqq p-|k|}\frac{e^{k\psi}u_{kpq}(z_0)(\log t)^q}{t^{p/2+ik\nu}}+O\left(\frac{(\log t)^N}{t^{(N+1)/2}}\right),$$

for any N *and all* $\frac{1}{M}\leqq z_0=\sqrt{\frac{-x}{12t}}\leqq M$.

(b) *The asymptotics in* (1.18) *can be differentiated term by term with respect to x and t.*

(c) $u_{kp}=0$ *for p even and* u_{kp} *can be determined recursively for p odd from*

$$u_{11}(z_0,t)=u_{110}(z_0)=\left(\frac{\nu}{12z_0}\right)^{1/2} e^{i\phi(z_0)}=\overline{u_{-11}(z_0,t)}\,, \tag{1.21}$$

where

$$\phi(z_0)=\arg\Gamma(i\nu)-\frac{\pi}{4}-\arg r(z_0)-\nu\log(192z_0^3)$$
$$+\frac{1}{\pi}\int_{-z_0}^{z_0}\log|s-z_0|\,d\log(1-|r(s)|^2)\,, \tag{1.22}$$

$$\nu=\nu(z_0)=-\frac{1}{2\pi}\log(1-|r(z_0)|^2)\,, \tag{1.23}$$

and $r(z)$ *is the reflection coefficient associated with* u_0 *through the inverse scattering method. For* $k>1$, u_{kp} *is determined by* $\{u_{k'p'}:(k',p')\prec(k,p)\}$ *and the reality condition* $u_{k_jp_j}=\overline{u_{-k_j,p_j}}$ *as follows:*

$$8i(k-k^3)z_0^3u_{kp}=\frac{1}{2}(k^2-p+2-2ik\nu+iz_0(k-k^3)\nu'\log t)u_{k,p-2}$$
$$+\frac{(k^2-1)z_0}{2}u'_{k,p-2}+t\dot{u}_{k,p-2}+\frac{k}{288z_0^3}(i+3iz_0k^2(\nu')^2(\log t)^2-3z_0^2\nu''k\log t)$$
$$-\frac{k}{96z_0}(iu''_{k,p-4}+2k\nu'(\log t)u'_{k,p-4})+t^{p/2+ik\nu}(t^{-(\frac{p-6}{2})-ik\nu}u_{k,p-6})_{xxx}$$
$$+4iz_0k\sum_{\substack{k_1+k_2+k_3=k\\|k_j|\leqq p_j,\,j=1,2,3\\p_1+p_2+p_3=p}}u_{k_1p_1}u_{k_2p_2}u_{k_3p_3}$$
$$+\frac{1}{4z_0}\sum_{\substack{k_1+k_2+k_3=k\\|k_j|\leqq p_j,\,j=1,2,3\\p_1+p_2+p_3=p-2}}(u'_{k_1p_1}-ik_1\nu'(\log t)u_{k_1p_1})u_{k_2p_2}u_{k_3p_3}\,. \tag{1.24}$$

For $k=1$, u_{1p} *is determined by* $\{u_{k'p'}:(k',p')\prec(1,p)\}$ *and the reality condition* $u_{k_j,p_j}=\overline{u_{-k_j,p_j}}$ *as follows: for* $0\leqq q\leqq p-1$, $p>1$,

$$u_{1pq}=\frac{4}{(p-1)^2}\left(\left(\frac{1-p}{2}-i\nu\right)(f_{pq}-(q+1)u_{1p,q+1})\right.$$
$$\left.-12iz_0u_{11}^2\,\overline{(f_{pq}-(q+1)u_{1p,q+1})}\right)\,, \tag{1.25}$$

where

$$\sum_{q=0}^{p-1}f_{pq}(\log t)^q$$
$$=\frac{-i}{96z_0}\left(\frac{u_{1,p-2}}{3z_0^2}+\frac{(\nu')^2(\log t)^2u_{1,p-2}}{z_0}\right.$$
$$\left.+2i\nu'(\log t)\,u'_{1,p-2}+i\nu''(\log t)u_{1,p-2}-ku''_{1,p-2}\right)$$

$$-t^{\frac{p+2}{2}+i\nu}\left(t^{-\frac{p-4}{2}-i\nu}u_{1,p-4}\right)_{xxx}$$

$$-4iz_0 \sum_{\substack{k_1+k_2+k_3=1 \\ |k_j|\leqq p_j<p,\ j=1,2,3 \\ p_1+p_2+p_3=p+2}} u_{k_1p_1}u_{k_2p_2}u_{k_3p_3}$$

$$-\frac{1}{4z_0} \sum_{\substack{k_1+k_2+k_3=1 \\ |k_j|\leqq p_j,\ j=1,2,3 \\ p_1+p_2+p_3=p}} \left(u'_{k_1p_1}-\frac{i}{3}\nu'(\log t)u_{k_1p_1}\right)u_{k_2p_2}u_{k_3p_3}\,. \tag{1.26}$$

$$u_{1pq}=0 \quad \textit{for } q\geqq p\,. \tag{1.27}$$

Here u_{1pq} is determined recursively in decreasing order of q starting with $q=p-1$.

The assumption in Theorems 1.10 and 1.17 that the initial data lie in Schwartz space, leads to the full asymptotic expansions (1.11) and (1.18) respectively. If the initial data has only a finite degree of smoothness and a finite order of decay, then the above method leads to asymptotic expansions of type (1.11) and (1.18), but only to a finite order in t^{-1}.

As opposed to previous authors,

(i) we do not require an ansatz for the asymptotic form of the solution,
(ii) we do not place any generic or nongeneric (cf. [BSa], [BSb]) restrictions on the Schwartz space initial data,
(iii) our method is general, systematic, and rigorous. We expand the solution $u(x, t)$ of the Cauchy problem directly by our method and the analytic origin of the logarithmic terms, as well as the analytic origin of the interaction of modes $e^{k\psi}$ (for MKdV), become transparent.

Parts (a) of the theorems will be proved in Sect. 2, parts (c) in Sect. 3, and parts (b) in Sect. 4.

Note finaly that as $u_p=0$ for p even, the expansion (1.11) for NLS reduces to (1.2), and as $u_{kp}=0$ for p even, the expansion (1.18) takes the form

$$u(x, t)\sim \sum_{k\,\mathrm{odd}} \frac{e^{k\psi}}{t^{ik\nu}} \sum_{\substack{p\geqq|k| \\ p\ \mathrm{odd}}} \frac{u_{kp}(z_0, t)}{t^{p/2}}\,. \tag{1.18$'$}$$

Furthermore substituting (1.18)′ in the Miura transform $u\to u_x+u^2$, we obtain the asymptotic expansion (1.6) for the KdV equation in the similarity region $M^{-1}\leqq z_0=\sqrt{\frac{-x}{12t}}\leqq M$.

2. Derivation of Asymptotic Forms

In this section we derive the asymptotic forms (1.11) for NLS and (1.18) for MKdV. Recursion formulae for the coefficients u_p and u_{kp} respectively, will be derived in the next section.

For the convenience of the reader we recall the solution procedure (see, for example, [BC]) for a RH problem on an oriented contour Σ. The RH problem on Σ is to find a $\nu \times \nu$ matrix-valued function $m(z)$ such that

$$\begin{cases} m(z) \text{ is analytic in } \mathbb{C}\setminus\Sigma \\ m_+(z)=m_-(z)v(z)\,, \quad z\in\Sigma\,, \\ m(z)\to I \text{ as } z\to\infty\,, \end{cases} \tag{2.1}$$

for a given jump matrix $v:\Sigma\to M_\nu(\mathbb{C})$, $v(z)\to I$ as $z\to\infty$. Here $m_\pm(z)$ refer to the boundary values of $m(z)$ taken from the left/right sides of Σ, respectively. Let

$$C_\pm f(z) = \lim_{\substack{z'\to z \\ z'\in\pm\text{side of }\Sigma}} \int_\Sigma \frac{f(s)}{s-z'}\frac{ds}{2\pi i} \tag{2.2}$$

denote the Cauchy operators on Σ. Suppose $v(z)$ has a factorization $v(z)=(I-\omega_-)^{-1}(I+\omega_+)$, $z\in\Sigma$, and introduce the operator C_ω on $L^2(\Sigma; M_\nu(\mathbb{C}))$,

$$C_\omega f = C_+(f\omega_-)+C_-(f\omega_+)\,, \quad f\in L^2(\Sigma; M_\nu(\mathbb{C}))\,. \tag{2.3}$$

Suppose that $\mu\in I+L^2(\Sigma; M_\nu(\mathbb{C}))$ solves the equation

$$(1-C_\omega)\mu = I\,.$$

Then

$$m(z) = I + \int_\Sigma \frac{\mu(s)(\omega_+(s)+\omega_-(s))}{s-z}\frac{ds}{2\pi i}\,, \quad z\in\mathbb{C}\setminus\Sigma\,, \tag{2.4}$$

is the solution of the RH problem (2.1). Also

$$m_+(z)=\mu(z)(I+\omega_+(z)),\ m_-(z)=\mu(z)(I-\omega_-(z)),\ z\in\Sigma\,. \tag{2.5}$$

Part 1: NLS. The NLS equation can be solved via a 2×2 matrix RH problem on $\mathbb{R}$ oriented from $-\infty$ to $+\infty$ as follows. Let $m(z)=m(z; x, t)$ be the solution the RH problem

$$\begin{cases} m_+(z)=m_-(z)v_{x,t}(z)\,,\ z\in\mathbb{R}\,, \\ m(z)\to I \text{ as } z\to\infty\,, \end{cases} \tag{2.6}$$

where

$$\begin{cases} v_{x,t}(z) = e^{-i(2tz^2+xz)\sigma_3}v(z)e^{i(2tz^2+xz)\sigma_3}\,,\ \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \\ \qquad \equiv e^{i(2tz^2+xz)\mathrm{ad}\sigma_3}v(z) \\ v = \begin{pmatrix}1-|r(z)|^2 & -\overline{r(z)}\\ r(z) & 1\end{pmatrix} = (I-\omega_-)^{-1}(I+\omega_+) \\ r(z) = \text{reflection coefficient associated with } u_0(x)\,. \end{cases} \tag{2.7}$$

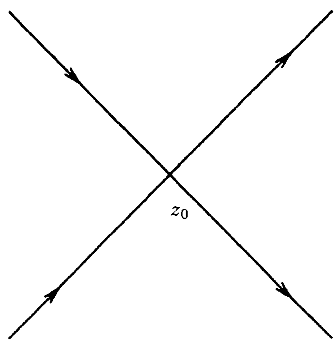
Then the solution $u(x, t)$ of the Cauchy problem (1.1) for NLS is given by

$$u(x,t) = 2\lim_{z\to\infty}(zm_{12}(z)) = -2\left(\int_\Sigma \mu(s)(\omega_{+,x,t}(s)+\omega_{-,x,t}(s))\frac{ds}{2\pi i}\right)_{12}\,, \tag{2.8}$$

where $\omega_{\pm,x,t}=e^{i(2tz^2+xz)\mathrm{ad}\sigma_3}\omega_\pm$.

In [DIZ] the authors show that in the similarity region $|z_0| = \left|\frac{-x}{4t}\right| < M$, the NLS equation can be solved to any fixed order $O\left(\frac{1}{t^n}\right)$, via an associated deformed RH problem ($\Sigma^{(1)}$, $v_{x,t}^{(1)}$) on a cross $\Sigma^{(1)}$

$$v_{x,t}^{(1)} = \delta^{\operatorname{ad}\sigma_3} e^{-it\theta \operatorname{ad}\sigma_3} \begin{pmatrix} 1 & r_2 \\ 0 & 1 \end{pmatrix} \qquad v_{x,t}^{(1)} = \delta^{\operatorname{ad}\sigma_3} e^{-it\theta \operatorname{ad}\sigma_3} \begin{pmatrix} 1 & 0 \\ r_1 & 1 \end{pmatrix}$$



$$v_{x,t}^{(1)} = \delta^{\operatorname{ad}\sigma_3} e^{-it\theta \operatorname{ad}\sigma_3} \begin{pmatrix} 1 & 0 \\ r_3 & 1 \end{pmatrix} \qquad v_{x,t}^{(1)} = \delta^{\operatorname{ad}\sigma_3} e^{-it\theta \operatorname{ad}\sigma_3} \begin{pmatrix} 1 & r_4 \\ 0 & 1 \end{pmatrix}$$

Fig. 2.9.

$$\begin{cases} m_+^{(1)}(z) = m_-^{(1)}(z) v_{x,t}^{(1)}(z), & z \in \Sigma^{(1)} \\ m^{(1)}(z) \to I \text{ as } z \to \infty \end{cases} \tag{2.10}$$

where

$$\begin{cases} \delta(z) = e^{\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\log(1-|r(s)|^2)}{s-z} ds} \equiv e^{\chi(z, z_0)} \\ \theta(z) = 2z^2 + (x/t)z \end{cases} \tag{2.11}$$

and $\{r_i(z)\}_{i=1}^4$ are rational functions which decay to zero as $z \to \infty$ on $\Sigma^{(1)}$. Indeed if

$$u^{(1)}(z, t) \equiv 2 \lim_{z \to \infty} (z m_{12}^{(1)}(z)), \tag{2.12}$$

then

$$u(x, t) = u^{(1)}(x, t) + 0\left(\frac{1}{t^n}\right) \quad \text{as } t \to \infty \tag{2.13}$$

for $|z_0| = \left|\frac{-x}{4t}\right| \leq M$.

Now note that by the signature of $\operatorname{Re}(i\theta(z))$ and the upper/lower triangular shape of $v_{x,t}^{(1)}$, $v_{x,t}^{(1)}(z) - I$ converges exponentially to zero as $t \to \infty$, uniformly for $z \in \Sigma^{(1)}$, outside any neighborhood of z_0. Using the elementary expansion for any N,

$$\chi(z,z_0)=i\nu(z_0)\log(z-z_0)+(\chi_1^{(1)}(z_0)\log(z-z_0)+\chi_1^{(2)}(z_0))(z-z_0)$$
$$+\cdots+(\chi_N^{(1)}(z_0)\log(z-z_0)+\chi_N^{(2)}(z_0))(z-z_0)^k$$
$$+O((z-z_0)^{N+1}\log(z-z_0)), \tag{2.14}$$

we find under the scaling $z\to(z/\sqrt{t})+z_0$

$$v_{x,t}^{(1)}\left(\frac{z}{\sqrt{t}}+z_0,z_0\right)=e^{\left(2itz_0^2-\frac{i\nu(z_0)}{2}\log t\right)\operatorname{ad}\sigma_3}v_{z_0,t}^{(2)}(z) \tag{2.15}$$

for $z\in\Sigma^{(1)}-z_0$, where

$$v_{z_0,t}^{(2)}=e^{-2iz^2\operatorname{ad}\sigma_3}z^{i\nu(z_0)\operatorname{ad}\sigma_3}\left[I+v_{00}^{(2)}+\frac{v_{10}^{(2)}+v_{11}^{(2)}\log t}{t^{1/2}}+\cdots\right.$$
$$\left.+\frac{v_{N0}^{(2)}+v_{N1}^{(2)}\log t+\cdots+v_{NN}^{(2)}(\log Z)^N}{t^{N/2}}\right]+E_v(z,t,z_0), \tag{2.16}$$

$$\|e^{-2i(\cdot)^2\operatorname{ad}\sigma_3}(\cdot)^{i\nu(z_0)\operatorname{ad}\sigma_3}v_{pq}^{(2)}(\cdot,z_0)\|_{L^1\cap L^\infty}\leqq C,\ \text{uniformly for } |z_0|\leqq M, \tag{2.17}$$

and

$$\|E_v(\cdot,t,z_0)\|_{L^1\cap L^\infty}=O\left(\frac{(\log t)^{N+1}}{t^{(N+1)/2}}\right),\quad \text{uniformly for } |z_0|\leqq M. \tag{2.18}$$

The estimate for E_v follows by a direct extension of the method in [DZ1, p. 332 et seq.].

Setting $\omega_-^{(2)}=0$, $\omega_+^{(2)}=v_{z_0,t}^{(2)}-I$, the operator $C_{\omega^{(2)}}$ in (2.3) on $\Sigma^{(1)}-z_0$ takes the form

$$C_{\omega^{(2)}}f=C_-(f(v_{z_0,t}^{(2)}-I)=\sum_{0\leqq q\leqq p}\frac{(\log t)^q}{t^{q/2}}C_{pq}f+E_c(t,z_0)f, \tag{2.19}$$

where

$$C_{pq}f=C_-(fe^{-2i(\cdot)^2\operatorname{ad}\sigma_3}(\cdot)^{i\nu(z_0)\operatorname{ad}\sigma_3}v_{pq}^{(2)}), \tag{2.20}$$

and

$$E_c(t,z_0)f=C_-(fE_v(\cdot,t,z_0)). \tag{2.21}$$

It follows from (2.17) and (2.18) that uniformly for $|z_0|\leqq M$,

$$\|C_{pq}\|_{L^2\cup L^\infty\to L^2}\leqq c_{pq} \tag{2.22}$$

and

$$\|E_c(t,z_0)\|_{L^2\cup L^\infty\to L^2}=0\left(\frac{(\log t)^{N+1}}{t^{(N+1)/2}}\right). \tag{2.23}$$

The computations for the leading order asymptotics in [DZ1] show that $(1-C_{00})^{-1}$ exists and is uniformly bounded,

$$\|(1-C_{00})^{-1}\|_{L^2\cup L^\infty\to L^2}\leqq c,\quad |z_0|\leqq M, \tag{2.24}$$

and hence $\mu^{(2)}=(1-C_{\omega^{(2)}})^{-1}I$ can be expanded in a Neumann series as $t\to\infty$,

$$\mu^{(2)}=I+\sum_{0\leqq q\leqq p}\frac{(\log t)^q}{t^{p/2}}\mu_{pq}(z,z_0)+E_\mu(z,t,z_0), \tag{2.25}$$

where

$$\|\mu_{pq}(\cdot, z_0)\|_{L^2} \leqq c_{pq}, \ \|E_\mu(\cdot, t, z_0)\|_{L^2} = 0\left(\frac{(\log t)^{N+1}}{t^{(N+1)/2}}\right), \tag{2.26}$$

uniformly for $|z_0| \leqq M$.

Taking into account the scaling and conjugation in (2.15),

$$\begin{aligned} m^{(1)}(z; x, t) &= e^{\left(2itz_0^2 - i\frac{\nu(z_0)}{2}\log t\right)\operatorname{ad}\sigma_3} m^{(2)}(\sqrt{t}(z-z_0), z_0) \\ &= e^{\left(2itz_0^2 - i\frac{\nu(z_0)}{2}\log t\right)\operatorname{ad}\sigma_3}\left(I + \int_{\Sigma^{(1)}-z_0} \frac{\mu^{(2)}(s)(v^{(2)}_{z_0,t}(s)-I)}{s-(\sqrt{t}(z-z_0))}\frac{ds}{2\pi i}\right), \end{aligned} \tag{2.27}$$

we obtain from (2.8)

$$u^{(1)}(x, t) = \frac{-2e^{(4itz_0^2 - i\nu(z_0)\log t)}}{\sqrt{t}}\left(\int_{\Sigma^{(1)}-z_0} \mu^{(2)}(s)(v^{(2)}_{z_0,t}(s)-I)\frac{ds}{2\pi i}\right)_{12}. \tag{2.28}$$

Inserting (2.25) we obtain the asymptotic series (1.11).

Part 2. MKdV. The MKdV equation can be solved via a similar RH problem to NLS. Let $m(z) = m(z; x, t)$ be the solution of the RH problem

$$\begin{cases} m_+(z) = m_-(z)v_{x,t}(z), \ z \in \mathbb{R}, \\ m(z) \to I \ \text{as} \ z \to \infty, \end{cases} \tag{2.29}$$

where

$$\begin{cases} v_{x,t}(z) = e^{-i(4tz^3 + xz)\operatorname{ad}\sigma_3} v(z) \\ v(z) = \begin{pmatrix} 1-|r(z)|^2 & -\overline{r(z)} \\ r(z) & 1 \end{pmatrix} \\ r(z) \text{ is the reflection coeffiicient associated with } u_0(z). \end{cases} \tag{2.30}$$

Then the solution $u(x, t)$ of the Cauchy problem (1.5) for MKdV is given by

$$u(x, t) = 2\lim_{z\to\infty}(zm_{21}) = \left(\left[\sigma_3, \int_{\mathbb{R}} \mu(s)\,(\omega_{+,x,t}(s) + \omega_{-,x,t}(s))\frac{ds}{2\pi i}\right]\right)_{21}, \tag{2.31}$$

where μ and $\omega_{\pm,x,t}$ are the analogs for MKdV of the quantities introduced above for NLS.

In [DZ1] the authors show that in the similarity region $M^{-1} \leqq z_0 = \sqrt{\frac{-x}{12t}} \leqq M$, the MKdV equation can be solved to any fixed order $O\left(\frac{1}{t^n}\right)$, via an associated RH problem $(\Sigma^A \cup \Sigma^B, v^{A\cup B}_{x,t}, m^{A\cup B}(z; x, t))$ on a union of two small crosses

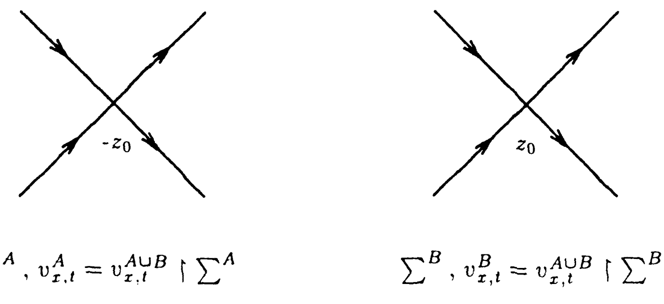


Fig. 2.32.

where $v^A_{x,t}$ and $v^B_{x,t}$ have analogous properties to those of $v^{(1)}_{x,t}$ in Fig. 2.9. Let m^A, m^B be the solutions for the RH problem $(\Sigma^A, v^A_{x,t})$, $(\Sigma^B, v^B_{x,t})$ respectively. Also let Γ^A, Γ^B be oriented, non-intersecting circles centered at z_0, $-z_0$ respectively,

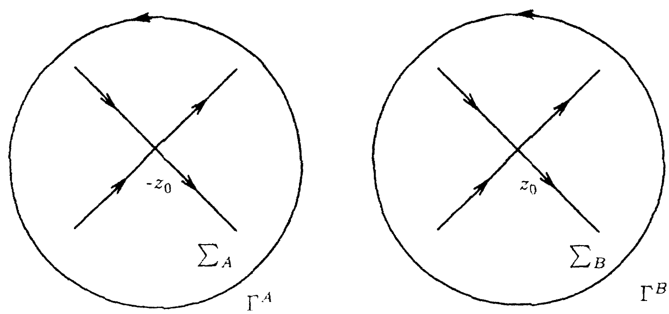


Fig. 2.33.

From the analogs of (2.25) and (2.27) we obtain

$$m^A(z; z_0, t) = e^{-\frac{\phi}{2}\mathrm{ad}\sigma_3}\left(I + \int\limits_{\sqrt{t}(\Sigma^A+z_0)} \frac{\mu^A(s)(v^A_{z_0,t}(s)-I)}{s-(\sqrt{t}(z+z_0))}\frac{ds}{2\pi i}\right)$$

$$= I + e^{-\frac{\phi}{2}\mathrm{ad}\sigma_3}\left(\sum_{0\leqq q\leqq p\leqq N} \frac{(\log t)^q}{t^{(p+1)/2}} m^A_{pq}(z, z_0)\right) + E^A_m(z, t, z_0)\,, \qquad (2.34)$$

where $v^A_{z_0,t}$ is the scaled version of $v^A_{x,t}$ analogous to (2.15), and

$$m^B(z; z_0, t) = I + e^{\frac{\phi}{2}\mathrm{ad}\sigma_3}\left(\sum_{0\leqq q\leqq p} \frac{(\log t)^q}{t^{(p+1)/2}} m^B_{pq}(z, z_0)\right) + E^B_m(z, t, z_0)\,, \qquad (2.35)$$

where

$$\phi = 16itz_0^3 - i\nu(z_0)\log t\,, \qquad (2.36)$$

$$\begin{cases} \|m^A_{pq}(\cdot, z_0)\|_{L^\infty(\Gamma^A)} \leqq c_{pq}\,, \ \|E^A_m(\cdot, t, z_0)\|_{L^\infty(\Gamma^A)} = 0\left(\dfrac{(\log t)^{N+1}}{t^{(N+2)/2}}\right), \\ \|m^B_{pq}(\cdot, z_0)\|_{L^\infty(\Gamma^B)} \leqq c_{pq}\,, \ \|E^B_m(\cdot, t, z_0)\|_{L^\infty(\Gamma^B)} = 0\left(\dfrac{(\log t)^{N+1}}{t^{(N+2)/2}}\right), \end{cases} \qquad (2.37)$$

uniformly for $M^{-1} \leqq z_0 = \sqrt{\dfrac{-x}{12t}} \leqq M$.

Set

$$m^{(3)}(z)=\begin{cases} m^{A\cup B}(z)\,, & z \text{ outside } \Gamma^A \text{ and } \Gamma^B\,,\\ m^{A\cup B}(z)(m^A(z))^{-1}\,, & z \text{ inside } \Gamma^A\,,\\ m^{A\cup B}(z)(m^B(z))^{-1}\,, & z \text{ inside } \Gamma^B\,. \end{cases} \tag{2.38}$$

(Note that $m^{(3)}(z)$ is analytic on the crosses Σ^A, Σ^B). The matrix $m^{(3)}(z)\to I$ as $z\to\infty$, solves the RH problem on $\Gamma^A\cup\Gamma^B$,

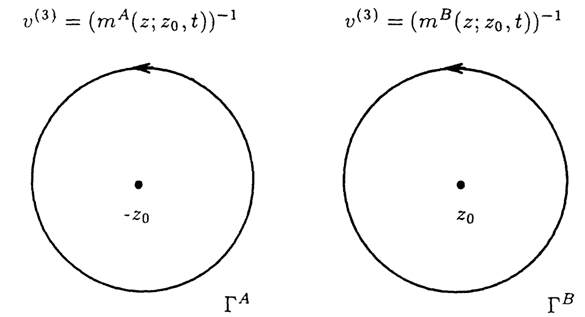


Fig. 2.39.

Let

$$\begin{cases} \hat{m}^A(z)=m^A(z;z_0,t) \text{ for } z\in\Gamma^A\\ \qquad =I \text{ for } z\in\Gamma^B\,,\\ \hat{m}^B(z)=m^B(z;z_0,t) \text{ for } z\in\Gamma^B\\ \qquad =I \text{ for } z\in\Gamma^A\,. \end{cases} \tag{2.70}$$

Then the operator $C^{(3)}$ for the inverse problem on $\Gamma^A\cup\Gamma^B$ is given by (take $\omega_-^{(3)}=0$, $\omega_+^{(3)}=v^{(3)}-I$)

$$C^{(3)}=A+B\,, \tag{2.41}$$

where

$$Af=C_-(f((\hat{m}^A)^{-1}-I))\,,\quad Bf=C_-(f((\hat{m}^B)^{-1}-I)) \tag{2.42}$$

and

$$\begin{aligned}\mu^{(3)}&=(1-C^{(3)})^{-1}I\\ &=I+\sum_{j=1}^{N}(A+B)^jI+E^{(3)}(z,t,z_0)\,,\end{aligned} \tag{2.43}$$

where

$$\|E^{(3)}(\cdot,t,z_0)\|_{L^2(\Gamma^A\cup\Gamma^B)}\leqq\frac{c}{t^{(N+2)/2}}\,, \tag{2.44}$$

uniformly for $M^{-1}\leqq z_0\leqq M$, by (2.37).

We say that a matrix $a=a(\phi)=\begin{pmatrix} a_{00} & a_{01}\\ a_{10} & a_{11}\end{pmatrix}$, respectively $b=b(\phi)=\begin{pmatrix} b_{00} & b_{01}\\ b_{10} & b_{11}\end{pmatrix}$, is of a-type, respectively b-type, if

$$a_{ij}(\phi)=\text{const. } e^{(i-j)\phi}\,,\quad b_{ij}(\phi)=\text{const. } e^{(j-i)\phi}\,,\quad 0\leqq i,j\leqq 1\,,$$

respectively. Note first that the a-type, respectively b-type, matrices form an algebra. Hence an arbitrary product of a-type and b-type matrices reduces to an alternating product . . . *abab* . . . . A simple computation shows that for such an alternating product,

$$(\ldots abab\ldots)_{01}=\sum_{j=-N_b}^{N_a-1} c_j e^{(2j+1)q}\,, \tag{2.45}$$

where N_a (respectively N_b) is the number of a-type (respectively) b-type matrices in the product.

From (2.34)–(2.37),

$$\begin{cases} A=\displaystyle\sum_{0\leqq q\leqq p\leqq N} \frac{(\log t)^q}{t^{(p+1)/2}}\, e^{\frac{-\phi}{2}\operatorname{ad}\sigma_3} A_{pq} e^{\frac{\phi}{2}\operatorname{ad}\sigma_3}+E_A(t,z_0)\,, \\ B=\displaystyle\sum_{0\leqq q\leqq p\leqq N} \frac{(\log t)^q}{t^{(p+1)/2}}\, e^{\frac{\phi}{2}\operatorname{ad}\sigma_3} B_{pq} e^{-\frac{\phi}{2}\operatorname{ad}\sigma_3}+E_B(t,z_0)\,, \end{cases} \tag{2.46}$$

where

$$\begin{cases} \|A_{pq}\|_{L^2\cup L^\infty\to L^2},\ \|B_{pq}\|_{L^2\cup L^\infty\to L^2}\leqq c_{pq} \\ \|E_A(t,z_0)\|_{L^2\cup L^\infty\to L^2},\ \|E_B(t,z_0)\|_{L^2\cup L^\infty\to L^2}=O\left(\dfrac{(\log t)^{N+1}}{t^{(N+2)/2}}\right), \end{cases}$$

uniformly for $M^{-1}\leqq z_0\leqq M$. As $\omega_-^{(3)}=0$, $\mu^{(3)}(z)=m_-^{(3)}(z)$ by (2.5), and hence by (2.31) and (2.38), for any fixed (and large) $n\geqq\dfrac{N+2}{2}$,

$$\begin{aligned} u^{(3)}(x,t)&=2\int_{\Gamma^A\cup\Gamma^B}(\mu^{(3)}(s)\omega_+^{(3)}(s))_{21}\,ds+O\left(\frac{1}{t^n}\right) \\ &=2\int_{\Gamma^A\cup\Gamma^B}(m_+^{(3)}(s)-\mu^{(3)}(s))_{21}\,ds+O\left(\frac{1}{t^n}\right) \\ &=-2\int_{\Gamma^A\cup\Gamma^B}\mu^{(3)}(s)\,ds+O\left(\frac{1}{t^n}\right) \\ &=-2\sum_{j=1}^{N}\int_{\Gamma^A\cup\Gamma^B}((A+B)^jI)_{21}(s)\,ds+O\left(\frac{1}{t^{(N+2)/2}}\right), \end{aligned} \tag{2.48}$$

where we have used (2.44). The asymptotic expansion (1.18) now follows easily by using (2.45) and (2.46). The condition $u_{kpq}(z_0)=\overline{u_{-kpq}(z_0)}$ in (1.19) follows from the reality of $u(x,t)$.

Remark 2.49. It is clear how to extend the above analysis to oscillatory RH problems with more than two points of stationary phase – simply solve the problem one point at a time, and then add in one circle for each point.

Remark 2.50. We see that analytically the $(\log t)^q$ terms arise in the asymptotic expansions from the factor $\delta(z)$, which is needed to control the decomposition of $v_{x,t}(z)$ into triangular factors (see [DZ1], pp. 300–301).

3. Determination of Coefficients

In this section we prove parts (c) of the theorems, assuming parts (b).

Part 1: NLS. Inserting (1.1) (and its x and t derivatives) into the NLS equation (1.1), we obtain for $|z_0|=\left|\frac{-x}{4t}\right|\leqq M$,

$$\sum_{p=1}^{\infty}\left(vu_p+it\dot{u}_p-\frac{i(p-1)}{2}u_p\right)\Big/t^{(p+2)/2}$$

$$+\frac{1}{16}\sum_{p=1}^{\infty}(u_p''-iv''(\log t)u_p-4iv'(\log t)u_p'-v'(\log t)^2u_p)/t^{(p+4)/2}$$

$$=2\sum_{p_j\geqq 1,j=1,2,3}\frac{u_{p_1}u_{p_2}\bar{u}_{p_3}}{t^{(p_1+p_2+p_3)/2}}, \tag{3.1}$$

where $u_p'(z_0,t)=\left.\frac{\partial}{\partial z_0}\right|_{t\,\mathrm{fixed}} u(z_0,t)$, $\dot{u}_p(z_0,t)=\left.\frac{\partial}{\partial t}\right|_{z_0\,\mathrm{fixed}} u_p(z_0,t)$ etc. Collecting terms of order $t^{-(p+2)/2}$, we obtain for $p=1$,

$$vu_1=2u_1|u_1|^2,\text{ or } v=2|u_1|^2 \text{ as in (1.3), and for } p>1\,, \tag{3.2}$$

$$vu_p+it\dot{u}_p-i((p-1)/2)u_p+\frac{1}{16}(u_{p-2}''-iv''(\log t)u_{p-2}-4iv'(\log t)u_{p-2}'$$

$$-v'(\log t)^2u_{p-2})=2\sum_{\substack{1\leqq p_j<p,j=1,2,3\\ p_1+p_2+p_3=p+2}}u_{p_1}u_{p_2}\bar{u}_{p_3}+2(u_1^2\overline{u_p}+2u_p|u_1|^2)\,. \tag{3.3}$$

Substituting (1.12) for u_p, we obtain the recursion relations (1.13), (1.14) by simple linear algebra.

Part 2: MKdV. Substituting (1.18) (and its x and t derivates) into the MKdV equation (1.7), and collecting terms of order $e^{\psi}t^{-(p+2)/2}$, we obtain for $p=1$,

$$|u_{11}|^2=v/12z_0,\text{ which agrees with (1.20)}\,, \tag{3.4}$$

and for $p>1$,

$$\left(\frac{1}{2}-\frac{p}{2}+\mathrm{i}v\right)u_{1p}+t\dot{u}_{1p}+12iz_0u_{11}^2\overline{u_{1p}}=\sum_{q=0}^{p-1}f_{pq}(\log t)^q\,, \tag{3.5}$$

where f_{pq} are given in (1.26). Inserting (1.19) for u_{1p}, we obtain the recursion formula (1.25) for u_{1pq}.

To determine u_{kp} for $k>1$, we collect terms of order $e^{k\psi}t^{-(p+2)/2}$ and we obtain (1.24).

4. Differentiation of the Asymptotic Series

In this section we show that the asymptotic series (1.11) and (1.18) are differentiable term by term with respect to x and t.

First we consider NLS. The calculations in [DIZ] show that the RH problem (2.6), (2.7) is equivalent to a RH problem $(\Sigma^{(4)}, v_{x,t}^{(4)})$,

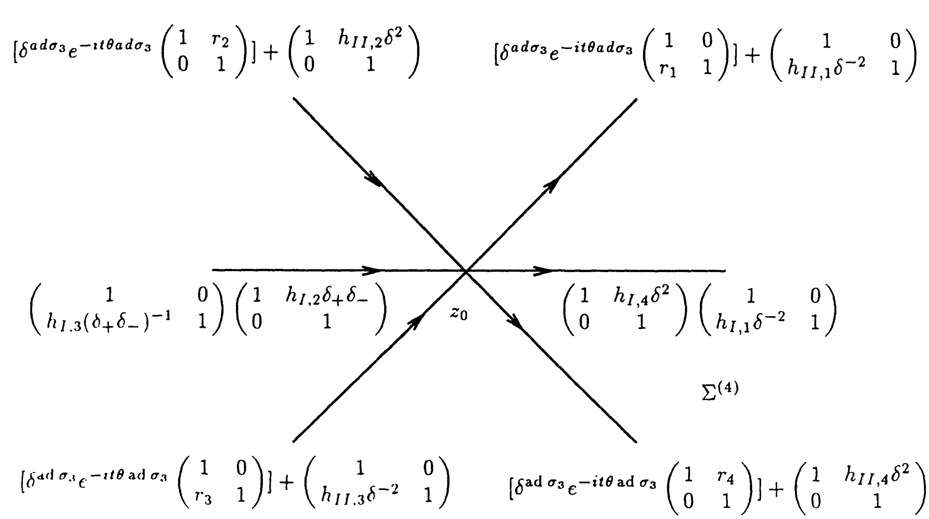


Fig. 4.1.

$$\begin{cases} m_+^{(4)}(z)=m_-^{(4)}(z)v_{x,t}^{(4)}(z),\ z\in\Sigma^{(4)}\,, \\ m^{(4)}(z)\to I \text{ as } z\to\infty\,. \end{cases} \tag{4.2}$$

Carrying the calculation in [DIZ] to higher order (see [DZ1]), one sees that for any p, L one can ensure that as $t\to\infty$, $|z_0|\leqq M$,

$$\left\|\frac{\partial^l}{\partial\xi^l}(h_{\alpha,i}\delta^{-2})(\cdot\,,z_0,t)\right\|_{L^1\cap L^\infty(\Sigma^{(4)})}\leqq c_{\alpha,i,l}t^{-p} \tag{4.3}$$

for $\alpha=I$ or II, $\xi=x$ or t, $1\leqq i\leqq 4$ and $0\leqq l\leqq L$. The associated operator $C^{(4)}$ $(\equiv C_\omega$, see (2.3)) acts on the space $L^2(\Sigma^{(4)})$, which clearly depends on z_0. In order to differentiate the operator we reduce the space to the fixed space $L^2(\Sigma^{(5)})=L^2(\Sigma^{(4)}-z_0)$. It turns out that the associated operator $C^{(5)}$ $((C^{(5)}f)(z)=(C^{(4)}f(\cdot+z_0))(z-z_0))$ on $L^2(\Sigma^{(5)})$ is not differentiable with respect to x and t from $L^2(\Sigma^{(5)})\to L^2(\Sigma^{(5)})$ because of the singularity of $\delta=e^{i\nu(z_0)\log(z-z_0)+\cdots}$ (see (2.14)) as $z\to z_0$ on $\Sigma^{(4)}$, or as $\zeta=z-z_0\to 0$ on $\Sigma^{(5)}$, so that $\dfrac{\partial\delta}{\partial t}\sim\log\zeta$, $\zeta=z-z_0$, which is not bounded on $\Sigma^{(5)}$. $\Big($For $\dfrac{\partial^l}{\partial\xi^l}(h_{\alpha,i}\delta^{\pm 2})$ in (4.3) this logarithmic divergence of δ is cancelled by the behavior of $h_{\alpha,i}(z)$ as $z \to z_0$ (see [DIZ, DZ1].) To avoid this difficulty, we consider the following equivalent RH problem $(\Sigma^{(6)}, v_{x,t}^{(6)})$

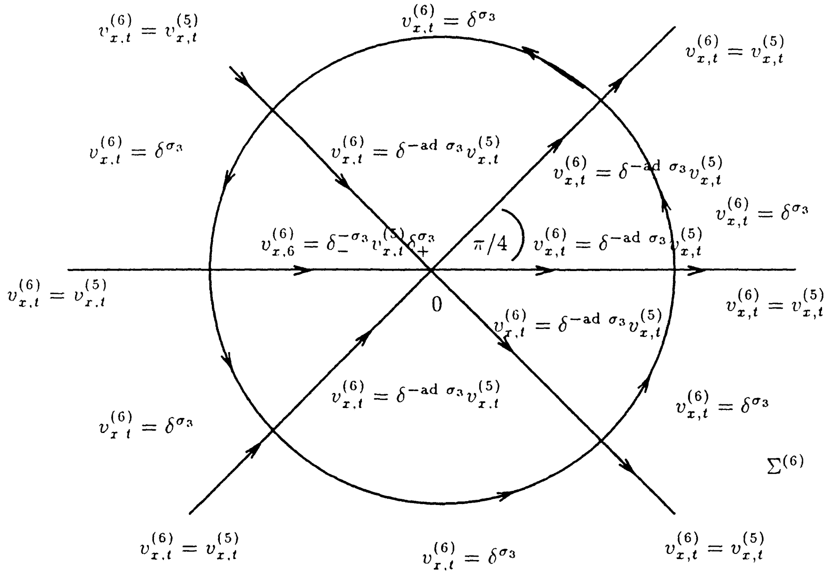


Fig. 4.4.

where the circle has fixed radius ρ, say $\rho = 1$. The RH problem is obtained from $(\Sigma^{(4)}, v_{x,t}^{(4)})$, or more properly $(\Sigma^{(5)}, v_{x,t}^{(5)})$, by setting $m^{(6)}(z) = m^{(5)}(z) = m^{(4)}(z + z_0)$ for $|z| > 1$, $m^{(6)}(z) = (m^{(5)}(z)\delta(z)^{\sigma_3}$ for $|z| < 1$. The singularity of δ at $z = 0$ is now absent. For example, for $z \in (0,1)e^{i\pi/4}$, $v_{x,t}^{(6)}(z) = e^{-it\theta\,\mathrm{ad}\,\sigma_3}\begin{pmatrix} 1 & 0 \\ r_1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ h_{\pi,1} & 1 \end{pmatrix}$ and for $z \in (-1, 0)$, $v_{x,t}^{(6)}(z) = \begin{pmatrix} 1 - |r(z)|^2 & h_{I,2}(z) \\ h_{I,3}(z) & (1 + h_{I,2}h_{I,3})(I - |r(z)|^2)^{-1} \end{pmatrix}$ as $\delta_+ = \delta_-(1 - |r(z)^2)$. The method in [DIZ, DZ1], together with (4.3), now implies that as $t \to \infty$,

$$\frac{\partial^l}{\partial \xi^l} u(x, t) = \frac{\partial^l}{\partial \xi^l} u^{(7)}(x, t) + O\left(\frac{1}{t^q}\right), \ |z_0| \leqq M ,$$

for any given $0 \leqq l \leqq l_1, q$, and again $\xi = x$ or t. Here $u^{(7)}(x, t)$ is the potential associated with $m^{(7)}(z; x, t)$,

$$u^{(7)}(x, t) = 2 \lim_{z \to \infty} z m_{12}^{(7)}(z; x, t) , \tag{4.5}$$

where $m^{(7)}(z; x, t)$ solves the RH problem $(\Sigma^{(7)}, v_{x,t}^{(7)})$,

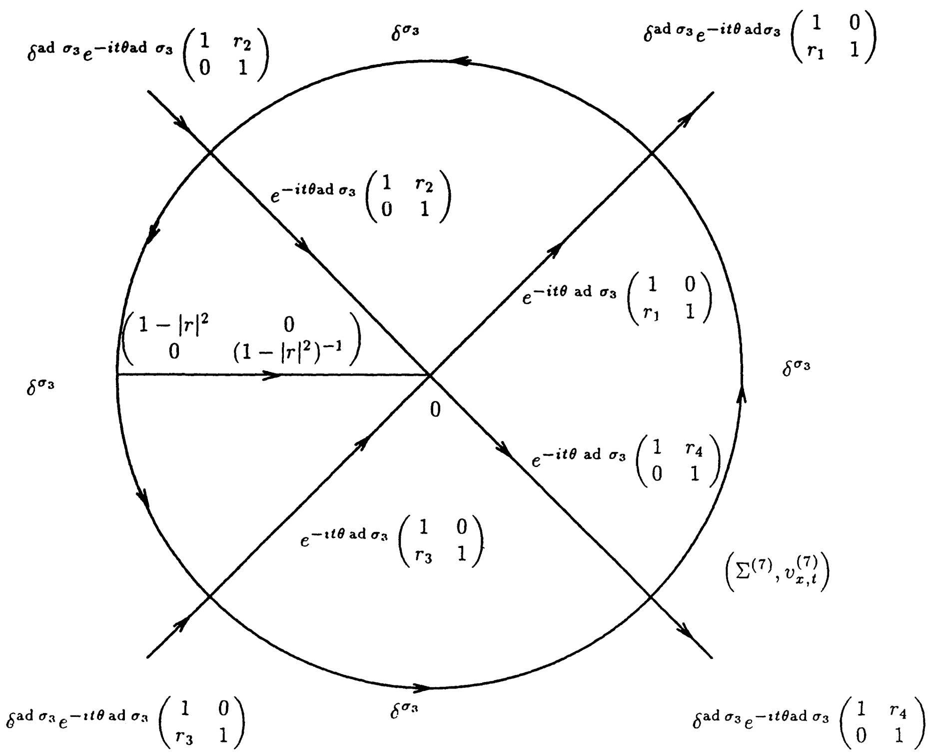


Fig. 4.6.

This RH problem is equivalent to the RH $(\Sigma^{(1)}, v_{x,t}^{(1)})$ in the sense that

$$\begin{cases} m^{(7)}(z;x,t) = m^{(1)}(z+z_0;x,t)\,, \; |z|>1\,, \\ m^{(7)}(z;x,t) = m^{(1)}(z+z_0;x,t)\delta^{\sigma_s}(z+z_0)\,, \; |z|<1\,, \end{cases} \tag{4.7}$$

and hence, using the analyticity of δ, is equivalent to a RH problem $(\Sigma^{(8)}, v_{x,t}^{(8)})$ on a contour $\Sigma^{(8)}$ of the same shape as before except the circle now has radius $t^{-1/2}$,

$$\begin{cases} m^{(8)}(z;x,t) = m^{(1)}(z+z_0;x,t)\,, \; |z|>t^{-1/2}\,, \\ m^{(8)}(z;x,t) = m^{(1)}(z+z_0;x,t)(\delta(z+z_0))^{\sigma_3}\,, \; |z|<t^{-1/2}\,. \end{cases} \tag{4.8}$$

Scaling $z \to z/\sqrt{t}$, we obtain a RH problem on $(\Sigma^{(9)} = \Sigma^{(7)}, v_{x,t}^{(9)})$ for $m^{(9)}(z;x,t) = m^{(8)}(z/\sqrt{t};x,t)$, with $v_{x,t}^{(9)}(z) = v_{x,t}^{(8)}(z/\sqrt{t})$. Observe that the RH problem for $m^{(9)}$ is given on a fixed contour and that δ occurs in $(\Sigma^{(9)}, v_{x,t}^{(9)})$ only on the fixed circle, $|z|=1$, which is away from the singularity of δ at zero. It follows that $m^{(9)}$ can be differentiated arbitrarily often with respect to x and t. Moreover, if one examines the analog for $m^{(9)}$ of the error term $E_v(z,t,z_0)$ in (2.16), for example, one sees easily that differentiation with respect to x or t cannot decrease the rate of decay with respect to t in the estimate (2.18). The same is then true for the analog of the operator bound in (2.23). Similar considerations also show that analog of the coefficients $v_{ij}^{(2)}$ in (2.16) and the operators C_{pq} in (2.19), can also be differentiated with respect to z_0, etc. This proves that the asymptotic series (1.11) can be differentiated term by term.

In the case of MKdV, δ has two singularities, at $-z_0$ and z_0 respectively. For each of these points we add in one circle, and proceed as in the case of NLS above.

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References

[BC] Beals R., Coifman, R.R.: Scattering and Inverse Scattering for First Order Systems. Comm. Pure Appl. Math. **37**, 39–90 (1984)

[BSa] Buslaev V.S., Sukhanov V.V.: Asymptotic Behavior of Solutions of the Korteweg-de Vries Equation. Proc. Sci. Seminar LOMI **120**, 32–50 (1982) (Russian); J. Sov. Math. **34**, 1905–1920 (1986) (in English)

[BSb] Buslaev V.S., Sukhanov V.V.: On the Asymptotic Behavior as $t \to \infty$ of the Solutions of the Equation $\psi_{xx} + u(x, t)\psi + (\lambda/4)\psi = 0$ With Potential u Satisfying the Korteweg-de Vries Equation, I. Prob. Math. Phys. **10**, M. Birman, ed., 70–102, (1982) (in Russian); Sel. Math. Sov. **4**, 225–248 (1985) (in English); II. Proc. Sci. Seminar LOMI **138**, 8–32 (1984) (in Russian); J. Sov. Math. **32**, 426–446 (1986) (in English); III. Prob. Math. Phys. **11**, M. Birman, ed., 78–113, (1986) (in Russian)

[DIZ] Deift P.A., Its A.R., Zhou X.: Long-time Asymptotics for Integrable Nonlinear Wave Equations. Important developments in Soliton theory. Fokas, A.S., Zakharov, V.E. (eds.) Berlin, Heidelberg, New York: Springer, 1993

[DZ1] Deift P.A., Zhou X.: A Steepest Descent Method for Oscillatory Riemann–Hilbert Problems. Asymptotics for the MKdV equation. Ann. of Math. **137**, 295–368 (1993)

[DZ2] Deift P.A., Zhou X.: Long-time Asymptotics for the Autocorrelation Function of the Transverse Ising Chain at the Critical Magnetic Field. To appear in the Proceedings of the Nato Advanced Research Workshop: Singular Limits of Dispersive Waves, Lyons, France, 8–12 July, 1992

[DZ3] Deift P.A., Zhou X.: Asymptotics for the Painlevè II equation. Announcement in Adv. Stud. Pure Math. **23**, 17–26 (1994). Full paper to appear in Comm. Pure Appl. Math, 1994

[DVZ] Deift, P.A., Venakides S., Zhou, X.: The Collisionless Shock Region for the Long-time Behavior of Solutions of the KdV equation. Comm. Pure Appl. Math. **47**, 199–206 (1994)

[K] Kamvissis, S.: Long-time Behavior of the Toda Lattice Under Initial Data Decaying at Infinity. Commun. Math. Phys. **153**, 479–519 (1993)

[N] Novokshenov, V.Y.: Asymptotics as $t \to \infty$ of the Solution of the Cauchy Problem for the Nonlinear Schrödinger Equation. Sov. Math. Dokl. **21**, 529–533 (1980)

[SA1] Segur, H., Ablowitz, M.J.: Asymptotics Solutions and Conservation Laws for the Non-linear Schrödinger equation. Part I, J. Math. Phys. **17**, 710–713 (1976)

[SA2] Segur, H., Ablowitz, M.J.: Asymptotic solutions of the Korteweg de Vries equation. Stud. Appl. Math. **57**(1), 13–44 (1977)

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