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New Jacobi-like Identities for Z_K Parafermion Characters

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Abstract. We state and prove various new identities involving the $\mathbf{Z}_K$ parafermion characters (or level-K string functions) c_n^l for the cases $K = 4$, $K = 8$, and $K = 16$. These identities fall into three classes: identities in the first class are generalizations of the famous Jacobi ϑ-function identity (which is the $K = 2$ special case), identities in another class relate the level $K > 2$ characters to the Dedekind η-function, and identities in a third class relate the $K > 2$ characters to the Jacobi ϑ-functions. These identities play a crucial role in the interpretation of fractional superstring spectra by indicating spacetime supersymmetry and aiding in the identification of the spacetime spin and statistics of fractional superstring states.

1. Introduction

$\mathbf{Z}_K$ parafermion theories [1] have recently found a new application as the basic worldsheet building blocks of fractional superstrings [2]. Fractional superstrings are generalizations of the traditional superstring and heterotic string, and are constructed essentially by replacing the worldsheet supersymmetry of the superstring with a *fractional* supersymmetry (parametrized by an integer K) which relates worldsheet bosons not to fermions but to $\mathbf{Z}_K$ *para*fermions. It is found that the critical spacetime dimensions of such string theories are less than ten, and are in fact given by the simple formula

$$D_c = 2 + \frac{16}{K}, \quad K \geqq 2 . \tag{1.1}$$

The special case $K = 2$ reproduces the usual superstring and heterotic string with critical dimension $D_c = 10$, and the cases with $K = 4$, $K = 8$, and $K = 16$ yield new

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fractional superstring theories with critical spacetime dimensions $D_c = 6$, $D_c = 4$, and $D_c = 3$ respectively.

The worldsheet field content of these string theories consists in general of bosons and $\mathbf{Z}_K$ parafermions; the special-case $\mathbf{Z}_2$ parafermions are equivalent to ordinary Majorana–Weyl fermions. The partition functions of fractional superstrings therefore involve the $\mathbf{Z}_K$ parafermion characters for $K > 2$, just as the partition functions of the superstring and heterotic string involve the ordinary "$\mathbf{Z}_2$" characters of the Majorana–Weyl fermions. Similarly, just as the usual fermion characters can be written in terms of the classical Jacobi theta-functions ϑ_i, the more general $\mathbf{Z}_K$ parafermion characters can be written in terms of the so-called "string functions" c^l_n originally introduced in the study of infinite-dimensional Lie algebras [3]. For $K = 2$ these string functions are equivalent to the Jacobi ϑ-functions, but such is of course not the case for $K > 2$.

As is well-known in the $K = 2$ superstring theory, the spacetime properties of the string spectrum are reflected at the partition-function level in the properties of these fermion characters. For example, any superstring or heterotic string spectrum exhibiting a spacetime supersymmetry gives rise to a partition function proportional to the factor

$$J = \vartheta_3{}^4 - \vartheta_2{}^4 - \vartheta_4{}^4 , \tag{1.2}$$

and the well-known Jacobi identity $J = 0$ is therefore responsible for the vanishing of such partition functions at all mass levels of the theory. This reflects the exact cancellation of spacetime bosonic states (which arise from the Neveu–Schwarz sector of the theory and yield the terms $\vartheta_3{}^4 - \vartheta_4{}^4$ within J) and spacetime fermionic states (which arise from the Ramond sector and yield the term $\vartheta_2{}^4$).

A similar situation exists for the $K > 2$ fractional superstrings, where once again we expect the spacetime properties of the fractional superstring spectrum to be reflected in the partition function through identities satisfied by the $\mathbf{Z}_K$ parafermion characters (i.e., by the string functions c^l_n). Indeed, since the consistency of fractional superstrings remains to be verified, the existence and use of such identities provides an important step towards that goal. In this paper we obtain and prove these new string-function identities, and discuss as well their relevance to fractional superstrings.

In particular, we state and prove three series of identities. First is a series of identities for the $\mathbf{Z}_K$ parafermion characters which are analogous to the $K = 2$ Jacobi identity and which can be considered to be its $K > 2$ generalizations. We will see that these new identities arise naturally in the partition functions of fractional superstrings, and their presence can therefore be interpreted as a signature of spacetime supersymmetry in fractional superstring spectra. For each value of K, we will find that the corresponding identity in this series involves a vanishing combination of only the $\mathbf{Z}_K$ parafermion characters; we will therefore, for reasons to become clear, refer to this series of identities as the $[K, 0]$ series. Second, just as the individual terms within the $K = 2$ Jacobi factor J can be recognized as arising from either spacetime bosonic or fermionic sectors, we will see that a similar self-consistent grouping of terms is possible within each of our new Jacobi-like identities for $K > 2$. This will result in a second series of new string-function identities, each of which (as we will see) relates the $K > 2$ characters to the $K = 2$ characters. We will therefore refer to this series of identities as the $[K, 2]$ series. Finally, we find a third series of identities which generalize another well-known ϑ-function identity $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$ (where η is the Dedekind η-function, the inverse of the boson character). Just as this identity relates the boson character η^{-1} to the fermion characters ϑ_i, our third series of identities relates the boson character η^{-1} to the $\mathbf{Z}_K$ *parafermion* characters c_n^l. We will therefore refer to this series of identities as the $[K, 1]$ series, since (as we will see) the $K = 1$ character is equal to the boson character η^{-1}. As our notation suggests, these are undoubtedly only three of many such $[K_1, K_2]$-type series of identities relating the characters of $\mathbf{Z}_{K_1}$ and $\mathbf{Z}_{K_2}$ parafermions to each other, and in this paper we also discuss how such general $[K_1, K_2]$ identities may be obtained and proven.

Clearly, many of these new identities may have interpretations in the theory of $\mathbf{Z}_K$ parafermions which are independent of fractional superstrings. As a striking example, our $[16, 2]$ identities indicate that the simple algebraic *differences* of many of the $\mathbf{Z}_{16}$ characters are nothing but the $\mathbf{Z}_2$ (fermion) characters; the implications of this fact for the relationship between ordinary fermions and $\mathbf{Z}_{16}$ parafermions are yet to be investigated. Similar relationships exist as well for other values of K. Therefore, we have organized this paper in such a way that it can be read without a detailed understanding of fractional superstrings *per se*. In Sect. 2.1 we provide a brief introduction to the $\mathbf{Z}_K$ parafermion theory and review the definitions and properties of the $\mathbf{Z}_K$ parafermion characters (or level-K string functions). In Sect. 2.2 we then discuss the role of the $\mathbf{Z}_K$ parafermion theory in the fractional superstring, along the way introducing some of our new identities and discussing the fractional-superstring contexts in which they arise. Section 3 is simply a list of all of these new identities, and in Sect. 4 we prove these identities using some powerful results from the theory of modular functions. The proofs of the $K > 2$ identities exactly mirror the proofs of the well-known $K = 2$ special cases, and accordingly we have kept the discussion in Sect. 4 sufficiently general so that it is clear how additional $[K_1, K_2]$ identities may be obtained and proven. A reader unconcerned with fractional superstrings can skip Sect. 2.2, but it seems that the fractional superstring framework provides interesting physical interpretations for many of these new identities.

2. $\mathbf{Z}_K$ Parafermions and Fractional Superstrings

In this section we first review the definitions and properties of the $\mathbf{Z}_K$ parafermion characters (or level-K string functions) which appear in our new identities. We then provide a brief but self-contained introduction to the fractional superstring idea, presenting many of these new identities in the physical contexts in which they arise and discussing some of their implications.

2.1. The $\mathbf{Z}_K$ Parafermion Characters. The $\mathbf{Z}_K$ parafermion theory [1, 4] is closely related to, and in fact can be derived from, the $SU(2)_K$ Wess–Zumino–Witten (WZW) theory [5]. As is well-known, the $SU(2)_K$ WZW theory can be viewed as a tensor product of two independent theories: the first is that of a free $U(1)$ boson compactified on a circle or radius $2\sqrt{K}$, and the remaining $SU(2)_K/U(1)$ coset theory is the $\mathbf{Z}_K$ parafermion theory. It therefore follows that the characters of these remaining $\mathbf{Z}_K$ parafermion fields can be obtained from the full $SU(2)_K$ characters by appropriately factoring out the $U(1)$ boson characters. We now review precisely how this is done.

We begin by considering the chiral $SU(2)_K$ WZW theory [5]. This theory consists of holomorphic primary fields $\Phi^j_m(z)$ which can be organized into $SU(2)$ representations labelled by $j \in \mathbf{Z}/2$, where $0 \leqq j \leqq K/2$ and $|m| \leqq j$ with $j - m \in \mathbf{Z}$. Since $SU(2)_K$ always has a $U(1)$ subgroup which can be bosonized as a free boson $\varphi(z)$ compactified on a circle of radius $2\sqrt{K}$, we can correspondingly factor the primary fields as

$$\Phi^j_m(z) = \phi^j_m(z) \exp\left\{\frac{m}{\sqrt{K}} i\varphi(z)\right\} . \tag{2.1}$$

The $\phi^j_m(z)$ are therefore primary fields of the coset $SU(2)_K/U(1)$ theory, i.e., of the $\mathbf{Z}_K$ parafermion theory. The parafermion fields ϕ^j_m have conformal dimensions h^{2j}_{2m}, where

$$h^l_n = \frac{l(l+2)}{4(K+2)} - \frac{n^2}{4K} \quad \text{for } |n| \leqq l , \tag{2.2}$$

and it is convenient to extend the definition of the parafermion fields outside the range $|m| \leqq j$ via the identifications

$$\phi^j_m = \phi^j_{m+K} = \phi^{K/2-j}_{-(K/2-m)} . \tag{2.3}$$

The fusion rules of these parafermion fields ϕ^j_m follow from those of the $SU(2)_K$ theory:

$$[\phi^{j_1}_{m_1}] \times [\phi^{j_2}_{m_2}] = \sum_{j=|j_1-j_2|}^{r} [\phi^j_{m_1+m_2}] , \tag{2.4}$$

where $r = \min(j_1 + j_2, K - j_1 - j_2)$ and where the sectors $[\phi^j_m]$ include the primary fields ϕ^j_m and their parafermion descendants. Note that a given field ϕ^j_m may appear multiple times in the theory.

The $SU(2)_K$ characters $\chi_l(\tau, z)$ for spin $j = l/2$ are given in Ref. [3]:

$$\chi_l(\tau, z) = \frac{\Theta_{l+1,K+2}(\tau, z) - \Theta_{-l-1,K+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} , \tag{2.5}$$

where the classical (Jacobi–Riemann) Θ-functions are defined by

$$\Theta_{n,L}(\tau, z) \equiv \sum_{s \in \mathbf{Z} + n/2L} \exp\{2\pi i L(s^2\tau - sz)\} \quad \text{for } n \in \mathbf{Z} \pmod{2L} . \tag{2.6}$$

These Θ-functions have relatively simple properties under modular transformations

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} , \tag{2.7}$$

where $\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of the homogeneous modular group $\tilde{\Gamma} \equiv SL(2, \mathbf{Z})$ if $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$. The modular group $\Gamma \equiv \tilde{\Gamma}/\{\pm 1\}$ is generated by $T: \tau \to \tau + 1$ and $S: \tau \to -1/\tau$, and under the action of these two generators we have

$$\Theta_{n,L}(\tau + 1, z) = \exp\left\{2\pi i \frac{n^2}{4L}\right\} \Theta_{n,L}(\tau, z) , \tag{2.8}$$

and

$$\Theta_{n,L}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\sqrt{\frac{-i\tau}{2L}}\exp\left\{\frac{\pi i L z^2}{\tau}\right\}\sum_{n'=0}^{2L-1}\exp\left\{\frac{\pi i n n'}{L}\right\}\Theta_{n',L}(\tau,z)\,. \quad (2.9)$$

Here the square root indicates the branch with non-negative real part. From these results the modular transformation properties of the full $SU(2)_K$ characters $\chi_l(\tau, z)$ can be obtained.

We are primarily interested in obtaining the characters Z^{2j}_{2m} of the parafermion sectors $[\phi^j_m]$, for the string functions c^l_n are simply related to these characters via

$$Z^l_n(\tau)=\eta(\tau)c^l_n(\tau)\,, \quad (2.10)$$

where $l \equiv 2j$, $n \equiv 2m$, and where η is the Dedekind η-function:

$$\eta(\tau)\equiv q^{1/24}\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=0}^{\infty}(-1)^n q^{3(n-1/6)^2/2} \quad (2.11)$$

with $q \equiv \exp\{2\pi i\tau\}$. However, since the $\mathbf{Z}_K$ parafermion theory is equivalent to the $SU(2)_K/U(1)$ coset theory, the parafermion characters Z^l_m can be obtained by expanding the full $SU(2)_K$ characters $\chi_l(\tau, z)$ in a basis of $U(1)$ characters [6]:

$$\chi_l(\tau,z)=\sum_{n=-l}^{2K-l-1}Z^l_n(\tau)\frac{\Theta_{n,K}(\tau,z)}{\eta(\tau)}=\sum_{n=-l}^{2K-l-1}c^l_n(\tau)\Theta_{n,K}(\tau,z)\,. \quad (2.12)$$

Here $\Theta_{n,K}(\tau, z)/\eta(\tau)$ are the characters of a $U(1)$ boson compactified at radius $2\sqrt{K}$. Equation (2.12), then, can be taken as a definition of the string functions c^l_n.

Explicit expressions for the c^l_n can be extracted from (2.5), (2.6), and (2.12). The original formula obtained by Kač and Peterson [6] is

$$c^l_n(\tau)=[\eta(\tau)]^{-3}\sum_{x,y}{}'\,\mathrm{sign}(x)q^{x^2(K+2)-y^2K}\,, \quad (2.13)$$

where the prime on the summation indicates that x and y must be chosen so that three conditions are satisfied: (1) $-|x| < y \leqq |x|$; (2) either x or $(\frac{1}{2}-x)$ must equal $(l+1)/[2(K+2)]$ modulo 1; and (3) either y or $(\frac{1}{2}+y)$ must equal $n/(2K)$ modulo 1. For many calculational purposes, however, a useful alternative expression is [7]

$$\begin{aligned}c^l_n(\tau)=q^{h^l_n+[4(K+2)]^{-1}}\eta^{-3}\sum_{r,s=0}^{\infty}(-1)^{r+s}q^{r(r+1)/2+s(s+1)/2+rs(K+1)}\\ \times\left\{q^{r(j+m)+s(j-m)}-q^{K+1-2j+r(K+1-j-m)+s(K+1-j+m)}\right\}\,, \quad (2.14)\end{aligned}$$

where $l-n\in 2\mathbf{Z}$ and where h^l_n are the highest weights given in (2.2). Note that the string functions exhibit the symmetries

$$c^l_n=c^l_{-n}=c^{K-l}_{K-n}=c^l_{n+2K}\,, \quad (2.15)$$

as a consequence of which for any K we are free to choose a "basis" of string functions c^l_n, where $0 \leqq l \leqq K$ and $0 \leqq n \leqq n_{\max}$, where $n_{\max}$ equals l if $l \leqq K/2$, and $l-2$ otherwise. Note also that the K-dependence of the string functions is suppressed in this notation. From (2.14), then, we see that the string functions c^l_n all take the general form $q^{H^l_n}(1+\ldots)$, where within the parentheses all powers of q are non-negative integers, and where

$$H^l_n\equiv h^l_n+\frac{1}{4(K+2)}-\frac{1}{8}=h^l_n-\frac{K}{8(K+2)}\,. \quad (2.16)$$

For arbitrary fixed K, the set of string functions forms an admissible representation of the modular group, i.e., they close under modular transformations. In fact, under T they transform as eigenfunctions:

$$c_n^l(\tau+1)=\exp\{2\pi i H_n^l\}\,c_n^l(\tau)\,, \tag{2.17}$$

and under S they mix among themselves [6]:

$$c_n^l(-1/\tau)=\frac{1}{\sqrt{-i\tau}}\frac{1}{\sqrt{K(K+2)}}\sum_{l'=0}^{K}\sum_{\substack{n'=-K+1\\ l'-n'\in 2\mathbf{Z}}}^{K} b(l,n,l',n')c_{n'}^{l'}(\tau)\,, \tag{2.18}$$

where the mixing coefficients $b(l,n,l',n')$ are

$$b(l,n,l',n')=\exp\left\{\frac{i\pi nn'}{K}\right\}\sin\left\{\frac{\pi(l+1)(l'+1)}{K+2}\right\}. \tag{2.19}$$

The first square root in (2.18) once again indicates the branch with non-negative real part. It is often convenient to define the linear combinations $d_n^{l\pm}\equiv c_n^l\pm c_n^{K-l}$ when l and n are even: if $K\in 4\mathbf{Z}$, then these $d^{\pm}$-functions are also eigenfunctions of T, and (2.18) implies that the d^+-functions close exclusively amongst themselves under S. The d^--functions, on the other hand, close exclusively with themselves and with the odd (l,n) string functions under S. We will see that for $K>2$ all of our new identities involve only the $d_n^l\equiv d_n^{l+}$ functions.

As expected, the string functions c_n^l reduce to the better-known Dedekind η-function (2.11) and the Jacobi ϑ-functions for the $K=1$ and $K=2$ special cases respectively. In particular, for $K=1$ there is only one independent string function c_0^0, and since the "$\mathbf{Z}_1$ parafermion" theory $SU(2)_1/U(1)$ consists of only the identity field $\phi_0^0=\mathbf{1}$, we immediately find

$$K=1:\quad Z_0^0=\eta c_0^0=1\Rightarrow c_0^0=\eta^{-1}\,. \tag{2.20}$$

Similarly, for $K=2$, there are precisely three independent string functions c_0^0, c_1^1, and c_0^2, and these can be expressed in terms of the three non-vanishing Jacobi ϑ-functions

$$\begin{aligned}\vartheta_2(\tau)&\equiv 2q^{1/8}\prod_{n=1}^{\infty}(1+q^n)^2(1-q^n)\,,\\ \vartheta_3(\tau)&\equiv\prod_{n=1}^{\infty}(1+q^{n-1/2})^2(1-q^n)\,,\\ \vartheta_4(\tau)&\equiv\prod_{n=1}^{\infty}(1-q^{n-1/2})^2(1-q^n)\,,\end{aligned} \tag{2.21}$$

via the relations

$$K=2:\quad\begin{cases}2(c_1^1)^2=\vartheta_2/\eta^3\\ (c_0^0+c_0^2)^2=\vartheta_3/\eta^3\\ (c_0^0-c_0^2)^2=\vartheta_4/\eta^3\,.\end{cases} \tag{2.22}$$

We thus see that for $K=1$ and $K=2$, the string functions reproduce the boson and fermion characters respectively; indeed, (2.22) reflects the fact that the $\mathbf{Z}_2$ parafermion theory is equivalent to that of a free fermion in two dimensions. For

$K > 2$, on the other hand, the string functions are the characters of *para*fermions, and as such they are not *a priori* related in these simple ways to the boson and fermion characters. However, we will see that our $[K, 1]$ and $[K, 2]$ series of identities nevertheless provide such unexpected relations. These new identities arise naturally in the partition functions of fractional superstrings, and it is in such theories that they find possible physical interpretations.

2.2. Fractional Superstrings. We now provide a brief introduction to the theory of fractional superstrings [2]. Since our purpose here is to motivate many of our new string-function identities and discuss some of the physical contexts in which they arise, our treatment will focus primarily on fractional-superstring partition functions. A fuller treatment of these and other aspects of the fractional superstring can be found in Refs. [2] and [8].

As indicated in the Introduction, the basic idea behind the construction of the fractional superstring is simple [2]. Let us first recall the special case of the superstring. The worldsheet structure of the superstring theory is closely related to the $SU(2)_2$ WZW theory [5]: the worldsheet superpartner of the coordinate boson X^μ is a Majorana fermion ψ^μ which can be described by the $SU(2)_2/U(1)$ coset theory, and the spacetime coordinate X^μ can be interpreted as the remaining $U(1)$ boson but with its radius of compactification relaxed to infinity. (The spacetime index μ runs from 0 to $D_c - 1$.) This boson-decompactification procedure destroys the $SU(2)_2$ symmetry of the original WZW model, but its superconformal symmetry survives and exists on the worldsheet.

The fractional superstring theory is related in the same way to the $SU(2)_K$ WZW theory for $K \geqq 2$. The coset theory $SU(2)_K/U(1)$ is the $\mathbf{Z}_K$ parafermion theory [1, 4], and once again we obtain the spacetime coordinate field X^μ by completely decompactifying the remaining WZW $U(1)$ boson. Replacing the supercurrent for $K > 2$ is a new chiral current [5, 9] whose conformal dimension is $(K+4)/(K+2)$; these new currents have fractional spin, and transform the bosonic X^μ fields to fractional-spin fields on the worldsheet. It is natural to refer to this remaining worldsheet symmetry as a fractional superconformal symmetry [10], and to the strings based on these worldsheet fractional supersymmetries as fractional superstrings.

We are interested in constructing the one-loop partition functions Z that such fractional superstring theories must have; in particular, we focus here on the "Type II" fractional superstring in which both the left-moving and right-moving worldsheet theories exhibit a level-K fractional supersymmetry. We can therefore consider, for simplicity, only the holomorphic components of Z; these are the terms arising from the left-moving worldsheet degrees of freedom. As discussed in Sect. 2.1, for each K such terms will be products of the characters of free worldsheet bosons X^μ and worldsheet $\mathbf{Z}_K$ parafermions $(\phi^j_m)^\mu$: each coordinate boson X^μ contributes to the partition function a factor of $1/\eta$ (the character of an infinite-radius boson), while its fractional superpartner (the corresponding worldsheet parafermion) contributes a factor of ηc^l_n. Thus, the net holomorphic contribution to the fractional-superstring partition function from each spacetime dimension is a factor of one string function c^l_n. Even though the fractional superstring theory is formulated in D_c spacetime dimensions (i.e., even though the full worldsheet structure of the fractional superstring is a tensor product of D_c copies of the individual boson/parafermion theories), the large gauge symmetry of critical string theory is expected to remove all time-like and longitudinal components, thereby leaving a spectrum of physical states arising from the excitations of fields corresponding to only the $D_c - 2$ transverse dimensions [2, 8]. This is completely analogous to the case of ordinary $K = 2$ superstring theory, in which there are only $D_c - 2 = 8$ "effective" dimensions giving rise to propagating fields (as is evident, for example, in light-cone gauge). The holomorphic components of K-fractional superstring partition functions therefore consist of $(D_c - 2)$ factors of level-K string functions, and take the general form $c^{D_c - 2}$.

Let us now consider the contribution to the partition function from some of the low-lying states of the fractional superstring; this will, as a byproduct, yield one method of determining the critical dimension D_c. Recall that in general an expansion $\sum_n a_n q^n$ of the holomorphic factors within the partition function Z indicates the net number a_n of states with (left-moving) mass $M^2 = n$ contributing to that term. As explained in Refs. [2] and [8], the left-moving bosonic vacuum state of the D_c-dimensional fractional superstring corresponds to $(c_0^0)^{D_c-2}$. Since (2.16) indicates that this term takes the form

$$(c_0^0)^{D_c-2} \sim q^{H_{\rm tach}}(1 + \ldots), \quad \text{where} \quad H_{\rm tach} = -\frac{(D_c - 2)K}{8(K+2)}, \tag{2.23}$$

we see that for $D_c > 2$ the bosonic ground state is tachyonic. This is analogous to the case of ordinary Type II superstrings, in which the bosonic ground state is tachyonic with $M^2 = H_{\rm tach} = -\frac{1}{2}$. Similarly, the left-moving vector state in the fractional superstring theory contributes to the first term in the expansion of

$$(c_0^0)^{D_c-3}(c_0^2) \sim q^{H_{\rm grav}}(1 + \ldots), \quad \text{where} \quad H_{\rm grav} = \frac{2}{K+2} - \frac{(D_c - 2)K}{8(K+2)}. \tag{2.24}$$

Since this vector state is the left-moving component of the graviton, the fractional superstring theory will therefore be a theory of gravity (i.e., contain a massless graviton) only if $H_{\rm grav} = 0$, or

$$D_c - 2 = \frac{16}{K}. \tag{2.25}$$

Thus, for $K = 2, 4, 8$, and 16 we have the integer critical spacetime dimensions $D_c = 10, 6, 4$, and 3 respectively.

Physically, we are interested only in fractional superstring theories which are tachyon-free. This requires that all tachyonic states be projected out of the physical spectrum, as occurs in physically sensible superstring and heterotic string theories. Hence, when constructing partition functions Z_K for the closed "Type II" K-fractional superstrings, we seek modular-invariant combinations of terms of the form $c^{D_c-2} = c^{16/K}$ in which the massless sector (2.24) is present but the tachyonic sector (2.23) is absent.

For the $K = 2$ (superstring) theory, of course, there exists such a unique tachyon-free modular-invariant solution:

$$Z_2(\tau) = \tau_2^{-4}|A_2|^2. \tag{2.26}$$

Here τ_2 is the imaginary part of τ, and A_2 is the modular-invariant combination

$$A_2 = 8(c_0^0)^7(c_0^2) + 56(c_0^0)^5(c_0^2)^3 + 56(c_0^0)^3(c_0^2)^5 + 8(c_0^0)(c_0^2)^7 - 8(c_1^1)^8. \tag{2.27}$$

This can be translated into a more familiar form by recalling the equivalences (2.22) between the $K = 2$ string functions and the Jacobi ϑ-functions; using these results, we find that A_2 can be rewritten as

$$A_2 = \frac{1}{2}\eta^{-12}(\vartheta_3{}^4 - \vartheta_4{}^4 - \vartheta_2{}^4) = \frac{1}{2}\Delta^{-1/2}J\,, \tag{2.28}$$

where $\Delta \equiv \eta^{24}$ and J is the vanishing Jacobi factor (1.2).

We can construct solutions satisfying our requirements for the $K > 2$ cases as well. For $K = 4$, we find one tachyon-free modular-invariant partition function [2]

$$Z_4(\tau) = \tau_2^{-2}(|A_4|^2 + 12|B_4|^2)\,, \tag{2.29}$$

where A_4 and B_4 are the combinations

$$\begin{aligned} A_4 &= 4(c_0^0 + c_0^4)^3(c_0^2) - 4(c_0^2)^4 - 4(c_2^2)^4 + 32(c_2^2)(c_2^4)^3\,, \\ B_4 &= 8(c_0^0 + c_0^4)(c_0^2)(c_2^4)^2 + 4(c_0^0 + c_0^4)^2(c_2^2)(c_2^4) - 4(c_0^2)^2(c_2^2)^2\,. \end{aligned} \tag{2.30}$$

Unlike A_2, which was by itself modular-invariant, A_4 and B_4 mix amongst themselves under modular transformations; under T we find that A_4 and B_4 transform as eigenfunctions with eigenvalues $+1$ and -1 respectively, whereas under S we find

$$S\colon \begin{pmatrix} A_4 \\ B_4 \end{pmatrix} \to -\frac{1}{\tau^2}\begin{pmatrix} 1/2 & 3 \\ 1/4 & -1/2 \end{pmatrix}\begin{pmatrix} A_4 \\ B_4 \end{pmatrix}. \tag{2.31}$$

Similarly, for the $K = 8$ closed "Type II" fractional superstring, we find the unique tachyon-free modular-invariant partition function [2]

$$Z_8(\tau) = \tau_2^{-1}(|A_8|^2 + |B_8|^2 + 2|C_8|^2)\,, \tag{2.32}$$

where we now have the three combinations

$$\begin{aligned} A_8 &= 2(c_0^0 + c_0^8)(c_0^2 + c_0^6) - 2(c_0^4)^2 - 2(c_4^4)^2 + 8(c_4^6 c_4^8)\,, \\ B_8 &= 4(c_0^0 + c_0^8)(c_4^6) + 4(c_0^2 + c_0^6)(c_4^8) - 4(c_0^4 c_4^4)\,, \\ C_8 &= 4(c_2^2 + c_2^6)(c_2^8 + c_6^8) - 4(c_2^4)^2\,. \end{aligned} \tag{2.33}$$

These combinations are eigenfunctions of T with eigenvalues $+1$, -1, and $-i$ respectively, and under S they mix as follows:

$$S\colon \begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix} \to \frac{i}{\tau}\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \end{pmatrix}\begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix}. \tag{2.34}$$

For the $K = 16$ fractional superstring there also exists a solution:

$$Z_{16} = \tau_2^{-1/2}(|A_{16}|^2 + 4|C_{16}|^2)\,, \tag{2.35}$$

where our combinations are

$$\begin{aligned} A_{16} &= c_0^2 + c_0^{14} - c_0^8 - c_8^8 + 2c_8^{14}\,, \\ C_{16} &= c_4^2 + c_4^{14} - c_4^8\,. \end{aligned} \tag{2.36}$$

These combinations have eigenvalues $+1$ and $-i$ under T, and under S they mix as follows:

$$S: \begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix} \to \frac{1}{\sqrt{-i\tau}} \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix} . \tag{2.37}$$

It turns out that each of the partition functions Z_K in this series has a remarkable property: viewed as a function of $q \equiv e^{2\pi i \tau}$, each vanishes identically. We shall prove this assertion in Sect. 4. In the $K = 2$ (superstring) case, we see from (2.26) and (2.28) that this vanishing is equivalent to the Jacobi identity

$$J \equiv \vartheta_3{}^4 - \vartheta_2{}^4 - \vartheta_4{}^4 = 0 , \tag{2.38}$$

and indeed it is well-known that the particle spectrum of this Type II superstring exhibits a spacetime supersymmetry. Thus, the famous Jacobi identity (2.38) is the reflection (at the partition-function level) of this underlying spacetime supersymmetry. In analogous fashion, we interpret the vanishing of each Z_K for $K > 2$ as a sign of spacetime supersymmetry in the *fractional*-superstring spectrum of states: the contribution to Z_K from each bosonic state at every mass level in the theory is cancelled by the (equal but opposite) contribution from a corresponding fermionic state. Since these $K > 2$ partition functions Z_K have been written as the sums of squares, it follows that the separate string-function combinations A_K, B_K, and C_K must each independently also vanish as functions of q:

$$A_4 = B_4 = A_8 = B_8 = C_8 = A_{16} = C_{16} = 0 . \tag{2.39}$$

These string-function identities (2.39) are therefore the $K > 2$ analogues of the ($K = 2$) Jacobi identity, and we shall prove this series of new Jacobi-like identities in Sect. 4.

We emphasize that in spite of (2.39), one-loop modular invariance continues to require that the individual terms $|A_K|^2$, $|B_K|^2$, and $|C_K|^2$ appear together in our partition functions Z_K in the combinations given in (2.29), (2.32), and (2.35). As in the superstring, one-loop modular invariance not only guarantees multi-loop modular invariance, but is also required for the internal consistency of the theory. In fact, since (2.39) is the reflection of spacetime supersymmetry in the fractional-superstring spectrum, such cancellations must be distinguished from those due to "internal" (GSO-like) projections which act between states with the *same* statistics and which thereby actually *remove* physical states from the theory.

Given that these partition functions Z_K vanish, we now turn to an examination of the individual terms within each Z_K. In particular, since this vanishing is a reflection of spacetime supersymmetry, we focus on determining the spacetime statistics of the particles contributing to each of the terms in Z_K.

Let us first recall the well-understood $K = 2$ case. In the $K = 2$ superstring partition function (where we are restricting ourselves to the left-moving holomorphic sector only), we have seen that the expression A_2 is proportional to $\Delta^{-1/2} J$, and it is well-known that within this factor $\Delta^{-1/2} J$ the terms $\Delta^{-1/2}(\vartheta_3{}^4 - \vartheta_4{}^4)$ represent the contributions from spacetime bosonic states (i.e., from the worldsheet Neveu–Schwarz sector). Likewise, the remaining term $\Delta^{-1/2}\vartheta_2{}^4$ within $\Delta^{-1/2} J$ represents the contributions from spacetime fermionic states (i.e., from the worldsheet Ramond sector). Using the relations (2.22) between the Jacobi ϑ-functions and the string functions, we see that the fermionic Ramond sector contributes only to the term $(c_1^1)^8$ within A_2, while the remaining terms within A_2 receive contributions from only the bosonic Neveu–Schwarz sector. Having thus distinguished these contributions in terms of the $K = 2$ string functions, we can now easily discern which of the worldsheet $\mathbf{Z}_2$ "parafermion" primary fields ϕ_m^j are responsible for bosonic or fermionic statistics: since the character of each field ϕ_m^j is ηc_{2m}^{2j}, we see that the parafermion fields giving rise to spacetime fermionic states are $\phi_{\pm 1/2}^{1/2}$, while those giving rise to spacetime bosonic states are ϕ_m^j with quantum number $m = 0$. Thus, in light-cone gauge, spacetime bosons will have vertex operators proportional to

$$B \sim \prod_{i=1}^{8} (\phi_0^{j_i}) , \tag{2.40}$$

where $j_i = 0$ or 1, and spacetime fermions will have vertex operators proportional to

$$F \sim \prod_{i=1}^{8} (\phi_{m_i}^{1/2}) , \tag{2.41}$$

where $m_i = \pm 1/2$. Note that we are suppressing the contributions to the vertex operators from the worldsheet bosons X^μ. Since the worldsheet bosons give rise only to states with spacetime bosonic statistics, this suppression does not affect the identification of the statistics of vertex operators. Similarly, any of the above parafermion primary fields ϕ_m^j may be replaced by one of its descendant fields without altering the statistics.

It is straightforward to demonstrate that the above vertex operator assignments satisfy a number of self-consistency checks. First, we can recall the correspondence between the $\mathbf{Z}_2$ parafermion fields and the free fermion fields of the Ising model

$$\begin{gathered}\phi_0^0 = \mathbf{1}, \quad \phi_0^1 = \psi , \\ \phi_{1/2}^{1/2} = \sigma, \quad \phi_{-1/2}^{1/2} = \sigma^\dagger ;\end{gathered} \tag{2.42}$$

here ψ is the Majorana fermion field and σ and $\sigma^\dagger$ are the spin fields. These worldsheet spin fields and their descendants create the (Ramond) spacetime fermionic states from the vacuum, and the Majorana and identity fields similarly create the (Neveu–Schwarz) spacetime bosonic states. This is therefore consistent with the above vertex-operator assignments. More compellingly, we can check that the $\mathbf{Z}_2$ parafermion algebra itself reproduces the correct spin-statistics selection rules. Recall that according to the fusion rules (2.4), the m-quantum number of the parafermion field is additive. From this and the field identifications (2.3), it follows that our vertex operators (2.40) and (2.41) satisfy the following selection rules under fusion:

$$\begin{aligned}F \times F &\to B , \\ F \times B &\to F , \\ B \times B &\to B .\end{aligned} \tag{2.43}$$

This is, of course, in accordance with the required spin-statistics connection.

These considerations can easily be generalized to the $K > 2$ cases: here the analogues of (2.40) and (2.41) are

$$B \sim \prod_{i=1}^{D_c - 2} (\phi_0^{j_i}) \quad \text{and} \quad F \sim \prod_{i=1}^{D_c - 2} (\phi_{\pm K/4}^{j_i}) , \tag{2.44}$$

and from the equivalences (2.3) and the fusion rules (2.4) we see that the selection rules (2.43) are again satisfied.

This identification of the spacetime spin and statistics of fractional-superstring states is compatible with the A_K parts of the partition functions Z_K given above, for each term within each A_K can readily be identified as the contribution from either a spacetime bosonic state or a spacetime fermionic state. It is therefore straightforward to decompose each A_K into two pieces (just as was done for the $K = 2$ Jacobi factor J), writing

$$A_K = A_K^b - A_K^f , \tag{2.45}$$

where A_K^b (respectively A_K^f) contains the terms in which all string functions are of the form c_0^l (respectively $c_{K/2}^l$):

$$\begin{aligned}
K = 2:& \quad A_2^b \equiv 8(c_0^0)^7(c_0^2) + 56(c_0^0)^5(c_0^2)^3 + 56(c_0^0)^3(c_0^2)^5 + 8(c_0^0)(c_0^2)^7 ,\\
& \quad A_2^f \equiv 8(c_1^1)^8 ,\\
K = 4:& \quad A_4^b \equiv 4(c_0^0 + c_0^4)^3(c_0^2) - 4(c_0^2)^4 ,\\
& \quad A_4^f \equiv 4(c_2^2)^4 - 32(c_2^4)^3(c_2^2) ,\\
K = 8:& \quad A_8^b \equiv 2(c_0^0 + c_0^8)(c_0^2 + c_0^6) - 2(c_0^4)^2 ,\\
& \quad A_8^f \equiv 2(c_4^4)^2 - 8(c_4^6)(c_4^8) ,\\
K = 16:& \quad A_{16}^b \equiv c_0^2 + c_0^{14} - c_0^8 ,\\
& \quad A_{16}^f \equiv c_8^8 - 2c_8^{14} .
\end{aligned} \tag{2.46}$$

This separation of terms is also consistent with our previous observations. First, the term we identified as containing a massless vector particle in (2.24) is of the form $\prod_i (c_0^{l_i})$, in agreement with the form of the boson vertex operators given in (2.44); note that it appears with a positive sign in A_K^b. Next, we observe that according to the above identification, massless fermions must correspond to the first term in the q-expansion of

$$(c_{K/2}^{K/2})^{16/K} \sim q^0(1 + \ldots) ; \tag{2.47}$$

fortunately this term is present within each of our expressions A_K^f, and indeed it can be shown [2] that the physical-state conditions on the massless state in (2.47) yield the massless Dirac equation. The Fermi statistics carried by these spacetime spinorial states is reflected in the negative sign with which this term appears in each A_K (or the positive sign within each A_K^f), and in fact the coefficient for this term in each case gives precisely the counting of physical states necessary for a Majorana and/or Weyl fermion in the critical spacetime dimension D_c of the fractional superstring. Furthermore, note that this left-moving spin-1/2 massless fermionic state in (2.47) can be tensored with the right-moving spin-1 massless bosonic state in (2.24) (and *vice versa*) to form a massless spin-3/2 gravitino. Consistency would then require that the closed fractional-superstring spectrum exhibit an $N = 2$ spacetime supersymmetry. The vanishing of each A_K, B_K, and C_K (or the equality of their respective bosonic and fermionic parts) is indeed consistent with this conclusion.

One new feature for $K > 2$ is the presence of additional terms within A_K^b and A_K^f which appear with the "wrong" signs (i.e., negative signs within the individual pieces A_K^b and A_K^f). Although these flipped signs might initially seem to contradict our identification of the statistics of states, it turns out that the expressions A_K^f as defined in (2.46) satisfy the following identities:

$$\begin{aligned} K=2:&\quad A_2^f = 8\left(\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}\right)^8 ,\\ K=4:&\quad A_4^f = 4\left(\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}\right)^4 ,\\ K=8:&\quad A_8^f = 2\left(\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}\right)^2 ,\\ K=16:&\quad A_{16}^f = \left(\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}\right) . \end{aligned} \tag{2.48}$$

These new identities will be proven in Sect. 4. Note that since $A_K = A_K^b - A_K^f = 0$, the bosonic terms A_K^b also have the above q-expansions. Since (2.48) implies that the coefficients in the q-expansions of each A_K^b and A_K^f are all positive, we are led to view the minus signs in A_K^b and A_K^f as "internal" projections (or cancellations) of degrees of freedom in the fractional superstrings. Furthermore, the pattern inherent in this series of identities (matching smoothly as it does onto the well-understood $K=2$ case) also suggests that our statistics identification is indeed correct.

From the definitions of η and ϑ_2, Eqs. (2.11) and (2.21) respectively, it follows that

$$\left(\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}\right)^2 = \frac{\vartheta_2}{2\eta^3} = (c_1^1)^2 , \tag{2.49}$$

where c_1^1 is a level $K=2$ string function. Thus the identities (2.48) can be viewed as relating the $K>2$ parafermion characters (string functions) to $K=2$ fermion characters (Jacobi ϑ-functions), and indeed these identities are just some of the identities in our $[K, 2]$ series. The full set is collected in Sect. 3.

One immediate consequence of (2.48) is that the level-by-level counting of physical degrees of freedom in each of the fermionic A_K^f sectors in the fractional superstring is identical to that of the Ramond sector of the usual superstring, except with spacetime dimension $D_c = 2 + 16/K$ instead of 10. At first glance, this observation might seem to imply that the A_K spectrum of the fractional superstring can itself be equivalently described by the worldsheet Majorana-fermion and free-boson theories of the ordinary ($K=2$) superstring, without any need for worldsheet parafermions. Indeed, the spectrum of the spacetime *fermionic* sectors A_K^f of the fractional superstring can be generated by $D_c - 2 = 16/K$ pairs of free worldsheet bosons X^μ and worldsheet Ramond fermions ψ^μ, as (2.48) and (2.49) jointly suggest, and this is of course simply the operator content of the Ramond sector in ordinary superstring theory. However, consider the counting of the spacetime *bosonic* states (Neveu–Schwarz sector) for the superstring

$$K=2:\quad A_2^b \equiv \frac{1}{2}\left[(c_0^0 + c_0^2)^8 - (c_0^0 - c_0^2)^8\right] = 8\left(\frac{\vartheta_3{}^4 - \vartheta_4{}^4}{16\eta^{12}}\right) . \tag{2.50}$$

Here $\sqrt{\vartheta_3/\eta}$ and $\sqrt{\vartheta_4/\eta}$ are the characters of worldsheet Majorana fermions obeying Neveu–Schwarz boundary conditions, and it is for this reason that we have written $\vartheta_3{}^4 - \vartheta_4{}^4$ rather than $\vartheta_2{}^4$ in (2.50). From (2.48), then, we can write the $K > 2$ generalizations of (2.50) as

$$\begin{aligned} K = 4: &\quad A_4^b = 4\left(\frac{\vartheta_3{}^4 - \vartheta_4{}^4}{16\eta^{12}}\right)^{1/2}, \\ K = 8: &\quad A_8^b = 2\left(\frac{\vartheta_3{}^4 - \vartheta_4{}^4}{16\eta^{12}}\right)^{1/4}, \\ K = 16: &\quad A_{16}^b = \left(\frac{\vartheta_3{}^4 - \vartheta_4{}^4}{16\eta^{12}}\right)^{1/8}. \end{aligned} \tag{2.51}$$

The right sides of (2.51) cannot be interpreted as the characters of any tensor products of bosons and Majorana fermions because of the presence of the fractional exponents for $K > 2$. Therefore, an attempted description of the A_K part of the fractional-superstring spectrum in terms of worldsheet bosons and fermions fails, and indeed it is only the introduction of *parafermions* on the worldsheet which allows us to generate the bosonic-sector partition functions given in (2.51).

Thus far we have focused exclusively on the A_K sectors of the fractional superstrings, and we have seen that obtaining the desired fractional exponents in (2.51) (i.e., reducing the critical dimension from $D_c = 10$ to $D_c = 2 + 16/K$) is achieved through the introduction of parafermions on the string worldsheet. There is, however, a price that must be paid. Let us now rewrite the entire expression $A_K = A_K^b - A_K^f$ in terms of Jacobi ϑ-functions: using (2.51) for A_K^b, and using (2.48) and (2.49) jointly for A_K^f, we find

$$A_K \propto \Delta^{-1/K}\{(\vartheta_3{}^4 - \vartheta_4{}^4)^{2/K} - \vartheta_2{}^{8/K}\}, \tag{2.52}$$

where $\Delta \equiv \eta^{24}$. Thus, we see that only for $K = 2$ is A_K by itself modular-invariant; for $K > 2$ we find that A_K does not close into itself under S, but rather requires the introduction of additional sectors (such as those giving rise to B_K and C_K) in order to achieve modular-invariant partition functions. This is the underlying reason why these additional expressions appeared naturally in our partition functions Z_K. Since these new sectors contain only massive states (i.e., states with masses at the Planck scale), we consider their introduction a small price to pay for the ability to decrease the critical spacetime dimension in string theory. Furthermore, we have seen that these additional expressions B_K and C_K also vanish as functions of q, and therefore the introduction of these additional sectors preserves the spacetime supersymmetry at all mass levels of the theory.

These additional sectors, however, seem to contain much of the spacetime physics which is intrinsically new to fractional superstrings. To see this, let us consider the spacetime statistics of the states appearing in these sectors. The fields corresponding to terms in the B_K sector are themselves products of parafermion fields, half of which have quantum number $m = 0$ and half of which have quantum number $m = K/4$. Therefore, according to our previous discussion, their vertex operators will have the following form in light-cone gauge:

$$Q_B \sim \prod_{i=1}^{D_c/2-1} (\phi_0^{j_i}\phi_{\pm K/4}^{j_i'}) . \tag{2.53}$$

According to (2.44), these states are therefore neither fermions nor bosons in D_c spacetime dimensions. A similar situation exists for the C_K sectors. These sectors consist of terms of the form $\prod_i (c^{j_i}_{K/4})$ [where we recall the identities (2.15)], and in light-cone gauge these naturally correspond to vertex operators of the form

$$Q_C \sim \prod_{i=1}^{D_c-2} (\phi^{j_i}_{\pm K/8}) \,. \tag{2.54}$$

Once again, such states cannot be interpreted as bosons or fermions in D_c space-time dimensions.

In order to gain some insight into the properties of these states, we can calculate the fusion rules that these additional vertex operators Q_B and Q_C satisfy. It turns out that these rules depend on the level K of the fractional superstring, since the number of parafermion fields within each vertex operator depends on the critical dimension D_c (and hence on K). For the $K = 4$ superstring, we have only the A_4 and B_4 sectors: the A_4 sector can be decomposed into bosonic and fermionic pieces satisfying the algebra (2.43), as we have seen, and the B_4 sector introduces the additional vertex-operator fusion rules:

$$\begin{aligned} Q_B \times B &\to Q_B \,, \\ Q_B \times F &\to Q_B \,, \\ Q_B \times Q_B &\to B \text{ or } F \text{ or } Q_B \,. \end{aligned} \tag{2.55}$$

These selection rules are therefore suggestive of "spin-quarter" statistics for B_4-sector particles, since the fusing of two identical B_4-sector particles can result in an A_4-sector fermion. Similarly, for the $K = 8$ superstring, we have three sectors: A_8, B_8, and C_8. While the A_8 sector again leads to the fusion rules (2.43), the B_8 sector now introduces the (slightly modified) fusion rules:

$$\begin{aligned} Q_B \times B &\to Q_B \,, \\ Q_B \times F &\to Q_B \,, \\ Q_B \times Q_B &\to B \text{ or } F \,, \end{aligned} \tag{2.56}$$

and the C_8 sector introduces the additional rules:

$$\begin{aligned} Q_C \times B &\to Q_C \,, \\ Q_C \times F &\to Q_C \,, \\ Q_C \times Q_B &\to Q_C \,, \\ Q_C \times Q_C &\to B \text{ or } F \text{ or } Q_B \,. \end{aligned} \tag{2.57}$$

Once again, therefore, the B_8-sector fusion rules suggest "spin-quarter" statistics, and the C_8-sector rules seem to indicate "spin-eighth" statistics. For the $K = 16$ string, on the other hand, there are only an A_{16} and a C_{16} sector. In this case the vertex operators contain only one parafermion field, and while the A_{16} sector again leads to (2.43), the C_{16} sector now yields the fusion rules:

$$\begin{aligned} Q_C \times B &\to Q_C \,, \\ Q_C \times F &\to Q_C \,, \\ Q_C \times Q_C &\to B \text{ or } F \,. \end{aligned} \tag{2.58}$$

Thus in this instance it is the C_{16} sector which seems to suggest "spin-quarter" statistics.

Even though this discussion has focused on only the left-moving sectors of the fractional superstrings, these considerations indicate that it is likely that the B_K and C_K sectors break the usual spin-statistics connection in the critical dimension. (This is not necessarily true of the $K = 16$ string, since fractional statistics are allowed in three spacetime dimensions.) This means that, for the fractional superstrings to be consistent, either Lorentz invariance, quantum mechanics, or locality must be broken in some way in the critical dimension.

An important point following from all of these fusion rules is that the tree-level scattering of particles in the A_K sector can involve only other fields in the A_K sector as intermediate states. Since only the A_K sectors contain the massless states, we see that the B_K and C_K sectors make no contribution to the semi-classical low-energy physics of the fractional superstrings. Therefore, since Lorentz invariance and the spin-statistics connection appear to hold in the A_K sector, the semi-classical low-energy physics predicted by the fractional superstrings will indeed be the familiar Yang–Mills and gravity theories (plus corrections proportional to powers of the string tension, as in the usual superstring). At the string loop level, however, fields in the B_K or C_K sectors can contribute to the scattering amplitudes of the massless particles in the A_K sector. Thus, it is only the quantum effects of the fractional superstring which render the massless (i.e., potentially observable) physics inconsistent with the traditional spin-statistics connection. Since the lowest-lying states in the B_K and C_K sectors have masses at the Planck scale, we expect the quantum corrections to the massless-sector gravity and Yang–Mills theories to be suppressed by factors of the Planck mass relative to the low-energy scale (at least for sufficiently weak string coupling).

These violations of the spin-statistics connection, though potentially weak, may nevertheless be important qualitative signals of stringy behavior. Indeed, there are many possible mechanisms through which these B_K and C_K sectors might lead to violations of the usual spin-statistics relation. First, as mentioned above, the spin-statistics connection can be invalidated by sacrificing locality. Since strings are extended objects, it is possible that the massive states in the B_K and C_K sectors correspond to extended states that cannot be interpreted as elementary particles. Indeed, the possibility of exotic statistics due to the extended nature of strings has been discussed in Ref. [12]. This proposal relies on the non-trivial nature of the motion group in string-configuration space in four spacetime dimensions. Another possibility is that the states in the B_K and C_K sectors are soliton-like objects, extended objects formed from the A_K-sector fields. The slow fall-off of massless scalar or vector fields could allow the violation of the usual statistics selection rules, similar to what occurs in monopole-fermion systems [13].

On the other hand, the spin-statistics theorem can be avoided at the expense of Lorentz invariance. Lorentz invariance is a symmetry known to hold only at distances much larger than the Planck scale. As we have seen, these fractional superstring theories lead to a breaking of Lorentz invariance only at the Planck scale and only at the quantum loop level. Such a breaking may actually exist, and simply not be observable experimentally at the present time.

Alternatively, even if the massive-sector breaking of Lorentz invariance in the critical dimension D_c is strong even at low energies, a lower-dimensional Lorentz invariance might still survive. In this scenario, the spacetime symmetry group of the B_K and C_K sectors would have a D-dimensional Lorentz subgroup with $D < D_c$, providing a mechanism for spontaneously compactifying from the critical dimension D_c down to D dimensions. Let us see explicitly how this might occur [8]. Recall the form of the B_K-sector vertex operators given in (2.53). While we have seen that these states cannot be interpreted as either bosons or fermions in D_c spacetime dimensions, they may well have bosonic or fermionic interpretations in spacetime dimensions $D < D_c$. For example, a bosonic interpretation for Q_B is possible if the parafermion fields with quantum number $m = 0$ within (2.53) are viewed as the (fractional) superpartners of the $D < D_c$ worldsheet coordinate bosons, with the remaining $m \neq 0$ parafermion fields viewed as part of an "internal" worldsheet theory resulting from spacetime compactification. In fact, it is shown in Ref. [8] that a theory fully consistent with Lorentz invariance and the spin-statistics connection is possible for the $K = 4$ and $K = 8$ cases if the spacetime dimensions are compactified respectively to $D = 4$ and $D = 3$. Furthermore, such a compactification scheme in the $K = 4$ case may also permit the construction of four-dimensional fractional-superstring models containing chiral spacetime fermions [8].

It is therefore evident that there is much potentially new physics to be discovered within fractional superstring theory, whether a possible Planck-scale breakdown of Lorentz invariance, the appearance of exotic statistics due to the extended (non-local) nature of strings, or a "self-compactification" required for internal consistency. While the interpretations of many of these effects have yet to be resolved, the important point is that fractional superstrings offer a unique and concrete framework in which to explore these issues.

3. List of New String-Function Identities

In this section we gather together all of the new string-function identities to be proven in Sect. 4. As discussed in the Introduction, we refer to an identity as a $[K_1, K_2]$ identity if it relates string functions c_n^l of level K_1 to string functions of level K_2. Recall that the level $K = 1$ string function is equivalent to the Dedekind η-function, and that the level $K = 2$ string functions are equivalent to the Jacobi ϑ_i-functions [these relations are given in (2.20) and (2.22)]. If we define the level $K = 0$ function $c_0^0 \equiv 0$, then our new identities come in three distinct series: these are the $[K, 0]$, $[K, 1]$, and $[K, 2]$ series, for $K = 2, 4, 8$, and 16. These identities are listed below.

3.1. First Series: The $[K, 0]$ *Identities.* This series of identities generalizes the famous Jacobi "supersymmetry" identity on fermion characters:

$$J = \vartheta_3{}^4 - \vartheta_2{}^4 - \vartheta_4{}^4 = 0 \,. \tag{3.1}$$

In terms of $K = 2$ string functions, (3.1) is equivalent to

$$\begin{aligned} A_2 &\equiv 8(c_0^0)^7(c_0^2) + 56(c_0^0)^5(c_0^2)^3 + 56(c_0^0)^3(c_0^2)^5 + 8(c_0^0)(c_0^2)^7 - 8(c_1^1)^8 \\ &= \frac{1}{2}\Delta^{-1/2}J = 0 \,, \end{aligned} \tag{3.2}$$

where $\Delta \equiv \eta^{24}$. The Jacobi identity (3.1) can therefore be regarded as the $[2, 0]$ special case, and the analogous $[K, 0]$ Jacobi-like "supersymmetry" identities for the parafermion characters at higher levels K are as follows. For $K = 4$, we define

$$A_4 \equiv 4(c_0^0 + c_0^4)^3(c_0^2) - 4(c_0^2)^4 - 4(c_2^2)^4 + 32(c_2^2)(c_2^4)^3 ,$$
$$B_4 \equiv 8(c_0^0 + c_0^4)(c_0^2)(c_2^4)^2 + 4(c_0^0 + c_0^4)^2(c_2^2)(c_2^4) - 4(c_0^2)^2(c_2^2)^2 , \tag{3.3}$$

and for $K = 8$, we define

$$A_8 \equiv 2(c_0^0 + c_0^8)(c_0^2 + c_0^6) - 2(c_0^4)^2 - 2(c_4^4)^2 + 8(c_4^6 c_4^8) ,$$
$$B_8 \equiv 4(c_0^0 + c_0^8)(c_4^6) + 4(c_0^2 + c_0^6)(c_4^8) - 4(c_0^4 c_4^4) ,$$
$$C_8 \equiv 4(c_2^2 + c_2^6)(c_2^8 + c_6^8) - 4(c_2^4)^2 . \tag{3.4}$$

Similarly, for $K = 16$, we define

$$A_{16} \equiv c_0^2 + c_0^{14} - c_0^8 - c_8^8 + 2c_8^{14} ,$$
$$C_{16} \equiv c_4^2 + c_4^{14} - c_4^8 . \tag{3.5}$$

Then each of these expressions also vanishes as a function of q:

$$A_2 = A_4 = B_4 = A_8 = B_8 = C_8 = A_{16} = C_{16} = 0 . \tag{3.6}$$

These new "Jacobi-like" identities therefore form the $[K, 0]$ series. Note that while the Jacobi identity is the *unique* such identity for $K = 2$, for each level $K > 2$ there are in fact *several* independent Jacobi-like $[K, 0]$ identities.

3.2. Second Series: The $[K, 2]$ *Identities.* This series of identities relates the $\mathbf{Z}_K$ parafermion characters to the ordinary fermion characters, and are the higher-K analogues of the $K = 2$ "identities" $\Delta^{-1/2}\vartheta_i{}^4 = \Delta^{-1/2}\vartheta_i{}^4$ for $i = 2, 3, 4$.

For $K = 4$, we define the following quantities:

$$A_4^b \equiv 4(d_0^0)^3 c_0^2 - 4(c_0^2)^4 ,$$
$$A_4^f \equiv 4(c_2^2)^4 - 32(c_2^4)^3 c_2^2 ,$$
$$C_4 \equiv 4c_0^2(c_2^2)^3 - 12d_0^0(c_2^4)^2 c_2^2 - 8c_0^2(c_2^4)^3 = 4q^{1/4}(1 + \ldots) ,$$
$$D_4 \equiv (d_0^0)^3 c_2^2 + 6(d_0^0)^2 c_0^2 c_2^4 - 4(c_0^2)^3 c_2^2 = q^{-1/4}(1 + \ldots) , \tag{3.7}$$

where $d_n^l \equiv c_n^l + c_n^{K-l}$. Note that $A_4^b - A_4^f = A_4$. Then our $[4, 2]$ identities, which contain the $[4, 0]$ Jacobi-like identities $A_4 = B_4 = 0$ as a subset, are as follows:

$$A_4^b = \vartheta_2{}^2/\eta^6 ,$$
$$A_4^f = \vartheta_2{}^2/\eta^6 ,$$
$$B_4 = 0 ,$$
$$C_4 = \frac{1}{2}(\vartheta_3{}^2 - \vartheta_4{}^2)/\eta^6 ,$$
$$D_4 = \frac{1}{2}(\vartheta_3{}^2 + \vartheta_4{}^2)/\eta^6 . \tag{3.8}$$

As a consequence of (3.8), we have

$$(A_4^b + A_4^f)^2 - 16C_4D_4 = 4\Delta^{-1/2}J = 0\,. \tag{3.9}$$

Similarly, for $K = 8$, we define

$$\begin{aligned}
A_8^b &\equiv 2d_0^0d_0^2 - \tfrac{1}{2}(d_0^4)^2\,,\\
A_8^f &\equiv \frac{1}{2}(d_4^4)^2 - 2d_4^6d_4^8\,,\\
E_8 &\equiv \frac{1}{2}d_2^4d_4^4 - d_2^2d_4^8 - d_4^6d_2^8 = 2q^{3/8}(1 + \dots)\,,\\
F_8 &\equiv d_0^0d_2^2 + d_0^2d_2^8 - \frac{1}{2}d_0^4d_2^4 = q^{-1/8}(1 + \dots)\,.
\end{aligned} \tag{3.10}$$

Once again $A_8^b - A_8^f = A_8$. Then our [8, 2] identities, which contain the [8, 0] Jacobi-like identities $A_8 = B_8 = C_8 = 0$ as a subset, are as follows:

$$\begin{aligned}
A_8^b &= \vartheta_2/\eta^3\,,\\
A_8^f &= \vartheta_2/\eta^3\,,\\
B_8 &= C_8 = 0\,,\\
E_8 &= \frac{1}{2}(\vartheta_3 - \vartheta_4)/\eta^3\,,\\
F_8 &= \frac{1}{2}(\vartheta_3 + \vartheta_4)/\eta^3\,.
\end{aligned} \tag{3.11}$$

As a consequence of (3.11), we have

$$(E_8 + F_8)^4 - (E_8 - F_8)^4 - 16(A_8^b + A_8^f)^8 = \Delta^{-1/2}J = 0\,. \tag{3.12}$$

Similarly, for $K = 16$, we define the five quantities:

$$\begin{aligned}
A_{16}^b &\equiv d_0^2 - \frac{1}{2}d_0^8\,,\\
A_{16}^f &\equiv \frac{1}{2}d_8^8 - d_8^{14}\,,\\
C_{16} &\equiv d_4^{14} - \frac{1}{2}d_4^8\,,\\
E_{16} &\equiv \frac{1}{2}d_6^8 - d_6^{14} = q^{7/16}(1 + \dots)\,,\\
F_{16} &\equiv d_2^2 - \frac{1}{2}d_2^8 = q^{-1/16}(1 + \dots)\,;
\end{aligned} \tag{3.13}$$

here too $A^b_{16} - A^f_{16} = A_{16}$. Then our [16, 2] identities, which likewise contain the [16, 0] Jacobi-like identities $A_{16} = C_{16} = 0$ as a subset, are as follows:

$$\begin{aligned} A^b_{16} &= \sqrt{\vartheta_2/(2\eta^3)} = c^1_1 \,, \\ A^f_{16} &= \sqrt{\vartheta_2/(2\eta^3)} = c^1_1 \,, \\ C_{16} &= 0 \,, \\ E_{16} &= \frac{1}{2}(\sqrt{\vartheta_3/\eta^3} - \sqrt{\vartheta_4/\eta^3}) = c^2_0 \,, \\ F_{16} &= \frac{1}{2}(\sqrt{\vartheta_3/\eta^3} + \sqrt{\vartheta_4/\eta^3}) = c^0_0 \,, \end{aligned} \tag{3.14}$$

where *on the right sides of these equations the string functions are at level* $K = 2$. As a consequence of (3.14), we have

$$(E_{16} + F_{16})^8 - (E_{16} - F_{16})^8 - \frac{1}{16}(A^b_{16} + A^f_{16})^8 = \Delta^{-1/2} J = 0 \,. \tag{3.15}$$

The identities (3.14) are in fact quite remarkable, for they are linear relations indicating that the differences of certain $K = 16$ string functions are nothing but the $K = 2$ string functions. The full consequences of these relations between the $\mathbf{Z}_2$ fermionic characters and the $\mathbf{Z}_{16}$ parafermionic characters are yet to be explored.

3.3. Third Series: The $[K, 1]$ *Identities.* This new series of identities generalizes the ϑ-function identity

$$\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3 \tag{3.16}$$

to levels $K > 2$, thereby relating the *parafermion* characters c^l_n to the boson character η.

This series of identities actually starts at $K = 1$, where we have the relation (2.20):

$$\eta c^0_0 = 1 \,. \tag{3.17}$$

For $K = 2$, as mentioned, we have the identity

$$\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3 \,, \tag{3.18}$$

which can be written in terms of the $K = 2$ string functions as

$$\eta^3 Q_2 = 1 \,, \tag{3.19}$$

where

$$Q_2 \equiv c^1_1[(c^0_0)^2 - (c^2_0)^2] = \eta^{-3} \sqrt{\frac{\vartheta_2 \vartheta_3 \vartheta_4}{2\eta^3}} \,. \tag{3.20}$$

[Thus (3.19) is in fact equivalent to the *square root* of (3.18), and thereby contains the extra information about the sign of the square root.] Equations (3.17) and (3.19) are the first two identities in our $[K, 1]$ series, each of the form $\eta^p \sum (c)^p = 1$ for some power p. The corresponding $[K, 1]$ identities for $K > 2$ are as follows. For $K = 4$, we have

$$\eta^2 Q_4 = 1 \,, \tag{3.21}$$

where

$$Q_4 \equiv (c_0^0 + c_0^4)c_2^2 - 2c_0^2 c_2^4 \,, \tag{3.22}$$

and for $K = 8$, we have

$$\eta^3 Q_8 = 1 \,, \tag{3.23}$$

where

$$\begin{aligned} Q_8 \equiv\ & c_4^4(c_0^2 + c_0^6)[(c_0^0 + c_0^8) - (c_2^0 + c_2^8)] \\ & + 2c_2^4[c_4^8(c_0^2 + c_0^6) - c_4^6(c_0^0 + c_0^8)] \\ & - 2c_0^4[c_4^8(c_2^2 + c_2^6) - c_4^6(c_2^0 + c_2^8)] \,. \end{aligned} \tag{3.24}$$

Note that this expression for Q_8 can be rewritten in a variety of forms due to the [8, 0] Jacobi-like identities $A_8 = B_8 = C_8 = 0$. Similarly, for $K = 16$, we define the string-function combinations $d_n^l \equiv c_n^l + c_n^{K-l}$. Then our corresponding [16, 1] identity is

$$\eta^3 Q_{16} = 1 \,, \tag{3.25}$$

where $2Q_{16}$ is the following quantity:

$$\begin{aligned} & -2d_0^0 d_2^4 d_2^{16} - 3d_0^0 d_2^6 d_2^6 + 3d_0^0 d_6^6 d_6^6 + 2d_0^0 d_6^{12} d_6^{16} - d_0^2 d_2^2 d_2^2 \\ & - 4d_0^2 d_2^2 d_2^8 - d_0^2 d_2^8 d_2^8 + d_0^2 d_6^8 d_6^8 + 4d_0^2 d_6^8 d_6^{14} + d_0^2 d_6^{14} d_6^{14} \\ & + 2d_0^4 d_2^4 d_2^6 - 3d_0^4 d_2^{16} d_2^{16} - 2d_0^4 d_6^6 d_6^{12} + 3d_0^4 d_6^{16} d_6^{16} + 3d_0^6 d_2^4 d_2^4 \\ & - 2d_0^6 d_2^6 d_2^{16} + 2d_0^6 d_6^6 d_6^{16} - 3d_0^6 d_6^{12} d_6^{12} - d_0^8 d_2^2 d_2^2 + d_0^8 d_6^{14} d_6^{14} \,. \end{aligned} \tag{3.26}$$

This expression for Q_{16} can also be rewritten in many different forms by using the [16, 0] Jacobi-like identities $A_{16} = C_{16} = 0$. Thus, defining $Q_1 \equiv c_0^0 = \eta^{-1}$ for $K = 1$, we have

$$Q_1{}^3 = Q_2 = Q_1 Q_4 = Q_8 = Q_{16} \,. \tag{3.27}$$

3.4. Comments on Other Identities. As indicated in the Introduction, these $[K, 0]$, $[K, 1]$, and $[K, 2]$ identities are undoubtedly only some of the general $[K_1, K_2]$ identities which exist. Of course, implicit in the above identities are $[K_1, K_2]$ relations which do *not* involve the $K = 1$ or $K = 2$ string functions; for example, we have the identities (3.9), (3.12), (3.15), and (3.27), as well as additional identities such as

$$\begin{aligned} \pm (C_4 \pm D_4) &= (E_8 \pm F_8)^2 = (E_{16} \pm F_{16})^4 \,, \\ \frac{1}{4}(A_4^b + A_4^f)^2 &= (C_8 + D_8)^4 + (E_{16} - F_{16})^8 \,, \\ (Q_{16})^6 &= \frac{1}{2}(A_8^b + A_8^f)(F_8{}^2 - E_8{}^2)(Q_4)^6 \,. \end{aligned} \tag{3.28}$$

However, these identities are not independent of those in the three preceding series.

We also point out that there can exist *several* distinct series of a given $[K_1, K_2]$ type. For example, although the above three series are all of the form

$$\sum (c_{K=K_1})^p = \sum (c_{K=K_2})^p \,, \tag{3.29}$$

where p is an arbitrary power, there also exist more general "mixed" identities of the form

$$\sum \{(c_{K=K_1})^p (c_{K=K_2})^q\} = \sum \{(c_{K=K_1})^p (c_{K=K_2})^q\} \tag{3.30}$$

which cannot be rewritten in the form (3.29) and which are indeed independent of all such identities. For example, consider the following three mixed identities involving the $K = 2$ and $K = 4$ string functions (where we have distinguished the two sets of string functions by writing the $K = 2$ string functions in terms of the Jacobi ϑ-functions):

$$\begin{aligned} A_4^b({\vartheta_3}^2 + {\vartheta_4}^2) &= 2D_4 {\vartheta_2}^2 , \\ [(d_2^2)^4 + 4d_2^2(d_2^4)^3]({\vartheta_3}^2 + {\vartheta_4}^2) &= [(d_0^2)^3 d_2^2 + 4(d_0^0)^3 d_2^2]\, {\vartheta_2}^2 , \\ [4(d_0^0)^2 d_2^2 d_2^4 - 2(d_0^2)^2(d_2^2)^2]({\vartheta_3}^2 - {\vartheta_4}^2) &= [-4(d_0^0)^2 d_0^2 d_2^4 - (d_0^2)^3 d_2^2]\, {\vartheta_2}^2 . \end{aligned} \tag{3.31}$$

While the first of these identities follows directly from (3.8), the remaining two are in fact *new* identities independent of any presented thus far.

It thus appears that there are many unexpected identities involving string functions at different levels K, implying a rich set of relations between the characters of different $\mathbf{Z}_K$ parafermions (and suggesting various as-yet-undiscovered relationships between the different $\mathbf{Z}_K$ parafermion theories themselves). In Sect. 4 we will prove the identities in our three series, and discuss how others, such as those in (3.31), might be obtained.

4. Proofs of the Identities

In this section we prove the series of identities listed in Sect. 3. We will find that the proofs of the new $K > 2$ cases exactly mirror the traditional proofs for the known $K = 2$ special cases, thus demonstrating that each series of identities shares the same underlying mathematical basis. Our proofs make use of some fundamental and powerful results from the theory of modular functions, suitably generalized so as to be appropriate for the $K > 2$ cases. In the first part of this section, therefore, we provide a review of these results from modular function theory, ultimately quoting a theorem upon which our proofs rest. The second part of this section then contains the proofs of our identities. We have kept our discussion sufficiently general throughout in the hope that methods of obtaining and proving additional $[K_1, K_2]$ identities will become self-evident.

4.1. Results from Modular Function Theory. We first provide a review of those aspects of modular function theory which will be relevant for the proofs of our identities. For more details, we refer the reader to any of the standard modular function theory references [14–16]; in particular, our approach is based upon those of Refs. [14] and [15].

The homogeneous modular group is $\tilde{\Gamma} \equiv SL(2, \mathbf{Z})$, the group formed by the set of 2×2 matrices with integer entries and unit determinant under matrix multiplication. The inhomogeneous modular group Γ (the so-called "modular group") is the quotient group $\Gamma \equiv PSL(2, \mathbf{Z}) \equiv \tilde{\Gamma}/\{\pm \mathbf{1}\}$, the subgroup of $\tilde{\Gamma}$ in which every matrix

$A \in \tilde{\Gamma}$ is identified with the matrix $-A$. We shall need to consider various subgroups of Γ. Three series of subgroups of Γ may be defined as follows:

$$\Gamma(N):\ \left\{ A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, a \overset{N}{=} d \overset{N}{=} 1,\, b \overset{N}{=} c \overset{N}{=} 0 \right\},$$

$$\Gamma_1(N):\ \left\{ A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, a \overset{N}{=} d \overset{N}{=} 1,\, c \overset{N}{=} 0 \right\},$$

$$\Gamma_0(N):\ \left\{ A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, c \overset{N}{=} 0 \right\}, \tag{4.1}$$

where $\overset{N}{=}$ signifies equality modulo N. We thus see that the elements of $\Gamma(N)$, $\Gamma_1(N)$, and $\Gamma_0(N)$ are respectively of the forms $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ modulo N (where the asterisk indicates the absence of any defining relation), and therefore $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma$ for $N > 1$. For $N = 1$ these groups each equal Γ. Such groups are called *congruence subgroups* of Γ, with $\Gamma(N)$ called the *principal* congruence subgroups; in general a congruence subgroup $\Gamma' \subset \Gamma$ is said to be of *level* N if $\Gamma(N) \subset \Gamma'$. (This level N bears no relation to the Kač–Moody level K.) Only the principal congruence subgroups $\Gamma(N)$ are normal subgroups of Γ, and in fact this series of subgroups will concern us the most. Note that $\Gamma(N)$ is isomorphic to $\Gamma/SL(2, \mathbf{Z}_N)$, where $\mathbf{Z}_N$ is the set of integers modulo N. In particular, $SL(2, \mathbf{Z}_1) \approx \{1\}$ since all elements of $SL(2, \mathbf{Z})$ are, modulo 1, isomorphic to the identity.

Each of the congruence subgroups Γ' in (4.1) has finite index in Γ:

$$[\Gamma : \Gamma'] \equiv \dim \Gamma/\Gamma' < \infty\ , \tag{4.2}$$

and straightforward number-theoretic arguments [14, 15] yield the values:

$$[\Gamma : \Gamma(N)] = \varepsilon_N N^3 \prod_{p|N} (1 - p^{-2})$$

$$[\Gamma : \Gamma_1(N)] = \varepsilon_N N^2 \prod_{p|N} (1 - p^{-2})$$

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}) \tag{4.3}$$

where the products are taken over all primes $p > 1$ dividing N and where

$$\varepsilon_N \equiv \begin{cases} 1 & \text{for } N = 1, 2 \\ 1/2 & \text{for } N > 2\ . \end{cases} \tag{4.4}$$

These factors of ε_N in (4.3) reflect the fact that $\mathbf{1} \overset{N}{=} -\mathbf{1}$ for $N = 1$, 2, but not for $N > 2$. From (4.3), therefore, we find

$$[\Gamma : \Gamma(N)] = \begin{cases} 1 & \text{for } N = 1 \\ 6 & \text{for } N = 2 \\ 24 & \text{for } N = 4\ . \end{cases} \tag{4.5}$$

For any congruence subgroups Γ' with finite index in Γ, we can identify two generators $\gamma_i \in \Gamma'$ $(i = 1, 2)$ and a set of coset representatives $\alpha_j \in \Gamma$ $(j = 1, \ldots, [\Gamma : \Gamma'])$. The γ_i are generators in the sense that every element of Γ' can be written as a "word" in γ_1 and γ_2. The set of coset representatives (also called a *transversal*) contains one element from each coset of Γ/Γ'; therefore only one of these coset representatives is in Γ' itself, and we are free to choose this representative to be $\alpha_1 = \mathbf{1}$. At level $N = 1$, we have only the full modular group Γ: its two generators are

$$S \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{4.6}$$

and its sole "coset" representative is of course $\alpha_1 = \mathbf{1}$. At higher levels there are more possibilities. At levels $N = 2$ and $N = 4$, for example, we find the following generators and transversals for the principal congruence subgroups:

$$\begin{aligned} \Gamma(2):\quad & \text{generators:} && T^2, ST^{-2}S \\ & \text{transversal:} && \{\mathbf{1}, S, T, ST, TS, T^{-1}ST\} \\ \Gamma(4):\quad & \text{generators:} && T^4, ST^4S \\ & \text{transversal:} && \{X, T^2X, ST^2SX, T^2ST^2SX\} \\ & && \text{where } X \equiv \{\mathbf{1}, S, T, ST, TS, TST\}\ , \end{aligned} \tag{4.7}$$

whereas for the Γ_0 subgroups we find:[1]

$$\begin{aligned} \Gamma_0(2):\quad & \text{generators:} && T, ST^2S \\ & \text{transversal:} && \{\mathbf{1}, S, ST\} \\ \Gamma_0(4):\quad & \text{generators:} && T, ST^{-4}S \\ & \text{transversal:} && \{\mathbf{1}, S, ST, ST^2, ST^3, ST^2S\}\ . \end{aligned} \tag{4.8}$$

Note that $\Gamma_0(N)$ and $\Gamma_1(N)$ contain T for all N; therefore one of the generators of these groups is always T. This is not the case for the principal subgroups $\Gamma(N)$, which contain T only for $N = 1$. As we shall see, we will be primarily interested in those subgroups for which T is *not* a generator.

The modular group Γ is isomorphic to the set of linear-fractional transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbf{Z}\ , \tag{4.9}$$

of $\tau \in H$ (where H is the complex upper half-plane); indeed, we can identify the transformation (4.9) with the group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. A fundamental domain $\mathscr{F} \equiv \mathscr{F}[\Gamma]$ for Γ, therefore, is a set of points $\tau \in H$ such that no two are related by a transformation of the form (4.9). It is conventional to choose this domain to be

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[1] These are actually *right* transversals (representatives of *right* cosets). For the $\Gamma(N)$ subgroups the right and left transversals coincide because $\Gamma(N)$ is a normal subgroup of Γ for every N

contiguous and symmetric about the τ_2 axis (where τ_1 and τ_2 are respectively the real and imaginary parts of τ):

$$\mathscr{F} \equiv \mathscr{F}[\Gamma] \equiv \left\{ \tau \in \mathbf{C} \,|\, \tau_2 > 0, |\tau_1| \leqq \frac{1}{2}, |\tau| \geqq 1 \right\} . \tag{4.10}$$

A fundamental domain $\mathscr{F}[\Gamma']$ corresponding to any *subgroup* $\Gamma' \subset \Gamma$ must therefore be *larger* than $\mathscr{F}[\Gamma]$, and is in fact the totality of points obtained by acting upon each point in $\mathscr{F}[\Gamma]$ with each of the coset representatives (including $\mathbf{1}$) of Γ':

$$\mathscr{F}[\Gamma'] \equiv \bigcup_{j=1}^{[\Gamma:\Gamma']} \alpha_j \mathscr{F}[\Gamma] . \tag{4.11}$$

One typically chooses the transversal $\{\alpha_j\}$ in such a way that with the choice (4.10), the domain $\mathscr{F}[\Gamma']$ in (4.11) is contiguous.

There are certain points in the complex upper half-plane H which are called *cusp points*: these are the point $\tau_\infty \equiv (0, \infty) \equiv i\infty$, along with the set $\mathbf{Q}$ (i.e., the points with $\tau_2 = 0$ and rational values of τ_1). We shall need to distinguish those cusp points which are Γ'-inequivalent for a given congruence subgroup $\Gamma' \subset \Gamma$ (i.e., those cusp points not related by a transformation in Γ'). At level $N = 1$ the only congruence subgroup is Γ itself, and indeed it is a simple matter to see that all of the cusp points are in the same Γ-equivalence class. The set of Γ-inequivalent cusp points therefore contains only one element, and the typical choice τ_∞ is consistent with the choice of fundamental domain (4.10). Given this choice, it follows that at any higher level $N > 1$, all Γ'-inequivalent cusps for $\Gamma' \subset \Gamma$ must be in the set $\{\alpha_i \tau_\infty\}$ where $\{\alpha_i, i = 1, \ldots, [\Gamma:\Gamma']\}$ is the transversal of Γ' in Γ. Such points $\alpha_i \tau_\infty$ for $i > 1$ are all $\in \mathbf{Q}$. However, even the points in this restricted set are not necessarily Γ'-inequivalent: in general two such points $\alpha_i \tau_\infty$ and $\alpha_j \tau_\infty$ are Γ'-inequivalent if and only if there does not exist an integer n such that $\alpha_j T^n \alpha_i^{-1} \in \Gamma'$. It turns out, for example, that for $\Gamma' = \Gamma_0(p)$ with p a prime number there are only *two* Γ'-inequivalent cusp points (0 and τ_∞), while for $\Gamma' = \Gamma_0(p^2)$ there are $p + 1$ such cusps points: 0, τ_∞, and $-1/(rp)$ for $r = 1, \ldots, p - 1$. A listing of the Γ'-inequivalent cusp points for the congruence subgroups at levels $N = 2$ and $N = 4$ which will be relevant to our later discussion is as follows:

$$\begin{aligned} \Gamma_0(2)\text{:} \quad & \text{cusps:} \quad 0, \tau_\infty , \\ \Gamma_0(4)\text{:} \quad & \text{cusps:} \quad 0, \tau_\infty, -1/2 , \\ \Gamma(2)\text{:} \quad & \text{cusps:} \quad 0, \tau_\infty, -1 , \\ \Gamma(4)\text{:} \quad & \text{cusps:} \quad 0, \tau_\infty, -1, -1/2, 1, 3/2, 2 . \end{aligned} \tag{4.12}$$

Such points are called cusp points of Γ' because in each case fundamental domains $\mathscr{F}[\Gamma']$ can be chosen whose shapes are cusp-like at each of these points (and "cusp"-like at τ_∞).

We can now rigorously define modular functions, modular forms, and cusp forms with respect to these general groups $\Gamma' \subseteq \Gamma$. First we consider the full modular group Γ (i.e., level $N = 1$). A *modular function* of weight $k \in 2\mathbf{Z}$ with respect to Γ is defined to be a function $f(\tau)$ satisfying two conditions. First, it must have an expansion in powers of $q \equiv \exp\{2\pi i\tau\}$ of the form

$$f(\tau) = \sum_{n\in\mathbf{Z}} a_n q^n , \tag{4.13}$$

where there exists an $m \in \mathbf{Z}$ such that $a_n = 0$ for all $n < m$ (i.e., there can be at most finitely many non-zero values of a_n with $n < 0$). This condition therefore ensures that $f(\tau)$ is meromorphic at $\tau = \tau_\infty$ (i.e., $q = 0$). Second, $f(\tau)$ must satisfy

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) \tag{4.14}$$

for all $\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in H$. It is convenient to define the *stroke operator* $[\alpha]$ for any $\alpha \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$: this operator $[\alpha]$ transforms a modular function f of weight k to $f[\alpha]$, where

$$(f[\alpha])(\tau) \equiv (c\tau + d)^{-k} f(\alpha\tau) . \tag{4.15}$$

With this notation, then, (4.14) becomes the requirement that

$$f[\gamma] = f \quad \text{for all } \gamma \in \Gamma , \tag{4.16}$$

and it is clear that any function f invariant under the two Γ-generators $[S]$ and $[T]$ therefore satisfies (4.16). Note that the set of modular functions of a given weight k forms a complex vector space, and that the product of two modular functions of weights k_1 and k_2 respectively is a modular function of weight $k_1 + k_2$.

Any such modular function f which additionally satisfies $a_n = 0$ for all $n < 0$ in (4.13) is said to be a modular *form* with respect to Γ. Modular forms are therefore holomorphic (rather than merely meromorphic) at $\tau = \tau_\infty$, remaining finite at $q = 0$. If in fact $a_0 = 0$ as well, so that the modular form f actually vanishes at $q = 0$, then f is called a *cusp form*. The complex vector spaces of modular forms and cusp forms of weight k with respect to Γ are respectively denoted $M_k[\Gamma]$ and $S_k[\Gamma]$.

It is straightforward to generalize these definitions to congruence subgroups at higher levels $N > 1$. For any such subgroup Γ', a modular function f of weight $k \in 2\mathbf{Z}$ with respect to Γ' must again satisfy two conditions. The natural generalization of (4.16) is the requirement that

$$f[\gamma] = f \quad \text{for all } \gamma \in \Gamma' , \tag{4.17}$$

and once again it is sufficient to demonstrate that $f[\gamma_i] = f$ for the two generators γ_i of Γ' in order to demonstrate (4.17). The generalization of (4.13), on the other hand, is a bit more subtle. Equation (4.13) was the requirement that f be meromorphic at $q = 0$ (i.e., at $\tau = \tau_\infty$, the cusp point of Γ). For subgroups Γ' at higher levels $N > 1$, we therefore analogously require that f be meromorphic at *each* of the Γ'-inequivalent cusp points of Γ'. This meromorphicity is not determined by evaluating $f(\tau)$ as τ approaches each of these cusp points, however; in fact, since each of these additional cusp points is $\in \mathbf{Q}$, we have $|q| = 1$ at these points and therefore a straightforward q-expansion of f does not converge. Instead, meromorphicity at the cusps is defined as follows. We have seen that this set of Γ'-inequivalent cusp points can be written as $\{\alpha_i\tau_\infty\}$, where these α_i are among (but not necessarily all of) the coset representatives of Γ' in Γ. Let s denote the set of these α_i (hence s is a subset of the transversal). The analogue of (4.13) is therefore the requirement that for each $\alpha_i \in s$, we can perform a q-expansion

$$f[\alpha_i] = \sum_{n \in \mathbf{Z}/N} a_n q^n , \tag{4.18}$$

where, as before, there exists an $m \in \mathbf{Z}/N$ such that $a_n = 0$ for all $n < m$. Note that n and m can now take values in the larger set $\mathbf{Z}/N$ (rather than $\mathbf{Z}$ itself). Also note that for $i > 1$ we have $\alpha_i \notin \Gamma'$, so (4.18) is in general quite stringent.

It is clear that at level $N = 1$, (4.18) reduces to (4.13), for in this case $s = \{\alpha_1\} = \{\mathbf{1}\}$. For $N > 1$, the $i = 1$ case of (4.18) implies meromorphicity at τ_∞, and the $i > 1$ cases imply meromorphicity at each of the remaining Γ'-inequivalent cusps $\in \mathbf{Q}$. Note that (4.18) is in fact sufficient to imply that f is actually meromorphic at *all* of the cusp points of Γ'.

Once again, if indeed $a_n = 0$ for all $n < 0$ and each $\alpha_i \in s$ (so that f is holomorphic at each Γ'-cusp, remaining finite), then f is deemed a modular *form* of weight k with respect to Γ'. If additionally $a_0 = 0$ for each $\alpha_i \in s$ (so that f vanishes at each cusp), then f is a *cusp form* with respect to Γ'. The complex vector spaces of such weight-k modular forms and cusp forms are denoted $M_k[\Gamma']$ and $S_k[\Gamma']$ respectively; note that $M_k[\Gamma] \subseteq M_k[\Gamma']$ for all $\Gamma' \subseteq \Gamma$, as well as the property $M_{k_1}[\Gamma'] M_{k_2}[\Gamma'] \subseteq M_{k_1+k_2}[\Gamma']$.

We are now in a position to state the fundamental theorem [15] upon which the proofs of our identities rest.

Theorem. *Let $\Gamma' \subseteq \Gamma$ be any level-$N \geq 1$ congruence subgroup of the modular group Γ, and let $M_k[\Gamma']$ denote the space of modular forms of weight $k \in 2\mathbf{Z}$ with respect to Γ'. Then the sizes of such spaces depend on k as follows*:

- $\dim M_k[\Gamma'] = 0$ *for all* $k < 0$.
- $\dim M_k[\Gamma'] = 1$ *for* $k = 0$.
- *For $k > 0$ a general formula exists as well. Although we will not require these $k > 0$ results for the proofs of our specific identities, we include the following two special cases which are likely to be useful in proving additional $[K_1, K_2]$ identities*:
  - *For level $N = 1$ (i.e., $\Gamma' = \Gamma$), we have*

$$\dim M_k[\Gamma] = \begin{cases} [k/12] & \text{for } k \stackrel{12}{=} 2 \\ [k/12] + 1 & \text{otherwise} \end{cases} , \tag{4.19}$$

  where $[x]$ is the greatest integer $\leq x$, and $\stackrel{12}{=}$ signifies equality modulo 12.
  - *For the principal congruence subgroups $\Gamma(N)$ at levels $N > 1$, we have instead*:

$$\dim M_k[\Gamma(N)] = \frac{(k-1)N + 6}{12N} [\Gamma : \Gamma(N)] , \tag{4.20}$$

  where the index $[\Gamma : \Gamma(N)]$ is given in (4.3).

Thus, for example, in our cases of interest this theorem tells us that for $\Gamma' = \Gamma$ we have

$$\dim M_k[\Gamma] = \begin{cases} 0 & \text{for } k < 0 \text{ or } k = 2 \\ 1 & \text{for } k = 0,\ 14 \text{ and } 4 \leq k \leq 10 , \end{cases} \tag{4.21}$$

whereas for $\Gamma' = \Gamma(N)$ we have

$$\dim M_k[\Gamma(N)] = \begin{cases} 0 & \text{for all } N > 1,\ k < 0 \\ 1 & \text{for all } N > 1,\ k = 0 \\ \frac{1}{2}k + 1 & \text{for } N = 2, k > 0 \\ 2k + 1 & \text{for } N = 4, k > 0\ . \end{cases} \tag{4.22}$$

These dimensions are important, for they tell us the number of "basis" modular forms in terms of which any modular form of weight k can be expressed as a polynomial. For example, since $\dim M_0[\Gamma'] = 1$ for all congruence subgroups $\Gamma' \subseteq \Gamma$, and since $f = 1$ is a valid Γ'-modular form of weight $k = 0$, *all* Γ'-modular forms of weight $k = 0$ must be constants:

$$M_0[\Gamma'] = \mathbf{C}\ , \tag{4.23}$$

where $\mathbf{C}$ is the space of complex numbers. Similarly, since $\dim M_k[\Gamma'] = 0$ for all $k < 0$, all Γ'-modular forms of negative weight must vanish identically:

$$f \in M_k[\Gamma'] \Rightarrow f = 0 \quad \text{for all } k < 0\ . \tag{4.24}$$

Likewise, for $k = 4$, 6, 8, 10, or 14, we see that all $f \in M_k[\Gamma]$ must be multiples of a single function E_k; these "basis functions" $E_k(\tau)$ form the so-called *Eisenstein series* [14–16].

Finally, we remark that equally powerful results can be obtained for the spaces of cusp forms $S_k[\Gamma']$, as well as for the cases when k is odd (and in fact half-integral). However, the above results will be sufficient for the proofs of our series of identities.

4.2. Proofs of the Identities. Given the theorem presented in the last subsection, it is relatively straightforward to prove each of the series of identities listed in Sect. 3. The basic idea of each proof is the same: we demonstrate that our string-function expressions are modular forms of a given weight with respect to an appropriate congruence subgroup $\Gamma' \subseteq \Gamma$, whereupon the theorem allows us to conclude the claimed identity. The primary subtleties involve properly formulating the identities and identifying the relevant congruence subgroups.

Since all of our identities involve combinations of the Dedekind η-function, the Jacobi ϑ_i-functions, and the string functions c^l_n, let us first recall how these functions transform under the modular group. Under T, their respective transformation rules take the forms:

$$\begin{aligned} \eta(\tau + 1) &= \alpha\eta(\tau)\ , \\ \vartheta_i(\tau + 1) &= \sum_j \alpha_{ij}\,\vartheta_j(\tau)\ , \\ c^l_n(\tau + 1) &= \alpha' c^l_n(\tau) \end{aligned} \tag{4.25}$$

(where α, α', and α_{ij} indicate various phases and mixing matrices), while under S we have

$$\begin{aligned} \eta(-1/\tau) &= \sqrt{\tau}\,\beta\eta(\tau)\ , \\ \vartheta_i(-1/\tau) &= \sqrt{\tau}\sum_j \beta_{ij}\,\vartheta_j(\tau)\ , \\ c^l_n(-1/\tau) &= (\sqrt{\tau})^{-1}\sum_{l',n'} \beta^{ll'}_{nn'}\,c^{l'}_{n'}(\tau) \end{aligned} \tag{4.26}$$

(where again the β's represent various phases and mixing matrices). Thus, we see that even though these functions themselves are not invariant under S and T, they each transform *covariantly* under the modular group, filling out (in the case of ϑ_i and c_n^l) representations of the modular group with dimensions greater than 1. Furthermore, we see from (4.25) and (4.26) that the η-function and the ϑ_i-functions transform with positive modular weight $k = 1/2$, while the string functions c_n^l transform with *negative* modular weight $k = -1/2$.

Given these observations, it is straightforward to determine the modular weights of each of our identities listed in Sect. 3. For the first series of identities, we see that each expression A_K, B_K, or C_K contains $16/K$ string-function factors; indeed, each of these identities takes the general form

$$\sum (c)^{16/K} = 0\,. \tag{4.27}$$

Thus, for any Kač–Moody level K, the level-K identity in the first series has modular weight $k = -8/K$. Similarly, the second series of identities involves combinations of all three of our functions (η, ϑ, and c); however, for each level K, the level-K identity always takes the general form

$$\sum (c)^{16/K} + \sum \left(\frac{\vartheta}{\eta^3}\right)^{8/K} = 0\,. \tag{4.28}$$

from which it follows once again that the level-K identity in this series has modular weight $k = -8/K$. A similar situation exists for the third series as well: each of these identities takes the general form

$$\eta^p \sum (c)^p = 1 \tag{4.29}$$

for some integer power p, and therefore for any K these identities have modular weight $k = 0$.

Hence it is clear that all of these identities can be made to follow from the theorem given in the last subsection, provided each can be rewritten in such a manner that their left sides are modular *forms* of appropriate weight k with respect to a congruence subgroup $\Gamma' \subseteq \Gamma$.

First Series–the $[K, 0]$ *Identities.* We begin by proving the $K = 2$ identity $A_2 = 0$, as given in Sect. 3. This is of course the famous Jacobi ϑ-function identity, and our subsequent proofs will be generalizations of this proof. Recall that A_2 can be rewritten in terms of ϑ and η functions as $\frac{1}{2}\Delta^{-1/2}J$, where $\Delta = \eta^{24}$ and $J \equiv \vartheta_3{}^4 - \vartheta_2{}^4 - \vartheta_4{}^4$. Thus it is clear that A_2 has modular weight $k = -8/K = -4$, and since $A_2[S] = A_2[T] = A_2$, we see that A_2 is a modular function with respect to the full modular group Γ. It is also straightforward to check that A_2 is *tachyon-free*: by this we mean that in a q-expansion

$$A_2 = \sum_n a_n q^n \tag{4.30}$$

we find $a^n = 0$ for all $n < 0$. It then follows that $A_2 \in M_{-4}[\Gamma]$, whereupon the theorem in the last subsection gives the result $A_2 = 0$. Note that since $A_2 = \frac{1}{2}\Delta^{-1/2}J$, and since it is easy to check that $\Delta^{-1/2} \neq 0$, this result implies the Jacobi identity $J = 0$. Note that it would have been more difficult to prove this latter identity directly, for even though the dimension of $M_2[\Gamma]$ is zero, J itself is *not* invariant under S and T (indeed, one finds $J[S] = J[T] = -J$). The extra factor of $\Delta^{-1/2}$, which appears naturally in the definition of A_2, absorbs this unwanted minus sign and leads to the simple proof $A_2 = 0$.

Let us now proceed to the $K = 2$ case: we wish to prove $A_4 = B_4 = 0$. Unlike the $K = 2$ case, A_4 and B_4 are not each invariant under the stroke operators $[S]$ and $[T]$; rather, they together fill out a *two*-dimensional representation of Γ:

$$\begin{pmatrix} A_4 \\ B_4 \end{pmatrix}[S] = e^{i\pi}\begin{pmatrix} 1/2 & 3 \\ 1/4 & -1/2 \end{pmatrix}\begin{pmatrix} A_4 \\ B_4 \end{pmatrix},$$
$$\begin{pmatrix} A_4 \\ B_4 \end{pmatrix}[T] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A_4 \\ B_4 \end{pmatrix}, \qquad (4.31)$$

and we wish to prove that A_4 and B_4 *individually* vanish. This is the reason it is necessary to consider the congruence subgroups $\Gamma' \subset \Gamma$.

Let us first prove $A_4 = 0$. It is clear that A_4 is not modular-invariant under $[S]$, so the full modular group Γ cannot be the relevant group in this case. Instead, let us consider $\Gamma_0(2)$. From the mixing matrices (4.31) we have $A_4[T] = A_4$, as well as

$$\begin{aligned} A_4[ST^2S] &= e^{i\pi}\left(\frac{1}{2}A_4 + 3B_4\right)[T^2S] \\ &= e^{i\pi}\left(\frac{1}{2}A_4 + 3B_4\right)[S] \\ &= (e^{i\pi})^2 A_4 = A_4\,. \end{aligned} \qquad (4.32)$$

The second equality follows from the fact that A_4 and B_4 are both invariant under T^2; note also that in general $f[\alpha\beta] = (f[\alpha])\,[\beta]$. Thus, the first condition (4.17) for A_4 to be a $\Gamma_0(2)$ modular form is satisfied: A_4 is invariant under all $\gamma \in \Gamma_0(2)$. We now must show that the second condition (4.18) is satisfied as well. We see from (4.12) that for $\Gamma' = \Gamma_0(2)$ there are only two Γ'-independent cusp points, τ_∞ and 0, and therefore the transversal subset s is only $\{\mathbf{1}, S\}$. It is clear that $A_4[\mathbf{1}] = A_4$ has a q-expansion of the proper form (4.18) (with $m = 0$), and similarly we see that $A_4[S] = e^{i\pi}(\frac{1}{2}A_4 + 3B_4)$ also has a q-expansion of the proper form (with $m = 0$, since *both* A_4 and B_4 are tachyon-free). It therefore follows that $A_4 \in M_{-2}[\Gamma_0(2)]$, whereupon we obtain the identity $A_4 = 0$.

Note that there are indeed other ways we might have obtained this result. For instance, let us consider the congruence subgroup $\Gamma' = \Gamma_0(4)$. It is straightforward to check that A_4 is invariant under both generators $[T^4]$ and $[ST^{-4}S]$, and for this congruence subgroup the cusp points are τ_∞, 0, and $-1/2$. We have already shown that A_4 has the proper behavior at the first two of these cusp points; let us therefore focus on the third. Note that $-1/2 = \alpha\tau_\infty$, where $\alpha \equiv \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} \in \Gamma$. We therefore must examine $A_4[\alpha]$. Since it turns out that $\alpha = ST^2ST^{-1}$, we have

$$\begin{aligned} A_4[ST^2ST^{-1}] &= e^{i\pi}\left(\frac{1}{2}A_4 + 3B_4\right)[T^2ST^{-1}] \\ &= e^{i\pi}\left(\frac{1}{2}A_4 + 3B_4\right)[ST^{-1}] \\ &= (e^{i\pi})^2 A_4[T^{-1}] = A_4\,. \end{aligned} \qquad (4.33)$$

Therefore A_4 has the same behavior at the cusp point $-1/2$ as it does at τ_∞, and since A_4 is tachyon-free (i.e., since A_4 remains finite at τ_∞), we find

$A_4 \in M_{-2}[\Gamma_0(4)]$. This again implies the conclusion $A_4 = 0$. In fact, one can similarly demonstrate that $A_4 \in M_{-2}[\Gamma(2)]$ and $A_4 \in M_{-2}[\Gamma(4)]$, each of which leads as well to this result.

Having proven $A_4 = 0$, we find that there are two ways to prove $B_4 = 0$. The first is indirect but simpler: since

$$A_4[S] = e^{i\pi}\left(\frac{1}{2}A_4 + 3B_4\right) = 0 , \tag{4.34}$$

we must have $B_4 = 0$. A more direct method (not relying on the identity $A_4 = 0$) is to construct an independent proof along the above lines. Note that since B_4 is not invariant under $[T]$, we cannot consider any subgroups for which T is a generator; we are therefore restricted to consideration of the *principal* congruence subgroups $\Gamma(N)$. It is straightforward to demonstrate that $B_4 \in M_{-2}[\Gamma(N)]$, where N is either 2 or 4, and therefore $B_4 = 0$. In fact, B_4 is a *cusp form* with respect to these groups $\Gamma(N)$, since B_4 *a priori* has a q-expansion of the form $q^h(1 + \ldots)$ where $h > 0$.

Let us now collect together the essential ingredients in these proofs, in order to frame a general argument. First, the set of string-function expressions $\{A_K, B_K, \ldots\}$ must be closed under $[S]$ and $[T]$, forming a multi-dimensional representation R_K of the modular group Γ; furthermore, a congruence subgroup $\Gamma' \subseteq \Gamma$ must be identified such that each member of the representation R_K is itself *invariant* under the generators of Γ' (i.e., each member must separately comprise a *one*-dimensional representation of Γ'). Second, this entire representation R_K must transform under the modular group with *negative* even modular weight k. Third, *all* members of R_K must have q-expansions with finitely many non-zero coefficients a_n with $n < 0$. This third condition is needed in order to insure that each element $\in R_K$ is meromorphic at *all* of the cusp points of Γ', for any one member of R_K will be meromorphic at *all* the cusp points of Γ' if and only if *all* the members of R_K are meromorphic at the *one* cusp point τ_∞ (because the set s always contains at least $\mathbf{1}$ and S). These three conditions then guarantee that each string-function expression in the representation R_K is itself a modular function with respect to Γ'. If each member of the representation is also tachyon-free, then each is a modular *form* with respect to Γ' and hence must vanish identically.

Let us now see how to apply this general argument to the $K = 8$ case as we attempt to prove $A_8 = B_8 = C_8 = 0$. The mixing matrices of these expressions under the stroke operators $[S]$ and $[T]$ are as follows:

$$\begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix}[S] = e^{i\pi/2}\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \end{pmatrix}\begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix},$$

$$\begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix}[T] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -i \end{pmatrix}\begin{pmatrix} A_8 \\ B_8 \\ C_8 \end{pmatrix}. \tag{4.35}$$

There are immediately two problems. First, we see that this $K = 8$ representation has an *odd* modular weight $k = -8/K = -1$; our theorem applies only to the cases

$k \in 2\mathbf{Z}$. Second, no member of this representation is invariant under *any* of the congruence subgroups: for example, under $\Gamma_0(4)$ we find $A_8[T] = A_8$ but

$$\begin{aligned} A_8[ST^{-4}S] &= e^{i\pi/2}\left(\frac{1}{2}A_8 + \frac{1}{2}B_8 + C_8\right)[T^{-4}S] \\ &= e^{i\pi/2}\left(\frac{1}{2}A_8 + \frac{1}{2}B_8 + C_8\right)[S] \\ &= (e^{i\pi/2})^2 A_8 = -A_8 \,. \end{aligned} \tag{4.36}$$

There are two ways to solve these difficulties. One possibility is to extend the theorem presented in Sect. 4.1 to apply to odd k and modular functions with so-called *multiplier systems* (i.e., phases such as the unwanted sign appearing above). Such extensions can indeed be made; in this relatively simple case, for example, we can instead choose to prove the modified identities $A_8/\eta^6 = B_8/\eta^6 = C_8/\eta^6 = 0$. These modified identities would then have modular weight $k = -2$, and the extra η-functions absorb the unwanted sign. A simpler approach, however, (and one which generalizes more easily to other situations) is to prove instead the identities $(A_8)^2 = (B_8)^2 = (C_8)^2 = 0$, for such identities also have an even modular weight $k = -2$ and simultaneously avoid such unwanted signs. (We are essentially enlarging our $K = 8$ representation: $R_8 \to R'_8 \equiv R_8 \otimes R_8$.) If these quadratic identities can be proven for all τ, then of course the linear results $A_8 = B_8 = C_8 = 0$ immediately follow. To prove these quadratic identities, we follow the procedure outlined above: either choice $\Gamma' = \Gamma_0(4)$ or $\Gamma(4)$ suffices for proving $(A_8)^2 = 0$, and $\Gamma(4)$ suffices for independently proving $(B_8)^2 = 0$ and $(C_8)^2 = 0$. It is of course possible to deduce $B_8 = C_8 = 0$ from the result $A_8 = 0$ as we did for the $K = 4$ case: in the present case the analogue of (4.34) is

$$A_8[S] = e^{i\pi/2}\left(\frac{1}{2}A_8 + \frac{1}{2}B + C_8\right) = 0 \,, \tag{4.37}$$

and this implies the weaker result $B_8 + 2C_8 = 0$. However, $B_8 \sim q^{1/2}(1 + \ldots)$ and $C_8 \sim q^{3/4}(1 + \ldots)$, where inside the parentheses all q-exponents are integral. Therefore, B_8 and C_8 must each vanish separately. Note that in this $K = 8$ case we are compelled to consider congruence subgroups of levels $N \in 4\mathbf{Z}$ only. This occurs because the $K = 8$ representation includes a sector C_8 with quarter-integer powers of q. Since in general the generators of $\Gamma(N)$ are of the form T^N and $ST^{\pm N}S$, the choice $N = 4$ is the smallest level N for which each element in the $K = 8$ representation R_8 is $\Gamma(N)$-invariant.

The same procedure applies for the $K = 16$ case as well; here we wish to prove $A_{16} = C_{16} = 0$, and the relevant mixing matrices are as follows:

$$\begin{aligned} \begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix}[S] &= e^{i\pi/4}\frac{1}{2\sqrt{2}}\begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix} \\ \begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix}[T] &= \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} A_{16} \\ C_{16} \end{pmatrix} \,. \end{aligned} \tag{4.38}$$

Once again we find that we must enlarge our representation

$$R_{16} \to R'_{16} \equiv R_{16} \otimes R_{16} \otimes R_{16} \otimes R_{16} \tag{4.39}$$

and prove instead the identities $(A_{16})^4 = (C_{16})^4 = 0$; similarly, we must choose the level $N = 4$ due to the presence of the C_{16} sector in R_{16}. As usual, the subgroup $\Gamma(4)$ suffices in general for proving that each member of R'_{16} vanishes, and we can instead make the choice $\Gamma_0(4)$ in the case of the A_{16} sector (which is invariant under T). Once again the proof that any one member of R_{16} vanishes is sufficient to prove that all vanish, provided each has a different eigenvalue under T. Note that instead of enlarging the representation as in (4.39), it would also have been possible in this case to divide our original representation by η^3; however, this would have necessitated constructing a proof using congruence subgroups of level $N = 8$.

Second Series – the $[K, 2]$ *Identities.* The second series of identities is closely related to the first and in fact contains the first series as a subset. Recall that for each Kač–Moody level $K \in \{2, 4, 8, 16\}$, there exists a string-function expression $A_K \sim q^0(1 + \ldots)$ which, according to the first series of identities, vanishes identically. In this second series of identities we show that the separate bosonic and fermionic pieces of A_K (denoted A_K^b and A_K^f respectively) can each be written in terms of Jacobi ϑ-functions; these ϑ-function expressions for A_K^b and A_K^f are of course equal, since $A_K = A_K^b - A_K^f = 0$. As a by-product, we also obtain additional string-function expressions $\{C_K, D_K, \ldots\}$ which can be easily expressed in terms of Jacobi ϑ-functions as well.

Let us first consider the $K = 4$ identities in this series, as given in (3.8): here we have separated A_4 into its separate bosonic and fermionic contributions A_4^b and A_4^f as in (3.7). While we know that A_4 and B_4 together fill out a two-dimensional representation of the modular group with weight $k = -2$, we find that the individual pieces A_4^b and A_4^f do not close separately into only themselves and B_4. Rather, in order to construct a representation of Γ containing A_4^b and A_4^f as separate members, we must introduce the two additional string-function expressions C_4 and D_4 defined in (3.7). Together, the set $\{A_4^b, A_4^f, B_4, C_4, D_4\}$ indeed fills out a complete representation R_c with weight $k = -2$; each member of R_c transforms as an eigenfunction under $[T]$:

$$\begin{pmatrix} A_4^b \\ A_4^f \\ B_4 \\ C_4 \\ D_4 \end{pmatrix} [T] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} A_4^b \\ A_4^f \\ B_4 \\ C_4 \\ D_4 \end{pmatrix}, \tag{4.40}$$

and under $[S]$ they close into each other:

$$\begin{pmatrix} A_4^b \\ A_4^f \\ B_4 \\ C_4 \\ D_4 \end{pmatrix} [T] = \frac{e^{i\pi}}{4} \begin{pmatrix} 1 & -1 & 6 & -4 & 4 \\ -1 & 1 & -6 & -4 & 4 \\ 1 & -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} A_4^b \\ A_4^f \\ B_4 \\ C_4 \\ D_4 \end{pmatrix}. \tag{4.41}$$

As a check, note that these matrices indeed contain the (A_4, B_4) mixing matrices (4.31) as the appropriate submatrices.

Unlike the previous representations we have considered, not all members of this representation R_c individually vanish, for while all are modular *functions* with respect to an appropriate $\Gamma' \subset \Gamma$, not all are modular *forms*. Since the quantity D_4 is not tachyon-free, it clearly has the wrong behavior at the cusp point $\tau = \tau_\infty$; furthermore, since the $[S]$-transforms of A_4^b, A_4^f, C_4, and D_4 each separately involve D_4, each of these quantities has incorrect (i.e., tachyonic) behavior at the Γ' cusp point $\tau = 0$. Indeed, only B_4 and the difference $A_4 = A_4^b - A_4^f$ are free of this tachyonic behavior at both cusp points $\tau = \tau_\infty$ and $\tau = 0$, so of the five quantities in the above representation R_c only B_4 is itself a proper Γ'-modular form. Therefore, in order to construct identities for these five individual quantities, it is necessary to build a *new* representation involving them in such a manner that *each* member is a tachyon-free modular form.

It turns out that this is not hard to do. As we have seen in (4.25) and (4.26), the ϑ- and η-functions fill out valid representations of the modular group, and indeed the three quantities $\{\eta^{-6}\vartheta_2{}^2, \eta^{-6}\vartheta_3{}^2, \eta^{-6}\vartheta_4{}^2\}$ fill out such a representation with modular weight $k = -2$. Let us take various linear combinations of these quantities, promoting them to the "five"-dimensional representation

$$R_\vartheta \equiv \left\{ \frac{\vartheta_2{}^2}{\eta^6}, \frac{\vartheta_2{}^2}{\eta^6}, 0, \frac{1}{2}\left(\frac{\vartheta_3{}^2 - \vartheta_4{}^2}{\eta^6}\right), \frac{1}{2}\left(\frac{\vartheta_3{}^2 + \vartheta_4{}^2}{\eta^6}\right) \right\} . \tag{4.42}$$

Written this way, this five-dimensional representation R_ϑ with weight $k = -2$ has two very important properties. First, its mixing matrices under $[S]$ and $[T]$ are (or can be chosen to be) the same as those in (4.40) and (4.41) for R_c; indeed, in this respect the two representations R_c and R_ϑ transform identically. More importantly, however, it is easy to verify that the tachyonic terms within the fifth member of the ϑ-function representation R_ϑ are the *same* as those of the fifth member D_4 of the string-function representation R_c; additionally, all of the other members of R_ϑ are themselves tachyon-free. Therefore, subtracting the two representations, i.e.,

$$R \equiv R_c - R_\vartheta = \begin{pmatrix} A_4^b - \vartheta_2{}^2/\eta^6 \\ A_4^f - \vartheta_2{}^2/\eta^6 \\ B_4 \\ C_4 - \frac{1}{2}(\vartheta_3{}^2 - \vartheta_4{}^2)/\eta^6 \\ D_4 - \frac{1}{2}(\vartheta_3{}^2 + \vartheta_4{}^2)/\eta^6 \end{pmatrix} , \tag{4.43}$$

yields a five-dimensional representation R in which *each* member is tachyon-free. It is then a straightforward matter to demonstrate that each member of this new representation R is indeed a modular form with respect to a congruence subgroup $\Gamma' \subset \Gamma$ of level $N \in 4\mathbf{Z}$ (e.g., each member is $\in M_{-2}[\Gamma(4)]$), whereupon it follows that each vanishes identically. This, then, establishes the identities (3.8).

Note that the existence of such identities relies on the existence of a ϑ-function representation R_ϑ with the desired modular weight k, the desired transformation matrices $[S]$ and $[T]$, and the required tachyonic behaviors of its members. Such a representation does not always exist. It is indeed fortuitous, however, that such representations do exist for each value of K, yielding all of the identities in this second series.

The derivations of the other identities in this series proceed in analogous fashion. For the $K = 8$ case, we find that A_8^b and A_8^f are members of the six-dimensional representation $R_c \equiv \{A_8^b, A_8^f, B_8, C_8, E_8, F_8\}$ [where these six quantities are defined in (3.4) and (3.10)]; under $[S]$ and $[T]$ quantities mix as follows:

$$\begin{pmatrix} A_8^b \\ A_8^f \\ B_8 \\ C_8 \\ E_8 \\ F_8 \end{pmatrix} [S] = \frac{e^{i\pi/2}}{4} \begin{pmatrix} 1 & -1 & 1 & 2 & -4 & 4 \\ -1 & 1 & -1 & -2 & -4 & 4 \\ 2 & -2 & 2 & -4 & 0 & 0 \\ 2 & -2 & -2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} A_8^b \\ A_8^f \\ B_8 \\ C_8 \\ E_8 \\ F_8 \end{pmatrix} \tag{4.44}$$

and

$$\begin{pmatrix} A_8^b \\ A_8^f \\ B_8 \\ C_8 \\ E_8 \\ F_8 \end{pmatrix} [T] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{3\pi i/4} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\pi i/4} \end{pmatrix} \begin{pmatrix} A_8^b \\ A_8^f \\ B_8 \\ C_8 \\ E_8 \\ F_8 \end{pmatrix} . \tag{4.45}$$

These matrices of course contain the (A_8, B_8, C_8) mixing matrices (4.35) as the appropriate submatrices, and these six quantities are again modular *functions* rather than modular *forms* due to the presence of the tachyonic sixth quantity F_8. To compensate for this, we introduce the corresponding six-dimensional R_ϑ representation with weight $k = -1$,

$$R_\vartheta \equiv \left\{ \frac{\vartheta_2}{\eta^3}, \frac{\vartheta_2}{\eta^3}, 0, 0, \frac{1}{2}\left(\frac{\vartheta_3 - \vartheta_4}{\eta^3}\right), \frac{1}{2}\left(\frac{\vartheta_3 + \vartheta_4}{\eta^3}\right) \right\}, \tag{4.46}$$

and construct the tachyon-free representation $R' \equiv (R_c - R_\vartheta) \otimes (R_c - R_\vartheta)$. It can be proven that each member of R' is a modular form of even weight $k = -2$ with respect to a congruence subgroups at level $N \in 8\mathbf{Z}$, whereupon the identities (3.11) immediately follow.

The $K = 16$ case is similar. Here the expressions A_{16}^b and A_{16}^f are members of the five-dimensional representation $R_c \equiv \{A_{16}^b, A_{16}^f, C_{16}, E_{16}, F_{16}\}$ [where these five quantities are defined in (3.13)]; under $[S]$ and $[T]$ these quantities mix as follows:

$$\begin{pmatrix} A_{16}^b \\ A_{16}^f \\ C_{16} \\ E_{16} \\ F_{16} \end{pmatrix} [S] = \frac{e^{i\pi/4}}{2\sqrt{2}} \begin{pmatrix} 1 & -1 & 2 & -2 & 2 \\ -1 & 1 & -2 & -2 & 2 \\ 1 & -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & \sqrt{2} & \sqrt{2} \\ 1 & 1 & 0 & \sqrt{2} & \sqrt{2} \end{pmatrix} \begin{pmatrix} A_{16}^b \\ A_{16}^f \\ C_{16} \\ E_{16} \\ F_{16} \end{pmatrix} \tag{4.47}$$

and

$$\begin{pmatrix} A^b_{16} \\ A^f_{16} \\ C_{16} \\ E_{16} \\ F_{16} \end{pmatrix} [T] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & e^{7\pi i/8} & 0 \\ 0 & 0 & 0 & 0 & e^{-\pi i/8} \end{pmatrix} \begin{pmatrix} A^b_{16} \\ A^f_{16} \\ C_{16} \\ E_{16} \\ F_{16} \end{pmatrix} . \tag{4.48}$$

These matrices of course contain the (A_{16}, C_{16}) mixing matrices (4.38) as the appropriate submatrices. Once again we find that only the fifth quantity F_{16} is tachyonic, and again there exists an appropriate compensating five-dimensional R_ϑ representation:

$$\begin{aligned} R_\vartheta &\equiv \left\{ \sqrt{\frac{\vartheta_2}{2\eta^3}}, \sqrt{\frac{\vartheta_2}{2\eta^3}}, 0, \frac{1}{2}\left(\sqrt{\frac{\vartheta_3}{\eta^3}} - \sqrt{\frac{\vartheta_4}{\eta^3}} \right), \frac{1}{2}\left(\sqrt{\frac{\vartheta_3}{\eta^3}} + \sqrt{\frac{\vartheta_4}{\eta^3}} \right) \right\} \\ &= \{c^1_1, c^1_1, 0, c^2_0, c^0_0\} , \end{aligned} \tag{4.49}$$

where in the second line the string functions are at level $K = 2$. This representation R_ϑ transforms under $[S]$ and $[T]$ with the same mixing matrices as R_c; in particular under $[S]$ the $K = 2$ string functions satisfy

$$\begin{pmatrix} c^0_0 \\ c^2_0 \\ c^1_1 \end{pmatrix} [S] = \frac{e^{i\pi/4}}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} c^0_0 \\ c^2_0 \\ c^1_1 \end{pmatrix} . \tag{4.50}$$

Since only c^0_0 is tachyonic (and in fact has the same tachyonic terms as F_{16}), the entire representation $R \equiv R_c - R_\vartheta$ is tachyon-free. The identities (3.14) then follow by building the tensor-product representation $R' \equiv R \otimes R \otimes R \otimes R$ and considering congruence subgroups with levels $N \in 16\mathbf{Z}$.

Third Series – the $[K, 1]$ *Identities.* This series of identities is actually the simplest to prove. Recall that the Dedekind η-function satisfies

$$\eta[S] = e^{-i\pi/4}\eta, \quad \eta[T] = e^{i\pi/12}\eta , \tag{4.51}$$

and that each identity in this series is of the form

$$\eta^p Q_K = 1 , \tag{4.52}$$

where p is a given power and Q_K is a sum of terms each containing p factors of level-K string functions. Since in each case Q_K satisfies

$$Q_K[S] = e^{ip\pi/4} Q_K, \quad Q_K[T] = e^{-ip\pi/12} Q_K , \tag{4.53}$$

and since in each case the product $\eta^p Q_K$ is tachyon-free, it follows that

$$\eta^p Q_K \in M_0[\Gamma] = \mathbf{C} . \tag{4.54}$$

Overall normalizations have been chosen in each case such that this constant is always 1. Note that this proof yields the familiar ϑ-function identity $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$

in the $K = 2$ special case, and just as easily yields its higher $K > 2$ string-function generalizations. Thus, we see once again how all of our series of identities provide the natural generalizations of their known $K = 2$ special cases.

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Note added in proof. We recently became aware of Ref. [17], and would like to briefly relate some of our fractional-superstring results in Sect. 2.2 to theirs. The authors of Ref. [17] attempted to contruct, from the bosonic string, string theories in lower dimensions by arbitrarily introducing non-integer modings for some of the 24 transverse bosonic fields X^{μ}, thereby effectively preventing the interpretation of the affected worldsheet fields as spacetime coordinates. They found that spacetime Lorentz anomalies could be avoided only in the cases for which the resulting spacetime dimension was equal to 26, 10, 6, 4, or 3; moreover, they observed that the partition functions of their theories were essentially equal to those of our A-sectors in Eq. (2.48), except with the parameter q replaced by the rescaled parameter $q^{1/2}$ (implying a rescaling of the energies in the theory). They found, however, that this superstring-like sector could not be the only sector of their theory, and remarked that additional sectors of their theory having different spacetime dimensions appeared to be necessary. Our approach, starting from an underlying *parafermionic* worldsheet theory, is undoubtedly quite different, yet we see in Eq. (2.48) that the fractional-superstring "internal projections" appear to reduce the A-sectors of our strings to superstring-like sectors resembling theirs (but without any rescaling of energies). Presumably, then, our new B- and C-sectors play the role of the additional sectors anticipated in Ref. [17], yet also have possible interpretations in the *same* number of spacetime dimensions as the A-sectors. Such interpretations for the B- and C-sectors are discussed in more detail in Refs. [18] and [19].

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