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Topological Representations of the Quantum Group $U_q(sl_2)$

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Abstract. We define a topological action of the quantum group $U_q(sl_2)$ on a space of homology cycles with twisted coefficients on the configuration space of the punctured disc. This action commutes with the monodromy action of the braid groupoid, which is given by the R-matrix of $U_q(sl_2)$.

0. Introduction

In the free field representation of conformal field theory based on $SU(2)$ one is led to consider integrals of the form [1, 2]

$$\begin{aligned} G_C(w_1, \dots, w_s) = \int_C & f(z_1, \dots, z_r, w_1, \dots, w_s) \\ & \times \prod_{i<j} (z_i - z_j)^{2\nu} \prod_{i,j} (z_i - w_j)^{(1-n_j)\nu} \prod_{i<j} (w_i - w_j)^{\frac{1}{2}(1-n_i)(1-n_j)\nu} \\ & \times dz_1 \wedge \dots \wedge dz_r . \end{aligned} \tag{0.1}$$

In this formula $n_1, \dots, n_s$ are positive integers, f is a single valued meromorphic function, symmetric under permutations of the z-variables, with poles on the hyperplanes $\{z_i = w_j\}$. The parameter ν is equal to $1/k+2$ for the WZW model on $SU(2)$ at level k and is equal to p'/p for minimal models with central charge $1-6(p-p')^2/pp'$.

For each integration cycle C in the r^{th} homology group with coefficients in the local system given by the monodromy of the differential form in (0.1), G_C is a many valued analytic function on the space $\mathscr{C}_{1,\dots,1}(\mathbf{C}) = \{(w_1, \dots, w_s) \in \mathbf{C}^s \mid w_i \neq w_j \ (i \neq j)\}$. To compute its transformation under analytic continuation along paths exchanging the punctures w_i, one needs to know the monodromy action of the braid

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groupoid on homology. Examples of this computation (by "contour deformation") have been worked out by several authors (among others [3, 1, 4–6]) in different languages. It generalizes the computation of Gauss for the hypergeometric function. It has become clear that the monodromy is described by the R-matrix (more precisely by the $6j$-symbols) of the quantum group $U_q(sl_2)$. The topological point of view we adopt here is closest to [6].

In this paper, we propose an "explanation" of this fact. It consists of two parts. First one considers a space of relative cycles on which $U_q(sl_2)$ acts. The action is described purely in topological terms and commutes with the monodromy action of the braid group. The absolute cycles are then given by the highest weight vectors in the space of relative cycles. We have then schematically the following dictionary between topological and algebraic entities:

Relative cycles	Elements of the tensor product of Verma modules $\bigotimes_i V_{n_i}$
Absolute cycles	Highest weight vectors in $\bigotimes_i V_{n_i}$
Intersection pairing	Covariant bilinear form
Monodromy action of the braid groupoid on relative cycles	R-matrix representation of the braid groupoid on $\bigotimes_i V_{n_i}$

Moreover, the quotient of the space of absolute cycles by the cycles in the null space of the intersection pairing is closed under braiding and is given by the fusion rule subquotient. More precise definitions and correspondences are explained in the bulk of this paper.

Our approach is rather elementary and based on the concept of "families of loops" rather than on the (in some sense more natural) homology groups directly. We expect that our construction extends to (locally finite) homology, but this would require a somewhat more sophisticated machinery.

Let us point out that part of our results can be understood as a topological version of results known in the literature on free field representation of conformal field theory ([7–9], and particularly [10]). The results in [9] suggest that our construction extends to groups of higher rank. In this paper we present the purely topological results in this subject, so that the paper can be read without knowledge in conformal field theory. See [11–13] for applications of these concepts to conformal field theory.

While this work was completed, we received some interesting preprints [14] where related results were obtained.

The paper is organized as follows: in Sect. 1 we introduce the concept of braid groupoid representations and local systems in a rather general context. In Sect. 2 we specialize to $SU(2)$, and explain the action of $U_q(sl_2)$ on relative cycles. Section 3 contains the discussion on intersection pairing. In Sect. 4 we show that the representation of $U_q(sl_2)$ on relative cycles is isomorphic to the tensor product of Verma modules, one for each puncture. In Sect. 5 we compute the monodromy action of the braid groupoid on relative cycles. The Appendix contains a summary of results on $U_q(sl_2)$.

1. Local Systems on Configuration Spaces

1.1. Colored Braid Groupoids. Let X be a connected two-dimensional manifold, possibly with boundary, k a positive integer (the number of colors), and $n_1, \ldots, n_k$ non-negative integers (the numbers of strands with given color). Set $n = \sum n_i$. Define the configuration spaces

$$\mathscr{C}_{(n_i)}(X) = \mathscr{C}_{n_1 \ldots n_k}(X) = X^n \Big\backslash \bigcup_{i<j} \{z_i = z_j\} / S_{n_1} \times \ldots \times S_{n_k}, \tag{1.1}$$

where the symmetric group S_{n_1} acts by permutations on the first n_1 variables, S_{n_2} on the subsequent n_2 variables, and so on. It is understood that the factors S_{n_i} with $n_i = 0$ should be omitted in (1.1). An element of $\mathscr{C}_{n_1 \ldots n_k}(X)$ can also be thought of as a sequence $(Z_1, \ldots, Z_k)$ of pairwise disjoint subsets of X with cardinalities $|Z_i| = n_i$.

Fix a base point x of $\mathscr{C}_{n_1 \ldots n_k}(X)$, and let O_x be the orbit of x under the symmetric group[1] S_n. Thus O_x can be identified with the right coset space

$$O_x = S_n / S_{n_1} \times \ldots \times S_{n_k}. \tag{1.2}$$

The colored braid groupoid $B_{n_1 \ldots n_k}(X, x)$ is the space of paths in X starting and ending in O_x, up to homotopies preserving endpoints, viewed as a subgroupoid of the fundamental groupoid of $\mathscr{C}_{n_1 \ldots n_k}(X)$. The groupoid $G = B_{n_1 \ldots n_k}(X, x)$ is indexed by O_x and has components labeled by the endpoints:

$$G = \bigcup_{\alpha, \beta \in O_x} G_{\alpha\beta}. \tag{1.3}$$

The multiplication law $G_{\alpha\beta} \times G_{\beta\gamma} \to G_{\alpha\gamma}$ is the composition of paths. Since X is connected, braid groupoids corresponding to different choices of base points are isomorphic. Any such isomorphism can be described as the composition with a homotopy class of paths connecting the base points. If $k = 1$, G is a group, the braid group on n strands on X. The groupoid $G = B_{n_1 \ldots n_k}(X, x)$ can be described in terms of the braid group $B_n(X, x)$. Let $h: B_n(X, x) \to S_n$ the canonical projection homomorphism. Then for $\alpha, \beta \in O_x$ there is a one-to-one map

$$\phi_{\alpha, \beta}: \{g \in B_n(X, x) \colon \alpha = h(g)\beta\} \to G_{\alpha\beta}, \tag{1.4}$$

such that $\phi_{\alpha\beta}(g)\phi_{\beta\gamma}(g') = \phi_{\alpha\gamma}(gg')$.

For $X \subset \mathbf{C}$, call $x \in \mathscr{C}_{n_1 \ldots n_k}(X)$ an *admissible base point* if x is the image of a point in $\mathbf{C}^n$ with

$$\mathrm{Re}(z_1) < \ldots < \mathrm{Re}(z_n).$$

Suppose now that $X = \mathbf{C}$. For any two admissible base points there is a unique homotopy class of paths in the space of admissible base points connecting them. Therefore, the corresponding colored braid groupoids can be uniquely identified, and we can omit the dependence on x in the notation, with the agreement that $G = B_{n_1 \ldots n_k}(\mathbf{C})$ is defined using any admissible base point.

An element α in O_x can be described by a color map

$$\bar{\alpha}: \{1, \ldots, n\} \to \{1, \ldots, k\} \tag{1.5}$$

such that $|\bar{\alpha}^{-1}(i)| = n_i$. The correspondence between α and $\bar{\alpha}$ is the following: Let $\alpha = \pi(x)$, $\pi \in S_n$. Then

$$\bar{\alpha}(i) = \lambda \quad \text{iff} \quad \pi^{-1}(i) \in \left\{ \sum_1^{\lambda-1} n_j + 1, \ldots, \sum_1^{\lambda} n_j \right\}. \tag{1.6}$$

[1] Acting as $\pi(z_1, \ldots, z_n) = (z_{\pi^{-1}(1)}, \ldots, z_{\pi^{-1}(n)})$

Let σ_i, $i=1,\ldots,n-1$, be the standard generator of $B_n(\mathbf{C})$, that exchanges the i^{th} strand with the $i+1^{\text{st}}$ one, and let $\tau_i=h(\sigma_i)$ denote the corresponding transposition. Then the system

$$\sigma_i^\alpha=\phi_{\tau_i\alpha,\alpha}(\sigma_i)\in G_{\tau_i\alpha,\alpha}, \qquad i=1,\ldots,n-1, \qquad \alpha\in O_x, \tag{1.7}$$

is a system of generators of G.

Let $a\in\mathbf{R}$. The inclusion $\mathscr{C}_{(n_i)}(\{\mathrm{Re}(z)<a\})\subset\mathscr{C}_{(n_i)}(\mathbf{C})$ induces an isomorphism $B_{(n_i)}(\{\mathrm{Re}(z)<a\})\to B_{(n_i)}(\mathbf{C})$. The same holds for the subset $\{\mathrm{Re}(z)>a\}$. Let $n_i=n_i'+n_i''$, $i=1,\ldots,k$. The inclusion

$$\begin{aligned}\phi:\mathscr{C}_{(n_i')}(\{\mathrm{Re}(z)<a\})\times\mathscr{C}_{(n_i'')}(\{\mathrm{Re}(z)>a\})&\to\mathscr{C}_{(n_i)}(\mathbf{C}),\\ ((Z_1',\ldots,Z_k'),(Z_1'',\ldots,Z_k''))&\mapsto(Z_1'\cup Z_1'',\ldots,Z_k'\cup Z_k'')\end{aligned} \tag{1.8}$$

induces an injective homomorphism of groupoids

$$\phi:B_{(n_i')}(\mathbf{C})\times B_{(n_i'')}(\mathbf{C})\to B_{(n_i)}(\mathbf{C}). \tag{1.9}$$

More precisely, we have a map $\phi:O'\times O''\to O$ defined by restriction to the orbits O', O'' of admissible base points, and maps (in an obvious notation)

$$\phi:G'_{\alpha'\beta'}\times G''_{\alpha''\beta''}\to G_{\alpha\beta}, \tag{1.10}$$

with $\alpha=\phi(\alpha',\alpha'')$ and $\beta=\phi(\beta',\beta'')$, compatible with the composition law. Intuitively, this homomorphism is simply the juxtaposition of colored braids.

1.2. R-Matrix Representations. A representation of a groupoid $G=\bigcup_{\alpha\beta\in I}G_{\alpha\beta}$ with index set I, on a family of complex vector spaces $(V_\alpha)_{\alpha\in I}$ is an index preserving homomorphism from G to the groupoid $\bigcup_{\alpha\beta\in I}\mathrm{Hom}^*(V_\beta,V_\alpha)$ of invertible linear maps of the vector spaces V_α. In other words, a representation ϱ of G is a family of maps

$$\varrho_{\alpha\beta}:G_{\alpha\beta}\to\mathrm{Hom}^*(V_\beta,V_\alpha) \tag{1.11}$$

such that $\varrho_{\alpha\beta}(g)\varrho_{\beta\gamma}(g')=\varrho_{\alpha\gamma}(gg')$. To simplify the notation, we will often omit the label $\alpha\beta$, thinking of $\varrho_{\alpha\beta}$ as the restriction of a map ϱ defined on G.

Definition. Let U_λ, $\lambda=1,\ldots,k$, be vector spaces and for each pair λ, μ let $R_{\lambda\mu}$ be an invertible element of $\mathrm{End}(U_\lambda\otimes U_\mu)$. An R-matrix representation of the groupoid $B_{(n_i)}(\mathbf{C})$ is a representation on the family of vector spaces, labeled by O_x,

$$V_\alpha=U_{\bar\alpha(1)}\otimes\ldots\otimes U_{\bar\alpha(r)}, \tag{1.12}$$

such that on generators

$$\varrho(\sigma_i^\alpha)=PR^{i,i+1}_{\bar\alpha(i)\bar\alpha(i+1)}, \qquad Pu\otimes v=v\otimes u, \tag{1.13}$$

where $PR^{ij}_{\lambda\mu}$ denotes $PR_{\lambda\mu}$ acting on the i^{th} and j^{th} factor in the tensor product.

Proposition 1.1. 1. *Let k be a fixed positive integer. A family of vector spaces U_λ, $\lambda=1,\ldots,k$, and a family $R_{\lambda\mu}$ of invertible elements of* $\mathrm{End}(U_\lambda\otimes U_\mu)$ *defines a representation $\varrho_{(n_i)}$ of $B_{n_1,\ldots,n_k}(\mathbf{C})$ for all $n_1,\ldots,n_k$ if and only if the Yang-Baxter equation*

$$R^{23}_{\mu\nu}R^{13}_{\lambda\nu}R^{12}_{\lambda\mu}=R^{12}_{\lambda\mu}R^{13}_{\lambda\nu}R^{23}_{\mu\nu} \tag{1.14}$$

holds on $U_\lambda\otimes U_\mu\otimes U_\nu$.

2. *Let ϕ be the homomorphism* (1.9),

$$B_{n'_1\dots n'_k}(\mathbf{C})\times B_{n''_1\dots n''_k}(\mathbf{C})\to B_{n_1\dots n_k}(\mathbf{C}), \qquad n_i=n'_i+n''_i, \tag{1.15}$$

and set $\varrho'=\varrho_{(n'_i)}$, $\varrho''=\varrho_{(n'')}$, $\varrho=\varrho_{(n_i)}$. Then, for all $\alpha',\beta'\in O'$, $\alpha'',\beta''\in O''$,

$$\varrho_{\alpha\beta}(\phi(g',g''))=\varrho_{\alpha'\beta'}(g')\otimes\varrho_{\alpha''\beta''}(g''), \tag{1.16}$$

where $\alpha=\phi(\alpha',\alpha'')$ and $\beta=\phi(\beta',\beta'')$.

Example 1. Let $U_\lambda=\mathbf{C}$, $\lambda=1,\dots,k$, and identify $V_\alpha=\mathbf{C}\otimes\dots\otimes\mathbf{C}$ with $\mathbf{C}$. Let $q_{\lambda\mu}$ be any non-zero complex numbers. Then $\varrho(\sigma_i^\alpha)=q_{\bar\alpha(i)\bar\alpha(i+1)}$ defines an R-matrix representation of $B_{n_1\dots n_k}(\mathbf{C})$.

Example 2. Let A be a quantum universal enveloping algebra [15] with universal R-matrix $R\in A\otimes A$, and ϱ_λ be finite dimensional representations of A on spaces U_λ. Then $R_{\lambda\mu}=\varrho_\lambda\otimes\varrho_\mu(R)$ defines an R-matrix representation of $B_{n_1\dots n_k}(\mathbf{C})$.

1.3. Local Systems. Let (M,x) be a topological space with base point, and $\hat M$ its universal covering space, with right action of $\pi_1(M,x)$. $\hat M$ is the space of homotopy classes of paths in M originating at x. For any representation $\varrho:\pi_1(M,x)\to GL(V)$ on a vector space V one defines a local system L as the vector bundle $(\hat M\times V)/\sim$ over M with the identification $(\hat m,\varrho(\eta)v)\sim(\hat m\eta,v)$, $\eta\in\pi_1(M,x)$, and projection $(\hat m,v)\mapsto m$, the covering projection on the first argument. Thus a local system is the same as a flat vector bundle with holonomy ϱ, and specified trivialization of the fiber over the base point.

This construction has the following slight generalization. Let O be a finite subset of M and G the subgroupoid of the fundamental groupoid of M consisting of homotopy classes of paths whose endpoints are in O. For $\alpha\in O$, let $\hat M_\alpha$ be the universal covering space of the space with base point (M,α). The groupoid G acts on the disjoint union $\coprod_\alpha \hat M_\alpha$ on the right by composition of paths, given by maps $\hat M_\alpha\times G_{\alpha\beta}\to\hat M_\beta$. Let ϱ be a representation of $G=\bigcup_{\alpha\beta\in O}G_{\alpha\beta}$ on a family of vector spaces $(V_\alpha)_{\alpha\in O}$. These data define a local system as the vector bundle

$$L=\coprod_\alpha \hat M_\alpha\times V_\alpha/\sim \tag{1.17}$$

with identification $(\hat m_\alpha,\varrho_{\alpha\beta}(\eta_{\alpha\beta})v)\sim(\hat m_\alpha\eta_{\alpha\beta},v_\beta)$, $\eta_{\alpha\beta}\in G_{\alpha\beta}$. Such a local system is the same as a flat vector bundle over M together with a family of vector spaces (V_α) and isomorphisms of the fibers over $\alpha\in O$ with V_α, such that parallel transport operators are given by ϱ. Local horizontal sections are continuous sections which locally can be written as $m\mapsto(\hat m,v)$, with constant v, and $\hat m$ covering m.

Let M_1,M_2 be topological spaces and $O_1\subset M_1$, $O_2\subset M_2$ be finite subsets. A homomorphism of local systems L_1 over M_1 to L_2 over M_2 is a map $L_1\to L_2$ mapping fibers to fibers linearly and sending local horizontal sections to local horizontal sections.

Lemma 1.2. *Let f be a map from M_1 to M_2 such that $f(O_1)\subset O_2$ and let $f_\alpha\in\mathrm{Hom}(V_\alpha\to V_{f(\alpha)})$, be linear maps indexed by O_1 such that the diagram*

$$\begin{array}{ccc} V_\alpha & \xrightarrow{\varrho_1(\eta)} & V_\beta \\ \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle f_\beta} \\ V_{f(\alpha)} & \xrightarrow{\varrho_2(f\circ\eta)} & V_{f(\beta)} \end{array} \tag{1.18}$$

is commutative for all $\alpha, \beta \in O_1$, $\eta \in G_{\alpha\beta}$. *Then f lifts uniquely to a homomorphism* $L_1 \to L_2$ *of the local systems associated to* ϱ_1, ϱ_2, *also denoted by f, which reduces to* f_α *on the fiber* V_α *over* $\alpha \in O_1$.

Let ϱ be a representation of $B_{n_1 \dots n_k}(\mathbf{C})$ and let L be the corresponding local system. Here is an explicit description of L in terms of transition functions. Fix an admissible base point x, and define the cells $C^\alpha_{(n_i)} \subset C_{(n_i)}(\mathbf{C})$ as follows: let $\alpha = \sigma x$, $\sigma \in S_n$ and define

$$C^\alpha_{(n_i)} = \{(z_1, \dots, z_n) \in \mathscr{C}_{(n_i)}(\mathbf{C}) \mid \mathrm{Re}(z_{\sigma^{-1}(1)}) < \dots < \mathrm{Re}(z_{\sigma^{-1}(n)})\}. \tag{1.19}$$

The cells $C^\alpha_{(n_i)}$ are pairwise disjoint, their union is dense in $\mathscr{C}_{(n_i)}(\mathbf{C})$, and each cell contains precisely one point in O_x, for any choice of admissible base point x.

Let $\bar{C}^\alpha_{(n_i)}$ be the closure of the cell $C^\alpha_{(n_i)}$. For $y \in \bar{C}^\alpha_{(n_i)} \cap \bar{C}^\beta_{(n_i)}$, let η be any path going from α to y in $C^\alpha_{(n_i)}$ and continuing from y to β in $C^\beta_{(n_i)}$. Define the locally constant transition function $g_{\alpha\beta}(y) = \varrho(\eta)$. Then L is the flat vector bundle over $\mathscr{C}_{(n_i)}(\mathbf{C})$,

$$L = \coprod_{\alpha \in O} (\bar{C}^\alpha_{(n_i)} \times V_\alpha)/\sim \tag{1.20}$$

with identification

$$(y, v_\alpha) \sim (y, v_\beta), \quad y \in \bar{C}^\alpha_{(n_i)} \cap \bar{C}^\beta_{(n_i)}, \quad v_\alpha \in V_\alpha, \quad v_\beta \in V_\beta, \tag{1.21}$$

if and only if $v_\alpha = g_{\alpha\beta}(y) v_\beta$.

Let η be any path whose endpoints lie in $\bigcup_{\alpha \in O} C^\alpha_{(n_i)}$. Then the parallel transport operator along η is an operator in $\mathrm{Hom}(V_\alpha, V_\beta)$, in the trivialization. Therefore, we have an extension of the definition of ϱ to all homotopy classes of paths with endpoints in $\bigcup_{\alpha \in O} C^\alpha_{(n_i)}$.

Let now $\varrho_{(n_i)}$ be the representations associated with a family of R-matrices, as in Proposition 1.1, and $L_{(n_i)}$ the corresponding local systems on $\mathscr{C}_{(n_i)}(\mathbf{C})$. Let $a \in \mathbf{R}$, $\mathbf{C}^+ = \{\mathrm{Re}(z) > a\}$, $\mathbf{C}^- = \{\mathrm{Re}(z) < a\}$. Denote by $L^<_{(n_i)}$ ($L^>_{(n_i)}$) the restriction of $L_{(n_i)}$ to $\mathscr{C}_{(n_i)}(\mathbf{C}^-)$ ($\mathscr{C}_{(n_i)}(\mathbf{C}^+)$, respectively). Let $L^<_{(n'_i)} \otimes L^>_{(n''_i)}$ be the flat bundle over $\mathscr{C}_{(n'_i)}(\mathbf{C}^-) \times \mathscr{C}_{(n''_i)}(\mathbf{C}^+)$ defined by taking the tensor products of the fibers.

Proposition 1.3. *The maps* $\phi: \mathscr{C}_{(n'_i)}(\mathbf{C}^-) \times \mathscr{C}_{(n''_i)}(\mathbf{C}^+) \to \mathscr{C}_{(n_i)}(\mathbf{C})$ *lifts to a homomorphism*

$$\phi: L^<_{(n'_i)} \otimes L^>_{(n''_i)} \to L_{(n_i)} \tag{1.22}$$

sending local horizontal sections to local horizontal sections. The lift is fixed by setting the homomorphisms of Lemma 1.2 equal to the canonical homomorphisms $V_\alpha \otimes V_\beta \xrightarrow{\sim} V_{\phi(\alpha, \beta)}$.

Proof. Let $C^{\alpha\,<}_{(n'_i)} = C^\alpha_{(n'_i)} \cap \mathscr{C}_{(n'_i)}(\mathbf{C}^-)$ and $C^{\alpha\,>}_{(n''_i)} = C^\alpha_{(n''_i)} \cap \mathscr{C}_{(n''_i)}(\mathbf{C}^+)$. Then $L^<_{(n'_i)} \otimes L^>_{(n''_i)}$ is the vector bundle

$$\coprod_{\alpha \in O', \beta \in O''} (\bar{C}^{\alpha\,<}_{(n'_i)} \times \bar{C}^{\beta\,>}_{(n''_i)}) \times (V_\alpha \otimes V_\beta)/\sim. \tag{1.23}$$

The map ϕ maps $\bar{C}^{\alpha\,<}_{(n')} \times \bar{C}^{\beta\,>}_{(n'')}$ to $\bar{C}^{\phi(\alpha,\beta)}_{(n_i)}$, and the transition functions are given by tensor products of transition functions. The claim follows then from Proposition 1.1 and Lemma 1.2. □

If $n_\lambda = 1$ and $n_\mu = 0$, $\mu \neq \lambda$, then $\mathscr{C}_{0,\dots,1,\dots,0}(\mathbf{C}) = \mathbf{C}$ and the fiber of $L_{0,\dots,1,\dots,0}$ over any point is canonically identified with U_λ. We have the following special case of the preceding proposition.

Proposition 1.4. *Let $a\in\mathbf{R}$, $\lambda\in\{1,\ldots,k\}$ and z_+,z_- be complex numbers with* $\mathrm{Re}(z_-)<a<\mathrm{Re}(z_+)$. *Then the maps*

$$\begin{aligned}\phi^\lambda_- &: \mathscr{C}_{n_1,\ldots,n_k}(\mathbf{C}^+)\to\mathscr{C}_{n_1,\ldots,n_\lambda+1,\ldots,n_k}(\mathbf{C}),\\ \phi^\lambda_+ &: \mathscr{C}_{n_1,\ldots,n_k}(\mathbf{C}^-)\to\mathscr{C}_{n_1,\ldots,n_\lambda+1,\ldots,n_k}(\mathbf{C}),\\ &(Z_1,\ldots,Z_k)\mapsto(Z_1,\ldots,Z_\lambda\cup\{z_\pm\},\ldots,Z_k)\end{aligned}\tag{1.24}$$

lift to homomorphisms

$$\begin{aligned}\phi^\lambda_- &: U_\lambda\otimes L^>_{n_1,\ldots,n_k}\to L_{n_1,\ldots,n_\lambda+1,\ldots,n_k},\\ \phi^\lambda_+ &: L^<_{n_1,\ldots,n_k}\otimes U_\lambda\to L_{n_1,\ldots,n_\lambda+1,\ldots,n_k}.\end{aligned}\tag{1.25}$$

These homomorphisms preserve horizontal sections and are isomorphisms on each fiber.

2. The Topological Action of $U_q(sl_2)$

2.1. The $SU(2)$ Case. Let us specialize the general discussion to the case of interest to us. Let D be the unit disc $\{|z|\leq 1\}$, and $w_1,\ldots,w_s$ be s distinct points in its interior. Define $X_r(w_1,\ldots,w_s)$ to be the fiber over $(w_1,\ldots,w_s)$ of the fibration $\mathscr{C}_{r,1,\ldots,1}(D)\to\mathscr{C}_{1,\ldots,1}(D)$. In other words, $X_r(w_1,\ldots,w_s)$ is the space of subsets of $D\backslash\{w_1,\ldots,w_s\}$ with r elements. Let $n_1,\ldots,n_s$ be positive integers, and $q\in\mathbf{C}\backslash 0$. The family of one-dimensional R-matrices

$$\begin{aligned}R_{11}&=-q^2,\\ R_{1j}&=R_{j1}=q^{1-n_j},\qquad j=2,\ldots,s+1,\end{aligned}\tag{2.1}$$

defines a representation of $B_{r,1,\ldots,1}(\mathbf{C})$, and a local system over $\mathscr{C}_{r,1,\ldots,1}(\mathbf{C})$ (and also on $\mathscr{C}_{r,1,\ldots,1}(D)$, by restriction). Let $L_r(w_1,\ldots,w_s)$ be the restriction of this local system to $X_r(w_1,\ldots,w_s)$. We will often omit the w dependence in the notation, and write X_r, L_r when no confusion arises.

In the following construction it is useful to choose also two points on the boundary of D. For definiteness, choose $P_+=1$, $P_-=-1$. Denote $X^\pm_r=\{Z\in X_r\,|\,Z\ni P_\pm\}$. By Proposition 1.4, the inclusions

$$\begin{aligned}X_r\backslash X^\pm_r&\to X^\pm_{r+1},\\ Z&\mapsto Z\cup\{P_\pm\}\end{aligned}\tag{2.2}$$

lift to homomorphisms $\phi_\pm: L_r|_{X_r\backslash X^\pm_r}\to L_{r+1}|_{X^\pm_{r+1}}$.

2.2. Families of Loops. In the following we fix s distinct points $w_1,\ldots,w_s$ in the interior of the unit disc, and denote by X the set $D\backslash\{w_1,\ldots,w_s\}$.

Definition. A non-intersecting family of loops in X, based at the point P_-, is a finite sequence $\gamma_0,\ldots,\gamma_{r-1}:[0,1]\to X$ of curves in X such that

(i) $\gamma_j(0)=\gamma_j(1)=P_-$; $\gamma_j(t)\neq P_-$ for $t\in]0,1[$.
(ii) If $t,s\in]0,1[$ and $\gamma_j(t)=\gamma_k(s)$, then $t=s$ and $j=k$.
(iii) For all j, the homotopy class of γ_j is non-trivial.

A non-intersecting family of loops can also be represented as a map Γ from the r-cube $]0,1[^r$ to X_r. It is the restriction of a continuous map $\bar{\Gamma}$ defined on the open

r-cube with open $r-1$-faces

$$Q_r =]0,1[^r \cup \bigcup_{i=1}^{r} (]0,1[\times \ldots \times \{0,1\} \times \ldots \times]0,1[) \tag{2.3}$$

defining an inclusion $\bar{\Gamma}: Q_r \to X_r$ of a closed subset of X_r.

Definition. A homotopy of non-intersecting families of loops is defined to be a homotopy $h:]0,1[^r \times [0,1] \to X_r$ such that for all $s \in [0,1]$, $h(\cdot, s)$ is a non-intersecting family of loops. Two families Γ, Γ' are said to be homotopic if there is a homotopy h such that $h(\cdot, 0) = \Gamma$ and $h(\cdot, 1) = \Gamma'$.

Consider the space $A_r = A_r(w_1, \ldots, w_r)$ of finite linear combinations

$$\sum_{\Gamma} \lambda_\Gamma [\Gamma], \tag{2.4}$$

where $[\Gamma] = [\gamma_0, \ldots, \gamma_{r-1}]$ are homotopy classes of families of loops and λ_Γ are horizontal sections of the pull-back bundle $\Gamma^* L_r$ over the contractible space Q_r, modulo the equivalence relations:

I. $\lambda[\Gamma] \sim \pm f^* \lambda[\Gamma \circ f]$, for any orientation preserving $(+)$ or reversing $(-)$ isometry f of the cube.

II. If, for some i, γ_i is homotopic to the composition $\gamma_i' * \gamma_i''$ with homotopy $\tilde{\gamma}_i: [0,1] \times [0,1] \to X$ and $\gamma_0, \ldots, \tilde{\gamma}_i(\cdot, s), \ldots, \gamma_{r-1}$ $(0 \leqq s < 1)$; $\gamma_0, \ldots, \gamma_i', \ldots, \gamma_{r-1}$; $\gamma_0, \ldots, \gamma_i'', \ldots, \gamma_{r-1}$ are all non-intersecting families of loops, then

$$\lambda[\gamma_0, \ldots, \gamma_{r-1}] \sim \lambda'[\gamma_0, \ldots, \gamma_i', \ldots, \gamma_{r-1}] + \lambda''[\gamma_0, \ldots, \gamma_i'', \ldots, \gamma_{r-1}], \tag{2.5}$$

where λ', λ'' are defined by restriction of λ.

It is understood that horizontal sections over homotopic families of loops are canonically identified by parallel transport, so that expressions (2.4) make sense.

Let ε be so small that the closed discs of radius ε centered at w_j are disjoint and contained in the interior of the unit disc. Let $X_r^\varepsilon, X_r^{\varepsilon-}$ be the spaces obtained from X_r, X_r^- by removing points $\{z_1, \ldots, z_r\}$ such that $|z_i - w_j| < \varepsilon$. Elements of A_r represent relative locally finite cycles in $H_r^{lf}(X_r^\varepsilon, X_r^{\varepsilon-}; L_r)$ with coefficients in the local system L_r. Thus we have a linear map

$$\varphi_r: A_r(w_1, \ldots, w_s) \to H_r^{lf}(X_r^\varepsilon, X_r^{\varepsilon-}; L_r). \tag{2.6}$$

On the other hand, we can also view a family Γ as a map $\tilde{\Gamma}$ from $]0,1[^r$ to $X_r^{\varepsilon-}$ by the formula

$$\tilde{\Gamma}: (t_0, \ldots, t_{r-1}) \mapsto \{-1, \gamma_0(t_0), \ldots, \gamma_{r-1}(t_{r-1})\}, \tag{2.7}$$

and a section λ of $\Gamma^* L_r$ is mapped under ϕ_- to a section of $\tilde{\Gamma}^* L_{r+1}$, and we also have a linear map

$$\psi_{r+1}: A_r(w_1, \ldots, w_s) \to H_{r+1}^{lf}(X_{r+1}^{\varepsilon-}; L_r). \tag{2.8}$$

2.3. Operators. We define a set of operators acting on $\bigoplus_0^\infty A_r$ and then compute their commutation relations.

Let γ be the path

$$\gamma: [0,1] \to X,$$
$$t \mapsto -e^{2\pi i t}.$$

Let $i:]0,1[^r \to]0,1[^{r+1}$ be the inclusion $(t_0,\dots,t_{r-1}) \mapsto (t_0,\dots,t_{r-1},1/2)$. Define a linear operator $F:A_r \to A_{r+1}$ that "adds a loop":

$$F:\lambda[\gamma_0,\dots,\gamma_{r-1}] \mapsto \lambda'[\gamma_0,\dots,\gamma_{r-1},\gamma], \tag{2.9}$$

where λ' is the section over Q_{r+1} such that $\phi_+\lambda = \lambda' \circ i$ on $]0,1[^r$. This definition makes sense since we can assume that the representative $\gamma_0,\dots,\gamma_{r-1}$ does not intersect γ except at the endpoints.

Introduce the face maps $[0,1]^r \to [0,1]^{r+1}$,

$$\begin{aligned} e^+_{i,r+1}(t_0,\dots,t_{r-1}) &= (t_0,\dots,t_{i-1},1,t_i,\dots,t_{r-1}), \\ e^-_{i,r+1}(t_0,\dots,t_{r-1}) &= (t_0,\dots,t_{i-1},0,t_i,\dots,t_{r-1}), \end{aligned} \tag{2.10}$$

and the linear operator that "kills a loop"

$$E:\lambda[\gamma_0,\dots,\gamma_{r-1}] \mapsto \sum_{i=0}^{r-1} (-1)^i \phi_-^{-1}(\lambda \circ e^+_{i,r} - \lambda \circ e^-_{i,r})[\gamma_0,\dots,\hat{\gamma}_i,\dots,\gamma_{r-1}] \tag{2.11}$$

($\hat{}$ denotes omission). The third operator is the diagonal operator K^2, defined on A_r as

$$K^2 = q^{\Sigma(n_i-1)-2r} 1_{A_r}. \tag{2.12}$$

The relation between E and the boundary operator is explained by the

Proposition 2.1. *The diagram*

$$\begin{array}{ccc} A_r & \xrightarrow{\varphi_r} & H^{lf}_r(X^\varepsilon_r, X^{\varepsilon-}_r; L_r) \\ \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle \partial_*} \\ A_{r-1} & \xrightarrow{\varphi_r} & H^{lf}_{r-1}(X^{\varepsilon-}_r; L_r) \end{array}$$

is commutative

2.4. Relations.

Theorem 2.2. *The operators E, F, K^2 obey the relations*

$$\begin{aligned} K^2E &= q^2EK^2, \\ K^2F &= q^{-2}FK^2, \\ EF - FE &= K^2 - K^{-2}. \end{aligned} \tag{2.14}$$

In other words, these operators define a representation of $U_q(sl_2)$ on $\bigoplus_r A_r(w_1,\dots,w_s)$.

Proof. The first two relations follow from the definition. The third relation is best checked in an explicit trivialization. We can assume that $\{\gamma_0(\frac12),\dots,\gamma_{r-1}(\frac12)\}$ is in some cell C^α_r. Denote by 1 the horizontal section of Γ^*L_r which takes the value 1 over the point $(\frac12,\dots,\frac12)$ in the trivialization over C^α_r. Let $\eta^\pm_i$ be the paths

$$t \mapsto \{\gamma_0(\tfrac12),\dots,\gamma_i(\tfrac12(1\pm t)),\dots,\gamma_{r-1}(\tfrac12)\}. \tag{2.15}$$

These paths go from the cell C^α_r to the cell containing the point $\{P_-,\gamma_0(\frac12),\dots,\gamma_{r-1}(\frac12)\}$. We have the explicit expressions

$$\begin{aligned} F1[\gamma_0,\dots,\gamma_{r-1}] &= 1[\gamma_0,\dots,\gamma_{r-1},\gamma], \\ E1[\gamma_0,\dots,\gamma_{r-1}] &= \sum_{i=0}^{r-1} (-1)^i(\varrho_r(\eta^+_i) - \varrho_r(\eta^-_i))1[\gamma_0,\dots,\hat{\gamma}_i,\dots,\gamma_{r-1}]. \end{aligned} \tag{2.16}$$

Denoting by $\eta^{\pm}$ the paths $t \mapsto \{\gamma_0(\frac{1}{2}), \ldots, \gamma_{r-1}(\frac{1}{2}), \gamma(\frac{1}{2}(1 \pm t))\}$ we have $\varrho_{r+1}(\eta^+) = q^{\Sigma(1-n_i)}(-q^2)^r$ and $\varrho_{r+1}(\eta^-) = \varrho_{r+1}(\eta^+)^{-1}$. We compute,

$$\begin{aligned} EF1[\gamma_0, \ldots, \gamma_{r-1}] &= E1[\gamma_0, \ldots, \gamma_{r-1}, \gamma] \\ &= \sum_{i=0}^{r-1} (-1)^i (\varrho_r(\eta_i^+) - \varrho_r(\eta_i^-)) 1[\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r-1}, \gamma] \\ &\quad + (-1)^r (\varrho_r(\eta^+) - \varrho_r(\eta^-)) 1[\gamma_0, \ldots, \gamma_{r-1}] \\ &= FE1[\gamma_0, \ldots, \gamma_{r-1}] \\ &\quad + (q^{\Sigma(n_i-1)-2r} - q^{-\Sigma(n_i-1)+2r}) 1[\gamma_0, \ldots, \gamma_{r-1}] . \end{aligned} \tag{2.17}$$

The proof is complete. □

From Proposition 2.1 and Theorem 2.2 follows:

Corollary 2.3. *Singular vectors in $A_r(w_1, \ldots, w_s)$ (i.e., vectors in* Ker$E$*) represent absolute cycles in $H_r^{lf}(X_r^{\varepsilon}; L_r)$.*

3. Intersection Pairing

3.1. Reflection and Duality. Let L'_r be the local system dual to L_r, i.e., the flat line bundle with holonomies $\varrho'(\eta) = \varrho(\eta)^{-1}$ (which is the representation obtained from ϱ by replacing q by its inverse) and θ the reflection sending $x+iy$ to $-x+iy$. The reflection θ maps orbits of admissible base points to orbits of admissible base points and preserves holonomies:

$$\varrho'(\theta \circ \eta) = \varrho(\eta) \tag{3.1}$$

and lifts therefore to an involutive homomorphism of local systems

$$\theta : L_r(w_1, \ldots, w_s) \to L'_r(\theta w_1, \ldots, \theta w_s) . \tag{3.2}$$

The lift is specified by setting the maps θ_α of Lemma 1.2 equal to the identity.

Denote by $A'_r(w_1, \ldots, w_s)$ the space of linear combinations $\sum \lambda_\Gamma [\Gamma]$ with $[\Gamma]$ homotopy classes of non-intersecting families of loops based at P_+, and λ_Γ horizontal sections of $\Gamma^* L'_r(w_1, \ldots, w_s)$, modulo the equivalence relations I and II. The reflection θ induces an isomorphism

$$\begin{aligned} \Theta : A_r(w_1, \ldots, w_s) &\to A'_r(\theta w_1, \ldots, \theta w_s) , \\ \lambda[\Gamma] &\mapsto \theta\lambda[\theta \circ \Gamma] \end{aligned} \tag{3.3}$$

which defines an action of $U_q(sl_2)$ on $\bigoplus A'_r$.

3.2. Intersection Pairing. In this subsection we assume that all families of curves are smooth maps on $]0,1[^r$.

Let Γ be a family of curves based at $P_- = -1$ and Γ' be a family of curves based at P_+. Suppose that Γ and Γ' intersect transversally in a finite number of points lying in the interior of X. Thus the set of (t, t') such that $\Gamma(t) = \Gamma'(t')$ is finite, contained in $]0,1[^r \times]0,1[^r$, and the tangent map $D\Gamma \times D\Gamma'$ is non-singular at any such (t, t'). The intersection index $\#(t, t')$ at (t, t') is then defined to be 1 if the tangent map preserves the orientation and -1 otherwise. The orientation of

$T_{\Gamma(t)}X_r = \mathbf{C}^r$ is conventionally defined via the identification

$$(x_1+iy_1,\ldots,x_r+iy_r) \equiv (x_1,\ldots,x_r,y_1,\ldots,y_r) \tag{3.4}$$

of $\mathbf{C}^r$ with $\mathbf{R}^{2r}$.

Definition. The intersection pairing is the complex bilinear form

$$(\ ,\ ): \bigoplus_r A_r(w_1,\ldots,w_r) \times \bigoplus_r A_r'(w_1,\ldots,w_r) \to \mathbf{C}$$

which is zero on $A_r \times A_{r'}'$, $r \neq r'$ and such that

$$\begin{aligned}&([\ ],[\ ])=1,\\ &(\lambda[\Gamma],\lambda'[\Gamma'])=(-1)^r \sum_{(t,t'):\Gamma(t)=\Gamma'(t')} \#(t,t')\langle\lambda(t),\lambda'(t')\rangle\end{aligned} \tag{3.5}$$

on $A_r \times A_r'$; $\langle\ ,\ \rangle$ denotes duality of fibers.

It is possible to give a more explicit formula for $(\ ,\ )$. Let $\dot{\gamma}(t)$ be the tangent vector at t to a smooth curve γ.

Proposition 3.1. *Suppose that $\Gamma=\gamma_0,\ldots,\gamma_{r-1}$ and $\Gamma'=\gamma_0',\ldots,\gamma_{r-1}'$ intersect transversally. Let $T_{ij}\subset]0,1[\times]0,1[$ be the set of (t,t') such that $\gamma_i(t)=\gamma_j'(t')$ and $\sigma_{ij}=\operatorname{sign}\operatorname{Im}(\bar{\dot{\gamma}}_i(t)\dot{\gamma}_j'(t'))$ be the intersection index of γ_i and γ_j' at (t,t'). Let for $\pi\in S_r$,*

$$T_\pi=\{(t,t')\in]0,1[^r\times]0,1[^r \mid (t_j,t_{\pi j}')\in T_{j\pi j},\ j=0,\ldots,r-1\}. \tag{3.6}$$

Then

$$(\lambda[\Gamma],\lambda'[\Gamma'])=(-1)^r\sum_{\pi\in S_r}\operatorname{sign}\pi\sum_{(t,t')\in T_\pi}\prod_{j=0}^{r-1}\sigma_{j\pi j}\langle\lambda(t),\lambda'(t')\rangle. \tag{3.7}$$

Proof. The condition $\Gamma(t)=\Gamma'(t')$ is equivalent to $\gamma_j(t_j)=\gamma_{\pi j}'(t_{\pi j}')$ for all j and some permutation π. Thus

$$\{(t,t') \mid \Gamma(t)=\Gamma'(t')\} = \bigcup_{\pi\in S_r} T_\pi, \tag{3.8}$$

and $T_\pi\cap T_{\pi'}=$ for $\pi\neq\pi'$, by property (ii) of non-intersecting families of curves. For $(t,t')\in T_\pi$, $\#(t,t')$ is the sign of the determinant of the matrix

$$\begin{pmatrix}\delta_{ij}\operatorname{Re}\dot{\gamma}_j(t_j) & \delta_{\pi i,j}\operatorname{Re}\dot{\gamma}_j'(t_j)\\ \delta_{ij}\operatorname{Im}\dot{\gamma}_j(t_j) & \delta_{\pi i,j}\operatorname{Im}\dot{\gamma}_j'(t_j)\end{pmatrix} \tag{3.9}$$

which is easily put in block form by permuting rows and column, and the result follows. $\square$

Theorem 3.2. *Fix $w_1,\ldots,w_s$, let $(\ ,\ )$ be the intersection pairing corresponding to $w_1,\ldots,w_s$ and let $(\ ,\ )_\theta$ be the intersection pairing corresponding to $\theta w_1,\ldots,\theta w_s$. Then*
(i) *For all $a\in A_r(w_1,\ldots,w_s)$, $b\in A_r(\theta w_1,\ldots,\theta w_s)$,*

$$(a,\Theta b)=(b,\Theta a)_\theta. \tag{3.10}$$

(ii) *Let T denote transposition with respect to $(\ ,\ )$. Then*

$$E^T=F,\quad F^T=E,\quad K^{2T}=K^2, \tag{3.11}$$

i.e., $(\ ,\ )$ is a covariant bilinear form.

Proof.
(i) Set $a=\lambda_1[\Gamma_1]$ and $b=\lambda_2[\Gamma_2]$. Looking at the definition of intersection pairing, we see that since θ preserves the pairing between fibers, it is sufficient to prove that

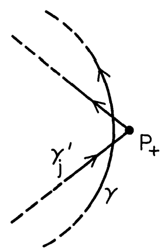

Fig. 1. The points of intersection of γ with γ_j'

the intersection index $\#(t_1, t_2)$ is the same on both sides of the equation. Let (t_1, t_2) be an intersection point of Γ_1 with $\theta\Gamma_2$. Identify the tangent space at a point of the unit r-cube in a canonical way with $\mathbf{R}^r$, and the tangent space at a point in X_r with $\mathbf{R}^{2r}$ as above. Then the intersection index occurring on the left-hand side is the sign of the determinant of the matrix $D\Gamma_1 \times \theta_* D\Gamma_2 : \mathbf{R}^r \times \mathbf{R}^r \to \mathbf{R}^{2r}$. The matrix θ_* is the diagonal matrix with entries $1, \ldots, 1, -1, \ldots, -1$. We have

$$\begin{aligned}\det(D\Gamma_1 \times \theta_* D\Gamma_2) &= (-1)^r \det(\theta_* D\Gamma_1 \times D\Gamma_2)\\ &= (-1)^{r^2+r} \det(D\Gamma_2 \times \theta_* D\Gamma_1)\\ &= \det(D\Gamma_2 \times \theta_* D\Gamma_1). \end{aligned} \tag{3.12}$$

The sign of the last determinant is the intersection index occurring on the right-hand side.

(ii) $K^{2T} = K^2$ follows immediately from the definition. Next, we show that $F^T = E$. The third relation follows then from (i). Let $\Gamma = \gamma_0, \ldots, \gamma_{r-1}$ be a family of loops based at P_- and $\Gamma' = \gamma_0', \ldots, \gamma_{r-1}'$ one based at P_+. Let us, as in the proof of Theorem 2.2, denote by 1 the section of $\Gamma^* L_r(w_1, \ldots, w_r)$ which takes the value 1 over the point $(\frac{1}{2}, \ldots, \frac{1}{2})$ in the trivialization over $\gamma_0(\frac{1}{2}), \ldots, \gamma_{r-1}(\frac{1}{2})$, which is assumed to be in a cell. Suppose that γ_i intersects γ_j' in a point which is in some cell C^α, and let t, t' the value of the parameters at the intersection. Denote by τ_{ij} the path $s \mapsto \gamma_i(\frac{1}{2}(1-s) + ts)$ and by τ_{ij}' the path $s \mapsto \gamma_i'(\frac{1}{2}(1-s) + t's)$. Then we have the explicit expression

$$(1[\Gamma], 1[\Gamma']) = (-1)^r \sum_{\pi \in S_r} \operatorname{sign} \pi \sum_{(t,t') \in T_\pi} \prod_{j=0}^{r-1} \sigma_{j\pi j} \varrho(\tau_{j,\pi j}) \varrho'(\tau_{j,\pi j}'). \tag{3.13}$$

Let γ be the path $t \mapsto e^{2\pi i t}$. We have to compute

$$(F[\gamma_0, \ldots, \gamma_{r-1}], 1[\gamma_0', \ldots, \gamma_r']) = (1[\gamma_0, \ldots, \gamma_{r-1}, \gamma], 1[\gamma_0', \ldots, \gamma_r']). \tag{3.14}$$

It can be assumed, by possibly applying a homotopy, that γ intersects each γ_j' at exactly two points, namely when the parameter t_j' of γ_j' is close to zero, with positive intersection index, and when t_j' is close to one, with negative index (see Fig. 1). In both cases the parameter t of γ is close to $\frac{1}{2}$. Therefore, the corresponding paths τ_{rj}, τ_{rj}', associated to these intersections, can be replaced by the trivial path and by the paths $\eta_j'^{\pm}$ defined by

$$t \to \{\gamma_0'(\tfrac{1}{2}), \ldots, \gamma_j'(\tfrac{1}{2}(1 \pm t)), \ldots, \gamma_r'(\tfrac{1}{2})\}. \tag{3.15}$$

We are in position to complete the calculation:

$$\begin{aligned}(F1[\gamma_0, \ldots, \gamma_{r-1}], 1[\gamma_0', \ldots, \gamma_{r-1}']) &= (-1)^{r+1} \sum_{i=0}^{r} \sum_{\pi \in S_r} (-1)^{r-i} \operatorname{sign}(\pi),\\ &\quad \sum_{(t,t') \in T_\pi} (\varrho_r'(\eta_i'^-) - \varrho_r'(\eta_i'^+)) \prod_{j=0}^{r-1} \sigma_{j,\pi j} \varrho(\tau_{j,\pi j}) \varrho'(\tau_{j,\pi j}')\\ &= (1[\gamma_0, \ldots, \gamma_{r-1}], E1[\gamma_0', \ldots, \gamma_{r-1}']). \quad \square\end{aligned} \tag{3.16}$$

4. Tensor Products and Coproduct

In this section we give explicitly the structure of $\bigoplus A_r(w_1, \ldots, w_s)$ as an $U_q(sl_2)$ module.

4.1. The Module $\bigoplus A_r(w_1, \ldots, w_s)$. The spaces $A_r(w_1, \ldots, w_s)$ constitute a complex vector bundle over $\mathscr{C}_{1,\ldots,1}(D)$. This bundle carries the flat Gauss-Manin connection induced by the connection of $L_{r,1,\ldots,1}$. The holonomy of this connection will be computed in the next section. Here we only notice that the spaces $A_r(w_1, \ldots, w_s)$ are all isomorphic (although not canonically isomorphic), and we can fix $w_1, \ldots, w_s$ as we like. For definiteness, choose $w_1, \ldots, w_s$ so that $\mathrm{Re}(w_1) < \ldots < \mathrm{Re}(w_s)$. To describe $A_r(w_1, \ldots, w_s)$ as a space, we choose a basis as follows. Fix a non-intersecting family of loops $\gamma_1, \ldots, \gamma_s$, so that γ_i loops around w_i as shown in Fig. 2a. Introduce the shortened notation

$$[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}] \tag{4.1}$$

to denote a homotopy class of non-intersecting families of loops, constructed as follows: Let $\gamma_i^{(j)}$ $(1 \leqq i \leqq s, 1 \leqq j \leqq r_i)$ be slight homotopic deformations of γ_i such that $\gamma_i^{(j)}$ lies inside $\gamma_i^{(j+1)}$ and such that $\gamma_1^{(1)}, \ldots, \gamma_1^{(r_1)}, \ldots, \gamma_s^{(1)}, \ldots, \gamma_s^{(r_s)}$ is a non-intersecting family of loops. The homotopy class of the latter is $[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}]$. Define a horizontal section denoted by 1 over this family to be the section which takes the value 1 with respect to the trivialization over a point with coordinates obeying

$$\begin{aligned} &\mathrm{Re}(w_1) < \mathrm{Re}(z_1) < \ldots < \mathrm{Re}(z_{r_1}) < \mathrm{Re}(w_2) \\ &\quad < \mathrm{Re}(z_{r_1+1}) < \ldots < \mathrm{Re}(z_{r_2}) < \mathrm{Re}(w_3) < \ldots . \end{aligned} \tag{4.2}$$

If $r_1, \ldots, r_s$ run over all non-negative integers with total sum r, $1[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}]$ are a basis of $A_r(w_1, \ldots, w_r)$.

Theorem 4.1. *The $U_q(sl_2)$ module $\bigoplus A_r(w_1, \ldots, w_s)$ is isomorphic to the tensor product of Verma modules*

$$V_{n_1} \otimes \ldots \otimes V_{n_s}$$

with action of $U_q(sl_2)$ given by the coproduct $\Delta^{(s)}$. If $\mathrm{Re}(w_1) < \ldots < \mathrm{Re}(w_s)$, *an isomorphism is explicitly given by*

$$1[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}] \mapsto F^{r_1} v_{n_1} \otimes \ldots \otimes F^{r_s} v_{n_s} . \tag{4.3}$$

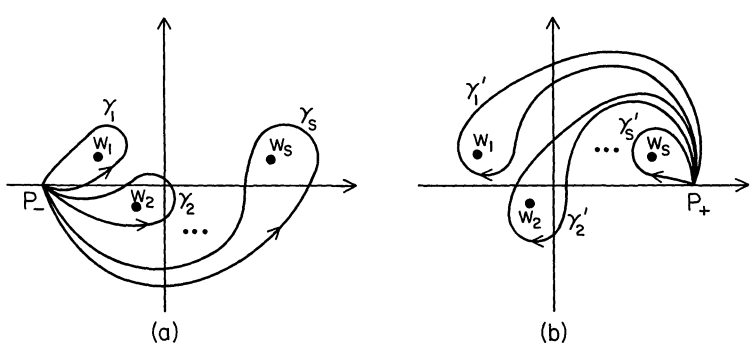

Fig. 2. The loops used to define a basis of $A_r(w_1, \ldots, w_s)$ and $A'_r(w_1, \ldots, w_s)$

Proof. For $s=1$, $1[\gamma_1^r]=F1[\gamma_1^{r-1}]$, by definition of F. For higher s we have to show that the action of the generators on the basis is indeed given by the coproduct. For the generators K^2, K^{-2} this follows from the definition. To compute the action of F we must deform the added loop γ to the composition of loops homotopic to $\gamma_s, \dots, \gamma_1$ using I of 2.2:

$$F1[\gamma_1^{r_1}, \dots, \gamma_s^{r_s}] = \sum_{i=1}^{s} \alpha_i 1[\gamma_1^{r_1}, \dots, \gamma_i^{r_i+1}, \dots, \gamma_s^{r_s}]. \tag{4.4}$$

The coefficient α_i is, up to a sign, the transition function we pick up by going from the point where the section 1 over $\gamma_1^{r_1}, \dots, \gamma_s^{r_s}, \gamma$ is trivialized to the point where the section 1 over $\gamma_1^{r_1}, \dots, \gamma_i^{r_i+1}, \dots, \gamma_s^{r_s}$ is trivialized. The sign is $(-1)^{\sum_{j>i} r_j}$, and comes from reordering the loops using rule I of 2.2. Thus $\alpha_i = q^{\sum_{j>i}(1-n_j-2r_j)}$ and we get the result

$$F = \sum_i 1 \otimes \dots \otimes 1 \otimes F \otimes K^{-2} \otimes \dots \otimes K^{-2}. \tag{4.5}$$

Similarly, by computing the contribution proportional to $1[\gamma_1^{r_1}, \dots, \gamma_i^{r_i-1}, \dots, \gamma_s^{r_s}]$ of $E1[\gamma_1^{r_1}, \dots, \gamma_s^{r_s}]$, we see that we get the same terms as in the computation of $E[\gamma_i^{r_i}]$ except for the factor $q^{-\sum_{j<i}(1-n_j-2r_j)}$ that we pick up by going from the vicinity of w_i to P_-, and we obtain the result:

$$E = \sum_i K^2 \otimes \dots \otimes K^2 \otimes E \otimes 1 \otimes \dots \otimes 1. \tag{4.6}$$

This concludes the proof. □

Remark. We see that the tensor product also has a topological interpretation: let S_+, S_- be the upper and lower halves of the unit circle. We can think of $D \backslash \{w_1, \dots, w_s\}$ (with $w_i \neq w_j$, $i \neq j$ and $w_i \in \operatorname{int} D$) as the result of glueing s punctured discs $D\backslash\{0\}$ in such a way that S_+ of the i^{th} disc is identified with S_- of the $i+1^{\text{st}}$ disc. This construction gives an identification of $A_r(w_1, \dots, w_s)$ with $A_1(0) \otimes \dots \otimes A_1(0)$ so that $1[\gamma_1^{r_1}, \dots, \gamma_s^{r_s}]$ is identified with $1[\gamma^{r_1}] \otimes 1[\gamma^{r_2}] \otimes \dots \otimes 1[\gamma^{r_s}]$.

The module $A_r'(w_1, \dots, w_s)$, being isomorphic to $A_r(\theta w_1, \dots, \theta w_s)$, also has the structure of a tensor product of Verma modules. In order to achieve compatibility between tensor product structure and bilinear form, one has to choose the isomorphism in a special way. Let γ_i' be the non-intersecting family depicted in Fig. 2b, and, as above, define $[(\gamma_1')^{r_1}, \dots, (\gamma_s')^{r_s}]$ and a horizontal section 1 taking the value 1 with respect to the trivialization over a point with

$$\operatorname{Re}(z_1) < \dots < \operatorname{Re}(z_{r_1}) < \operatorname{Re}(w_1) < \operatorname{Re}(z_{r_1+1}) < \dots < \operatorname{Re}(z_{r_2}) < \operatorname{Re}(w_2) < \dots . \tag{4.7}$$

Let furthermore λ be the automorphism of $U_q(sl_2)$ defined on generators by

$$\lambda(H) = H, \quad \lambda(E) = K^{-2}E, \quad \lambda(F) = FK^2. \tag{4.8}$$

The following dual version of Theorem 4.1 is proven exactly as Theorem 4.1.

Theorem 4.2. *The $U_q(sl_2)$ module $A_r'(w_1, \dots, w_s)$ is isomorphic to the tensor product of Verma modules*

$$V_{n_1} \otimes \dots \otimes V_{n_s} \tag{4.9}$$

with action of $U_q(sl_2)$ given by the twisted coproduct $\lambda^{-1} \circ \Delta^{(s)} \circ \lambda$.

If $\mathrm{Re}(w_1) < \ldots < \mathrm{Re}(w_s)$, *an isomorphism is explicitly given by*

$$1[(\gamma'_1)^{r_1}, \ldots, (\gamma'_s)^{r_s}] \mapsto F^{r_1} v_{n_1} \otimes \ldots \otimes F^{r_s} v_{n_s}. \tag{4.10}$$

4.2. Tensor Products and Intersection Pairing. The isomorphisms described in the preceding Theorems define intersection pairing as a bilinear form $(\ ,\ )$ on $V_{v_1} \otimes \ldots \otimes V_{n_s}$.

Theorem 4.3. *The intersection pairing coincides with the product of the Shapovalov bilinear forms on V_{n_i}. In particular, it is symmetric and degenerate. It reduces to a non-degenerate symmetric bilinear form on the fusion rule subquotient.*

Proof. For $s=1$, the highest weight vector v_n of V_n has $(v_n, v_n)=1$ and one has $E^T = F$, $F^T = E$, $K^{2T} = K^2$, which are the characterizing properties of the Shapovalov bilinear form on the Verma module V_n. The choice of identification of A_r, A'_r with the product of Verma modules is chosen in such a way that the weight $(-1)^r \#(t, t') \langle \lambda(t), \lambda'(t') \rangle$ of each intersection point factorizes into s factors equal to the weights of the corresponding intersections in $([\gamma_i^{r_i}], [\gamma_i'^{r_i}])$. $\square$

4.3. The Local System. The result of Theorem 4.1 can be cast into the formalism of 1.3. To be more precise, introduce the dependence of the labels $n_1, \ldots, n_s$ explicitly in the notation:

$$A_r = A_r(w_1, \ldots, w_s \mid n_1, \ldots, n_s). \tag{4.11}$$

Fix a base point $(w_1^0, \ldots, w_s^0)$ such that $\mathrm{Re}(w_1) < \ldots < \mathrm{Re}(w_s)$. The spaces $A_r(w_1, \ldots, w_s \mid n_1, \ldots, n_s)$ define a flat vector bundle over $\mathscr{C}_{1,\ldots,1}(D)$ with (Gauss-Manin) connection induced by the connection on $L_{r,1,\ldots,1}$. The fiber over $(w_1^0, \ldots, w_s^0)$ is identified with $V_{n_1} \otimes \ldots \otimes V_{n_s}$ by the explicit isomorphism of Theorem 4.1. Similarly, for any permutation $\alpha \in S_s$ we can identify the fiber over $\alpha(w_1^0, \ldots, w_s^0)$ with $V_{n_{\alpha(1)}} \otimes \ldots \otimes V_{n_{\alpha(s)}}$ using the trivial identification

$$A_r(w^0_{\alpha^{-1}(1)}, \ldots, w^0_{\alpha^{-1}(s)} \mid n_1, \ldots, n_s) = A_r(w_1^0, \ldots, w_s^0 \mid n_{\alpha(1)}, \ldots, n_{\alpha(s)}). \tag{4.12}$$

5. Monodromy Action of the Braid Groupoid and Universal R-Matrix

In the following we will consider the configuration space $\mathscr{C}_{n_1,\ldots,n_{s+1}}(D)$ with $D = \{z \in \mathbf{C} \mid |z| \leqq 1\}$, $n_1 = r$ and $n_2 = \ldots = n_{s+1} = 1$.

Let $p : \mathscr{C}_{r,1,\ldots,1}(D) \to \mathscr{C}_{1,\ldots,1}(D)$ be the projection given by omitting the first r entries of $(z_1, \ldots, z_r, w_1, \ldots, w_s)$. p defines a fiber bundle over $\mathscr{C}_{1,\ldots,1}(D)$ with fibers $p^{-1}(w_1, \ldots, w_s) = X_r(w_1, \ldots, w_s)$. In particular, $X_1(w_1, \ldots, w_s) = D \setminus \{w_1, \ldots, w_s\}$ is the punctured unit disc. We will restrict our attention to $\{w_1, \ldots, w_s\} \subset \mathrm{int}\, D$.

Fix a base point $x \in \mathscr{C}_{1,\ldots,1}(D)$, $x = (w_1, \ldots, w_s)$ with $\mathrm{Re}(w_1) < \ldots < \mathrm{Re}(w_s)$.

In the following we will construct a non-abelian representation ϱ of the colored braid groupoid $B_{1,\ldots,1}(D, x) = G$. Note that $O_x = S_s x$ and $G = \bigcup_{\alpha, \beta \in S_s} G_{\alpha,\beta}$. G is generated by $[\sigma_i^\alpha]$, $i \in \{1, \ldots, s-1\}$ and $\alpha \in S_s$. Here $\sigma_i^\alpha : [0,1] \to \mathscr{C}_{1,\ldots,1}(D)$ is a smooth parametrized curve with $\sigma_i^\alpha(0) = \alpha x$ and $\sigma_i^\alpha(1) = \tau_i \alpha x$, which implements a counterclockwise exchange of $w_{\alpha^{-1}(i)}$ and $w_{\alpha^{-1}(i+1)}$.

Let the representation space V_α associated with $\alpha \in S_s$ be $A_r(w_{\alpha^{-1}(1)}, \ldots, w_{\alpha^{-1}(s)})$. In a self-explanatory notation (see 4.3),

$$A_r(w_{\alpha^{-1}(1)}, \ldots, w_{\alpha^{-1}(s)}) = \bigoplus \mathbf{C} 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha. \tag{5.1}$$

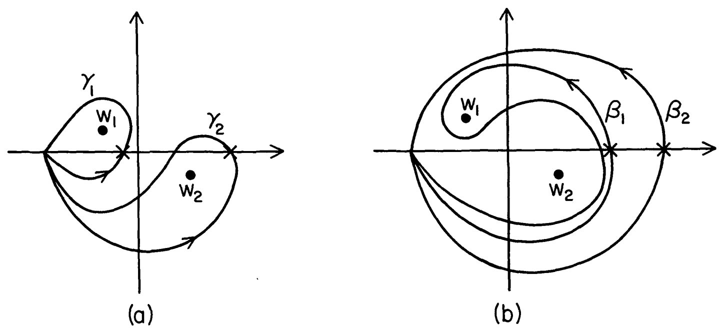

Fig. 3. The loops appearing in the proof of Proposition 5.1. The points marked with a cross are the points used to define the section 1

The sum is over $(j_1, \ldots, j_s) \in \{0, \ldots, p-1\}^s$ such that $j_1 + \ldots + j_s = r$. Thus we have an identification $A_r(w_{\alpha^{-1}(1)}, \ldots, w_{\alpha^{-1}(s)}) \cong \mathbf{C}^{N(r,s)}$ with $N(r,s) = \binom{r+s}{s-1}$. The simplest nontrivial case is $s=2$ with $N(r,s) = r+1$. Let $[\sigma_i^\alpha]$ be represented by the deformation homomorphism $\varrho([\sigma_i^\alpha]): V_\alpha \to V_{\tau_i\alpha}$ associated with σ_i^α. Introduce the q-number notation $[n]_q = q^n - q^{-n}$, and, for $k = 0, \ldots, n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q [n-1]_q \ldots [n-k+1]_q}{[k]_q [k-1]_q \ldots [1]_q}. \tag{5.2}$$

Proposition 5.1.

1)
$$\begin{aligned} &\varrho([\sigma_i^\alpha]) 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha \\ &= \sum_{k=0}^{j_{i+1}} 1[(\gamma_1)^{j_1} \ldots (\gamma_i)^{j_{i+1}-k} (\gamma_{i+1})^{j_i+k} \ldots (\gamma_s)^{j_s}]_{\tau_i\alpha} \\ &\quad \times q^{\frac{1}{2}k(k-1)} q^{\frac{1}{2}(n_{\alpha(i+1)} - 1 - 2(j_{i+1} - k))(n_{\alpha(i)} - 1 - 2(j_i + k))} \\ &\quad \times \begin{bmatrix} j_{i+1} \\ k \end{bmatrix}_q \prod_{l=0}^{k-1} [n_{\alpha(i+1)} - (j_{i+1} - k)]_q, \end{aligned} \tag{5.3}$$

2)
$$\begin{aligned} &\varrho([\sigma_i^\alpha]^{-1}) 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha \\ &= \sum_{k=0}^{j_i} 1[(\gamma_1)^{j_1} \ldots (\gamma_i)^{j_{i+1}+k} (\gamma_{i+1})^{j_i-k} \ldots (\gamma_s)^{j_s}]_{\tau_i\alpha} \\ &\quad \times (-1)^k q^{-\frac{1}{2}k(k-1)} q^{-\frac{1}{2}(n_{\alpha(i+1)} - 1 - 2j_{i+1})(n_{\alpha(i)} - 1 - 2j_i)} \\ &\quad \times \begin{bmatrix} j_i \\ k \end{bmatrix}_q \prod_{l=0}^{k-1} [n_{\alpha(i)} - (j_i - k)]_q. \end{aligned} \tag{5.4}$$

Proof. Without loss of generality we can restrict the proof to the case $s=2$, $i=1$, and $\alpha = \mathrm{id}$. The loops used in this proof are represented in Fig. 3. The matrix representation of $[\sigma]$ is computed by consecutive deformations and subdivisions of the individual loops in

$$\varrho([\sigma]) 1[(\gamma_1)^{j_1}(\gamma_2)^{j_2}] = q^{\frac{1}{2}(n_1 - 1 - 2j_1)(n_2 - 1 - 2j_2)} q^{j_2(n_1-1) - 2j_1 j_2} 1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau. \tag{5.5}$$

Subdivide the last β_1-loop in a γ_2- and a β_2-loop to obtain

$$1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau = -q^{-2(n_2-1)+2(j_2-1)}1[(\gamma_2)^{j_1+1}(\beta_1)^{j_2-1}]_\tau + 1[(\gamma_2)^{j_1}(\beta_1)^{j_2-1}\beta_2]_\tau . \tag{5.6}$$

Iterate this subdivision until there is no β_1-loop left over. The result is

$$1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau = \sum_{k=0}^{j_2}(-1)^k q^{-2k(n_2-1)} \sum_{0\leqq i_1<\ldots<i_k\leqq j_2-1} q^{2\sum_{l=1}^{k} i_l} \times 1[(\gamma_2)^{j_1+k}(\beta_2)^{j_2-k}]_\tau . \tag{5.7}$$

The sum over ordered k-tuples is performed with Gauss' formula

$$\sum_{0\leqq i_1<\ldots<i_k\leqq j_2-1} q^{2\sum_{l=1}^{k} i_l} = q^{k(j_2-1)}\begin{bmatrix} j_2 \\ k\end{bmatrix}_q . \tag{5.8}$$

Then subdivide the β_2-loops in γ_1- and γ_2-loops. This decomposition yields

$$1[(\gamma_2)^{j_1+k}(\beta_2)^{j_2-k}]_\tau = \sum_{l=0}^{j_2-k} q^{-(j_2-k-l)(n_1-1-2(j_1+k)-l)}\begin{bmatrix} j_2-k \\ l\end{bmatrix}_q \times 1[(\gamma_1)^{j_2-k-l}(\gamma_2)^{j_1+k+l}]_\tau . \tag{5.9}$$

Insert (5.8) and (5.9) into (5.7), and reorder the double sum to obtain

$$1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau = \sum_{k=0}^{j_2}(-1)^k q^{k(j_2-2n_2+1)-(j_2-k)(n_1-1-2(j_1+k))}\begin{bmatrix} j_2 \\ k\end{bmatrix}_q \times \sum_{l=0}^{k}(-1)^l q^{-l(2j_2-2n_2-k+1)}\begin{bmatrix} k \\ l\end{bmatrix}_q 1[(\gamma_1)^{j_2-k}(\gamma_2)^{j_1+k}]_\tau . \tag{5.10}$$

Then perform the second sum with the q-binomial formula

$$\sum_{l=0}^{k}(-1)^l q^{-l(2j_2-2n_2-k+1)}\begin{bmatrix} k \\ l\end{bmatrix}_q = (-1)^k q^{k(n_2-j_2)+\frac{1}{2}k(k-1)}\prod_{l=0}^{k-1}[n_2-(j_2-l)]_q . \tag{5.11}$$

Insert (5.10) and (5.11) into (5.5) to find (5.3). The matrix representation of $[\sigma]^{-1}$ is computed following the same lines. □

The important consequence of Proposition 5.1 is that the deformation homomorphism $\varrho([\sigma_i^\alpha]): V_\alpha \to V_{\tau_i\alpha}$, written as an operator, has a universal form which resembles the universal R-matrix of the quantum group algebra $U_q(sl_2)$. Let $p\in \mathbf{N}\cup\{\infty\}$ be the smallest positive integer such that $q^{2p}=1$.

Theorem 5.2. *Denote by X_i the operator $1\otimes\ldots\otimes 1\otimes X\otimes 1\otimes\ldots\otimes 1$ acting on the i^{th} factor of $V_{n_1}\otimes\ldots\otimes V_{n_s}$. Suppose that $1\leqq n_1,\ldots,n_s\leqq p-1$.*

(i) $$\varrho([\sigma_i^\alpha]) = \sum_{k=0}^{p-1} q^{\frac{1}{2}k(k-1)}\frac{(q-q^{-1})^k}{[k]_q!} q^{\frac{1}{2}H_iH_{i+1}}E_i^k F_{i+1}^k \tau_i , \tag{5.12}$$

(ii) $$\varrho([\sigma_i^\alpha]^{-1}) = \sum_{k=0}^{p-1}(-1)^k q^{-\frac{1}{2}k(k-1)}\frac{(q-q^{-1})^k}{[k]_q!} F_i^k E_{i+1}^k q^{-\frac{1}{2}H_iH_{i+1}}\tau_i . \tag{5.13}$$

Proof. Recall the definition of the operators E_i, F_i, H_i, and τ_i. They act as

$$E_i 1[(\gamma_1)^{j_1} \dots (\gamma_s)^{j_s}]_\alpha = \frac{[j_i]_q [n_{\alpha(i)} - j_i]_q}{q - q^{-1}} 1[(\gamma_1)^{j_1} \dots (\gamma_i)^{j_i - 1} \dots (\gamma_s)^{j_s}]_\alpha ,$$
$$F_i 1[(\gamma_1)^{j_1} \dots (\gamma_s)^{j_s}]_\alpha = 1[(\gamma_1)^{j_1} \dots (\gamma_i)^{j_i + 1} \dots (\gamma_s)^{j_s}]_\alpha , \tag{5.14}$$
$$H_i 1[(\gamma_1)^{j_1} \dots (\gamma_s)^{j_s}]_\alpha = (n_{\alpha(i)} - 1 - 2j_i) 1[(\gamma_1)^{j_1} \dots (\gamma_s)^{j_s}]_\alpha$$

and

$$\tau_i 1[(\gamma_1)^{j_1} \dots (\gamma_s)^{j_s}]_\alpha = 1[(\gamma_1)^{j_1} \dots (\gamma_s)^{j_s}]_{\tau_i \alpha} . \tag{5.15}$$

Compare (5.3) and (5.4) with (5.14) and (5.15) to conclude (5.12) and (5.13). □

The content of Theorem 5.1 exceeds pure nomenclature since the operators E_i, F_i, and H_i have a topological interpretation. Recall that they satisfy the commutation relations

$$[H_i, E_j] = 2E_j \delta_{j,i} ,$$
$$[H_i, F_j] = -2F_j \delta_{j,i} , \tag{5.16}$$
$$[E_i, F_j] = [H_j]_q \delta_{j,i} .$$

We have identified $A_r(w_{\alpha(1)}, \dots, w_{\alpha(s)})$ with the tensor product $V_{n_{\alpha(1)}} \otimes \dots \otimes V_{n_{\alpha(s)}}$ of $U_q(sl_2)$ Verma modules, the identification being

$$1[(\gamma_1)^{j_1} \dots (\gamma_s)^{j_s}] \mapsto F^{j_1} v_{n_{\alpha(1)}} \otimes \dots \otimes F^{j_s} v_{n_{\alpha(s)}} .$$

Moreover, we have identified $E_i \in \mathrm{Hom}(A_r(w_{\alpha(1)}, \dots, w_{\alpha(s)}), A_{r-1}(w_{\alpha(1)}, \dots, w_{\alpha(s)}))$ with the element

$$E_i \mapsto 1 \otimes \dots \otimes E \otimes \dots \otimes 1$$

of $U_q(sl_2)^{\otimes s}$. Here E stands in the i^{th} entry. Similarly we have proceeded with F_i and E_i. τ_i is identified with the i^{th} transposition. We have proved that this identification is a quantum group algebra homomorphism and a module isomorphism.

The observation of this section is that

$$\varrho([\sigma_i^\alpha]^{\pm 1}) \mapsto 1 \otimes \dots \otimes (RP)^{\pm 1} \otimes \dots \otimes 1$$

with $R \in U_q(sl_2)^{\otimes 2}$ the universal R-matrix in an obvious normalization and R^{-1} its inverse acting on the i^{th} and $(i+1)^{\text{st}}$ entry.

It follows that ϱ defines an R-matrix representation of $B_{1,\dots,1}(X)$, the Yang-Baxter equations following from the properties of the universal R-matrix.

Having constructed an $N(r,s)$-dimensional R-matrix representation of $B_{1,\dots,1}(D)$, we also have a rank $N(r,s)$ local system $L^r_{1,\dots,1}(D)$ over $\mathscr{C}_{1,\dots,1}(D)$. Let $C^\alpha_{1,\dots,1}(D)$ be the intersection of the cell $C^\alpha_{1,\dots,1}$ of 1.3 with $\mathscr{C}_{1,\dots,1}(D)$,

$$L^r_{1,\dots,1}(D) = \bigcup_{\alpha \in S_s} \overline{C^\alpha_{1,\dots,1}(D)} \times \mathbf{C}^{N(r,s)} / \sim ,$$

with equivalence relation over $\overline{C^\alpha_{1,\dots,1}(D)} \cap \overline{C^{\tau_i \alpha}_{1,\dots,1}(D)}$ given by multiplication with the matrices (5.3) and (5.4), respectively. In this local system the fiber is $p^{-1}(w_{\alpha^{-1}(1)}, \dots, w_{\alpha^{-1}(s)}) = A_r(w_{\alpha^{-1}(1)}, \dots, w_{\alpha^{-1}(s)})$. The parallel transport matrix associated with σ_i^α in the basis (5.2) is the universal R-matrix in the representation $n_{\alpha(i)} \otimes n_{\alpha(i+1)}$.

This representation of the braid groupoid is not irreducible in general. In particular, it has as invariant subspaces the null space of the bilinear form (,), which defines a subbundle of our flat bundle, invariant under parallel transport. These subspaces are described explicitly in the Appendix.

6. Locally Finite Homology

In this paper, we have worked on the spaces A_r rather than on homology groups directly. We now formulate some conjecture on the relation to homology, and the structure of the corresponding locally finite homology groups. These conjectures follow from the assumption that our quantum group action extends to an action on homology, and by applying the computations of [11, 12], which are not completely rigorous, to the situation studied here. As usual, we assume that s distinct points $w_1, \dots, w_s$ in the interior of the unit circle, s positive integers $n_1, \dots, n_s$, and a complex number $q \neq -1, 0, 1$ are given. If q is a root of unity, we furthermore assume that $1 \leqq n_i \leqq p-1$, where p is the smallest positive integer such that $q^{2p}=1$. For ε small enough, the locally compact spaces $X_r^\varepsilon \supset X_r^{\varepsilon-}$ are defined as in 2.2, and we have a local system L_r over X_r^ε.

Conjecture 6.1. *If q is not a root of unity, the map*

$$\varphi_r : A_r(w_1, \dots, w_s) \to H_r^{lf}(X_r^\varepsilon, X_r^{\varepsilon-}; L_r), \tag{6.1}$$

is an isomorphism of vector spaces.

If q is a root of unity, let $U_q^L(sl_2)$ be Lusztig's version of $U_q(sl_2)$ [18], with generators $H, E, F, E^p/[p]!, F^p/[p]!$. Let V_n^L be the Verma module over $U_q^L(sl_2)$ with vacuum vector v_n, so that $Hv_n=(n-1)v_n$, and $Ev_n=E^p/[p]!\,v_n=0$. There is a canonical Hopf algebra homomorphism $U_q(sl_2) \to U_q^L(sl_2)$, so that V_n^L is also an $U_q(sl_2)$ module. For any H-diagonalizable $U_q(sl_2)$ module M, denote by $(M)_n$ the eigenspace of H to the eigenvalue n.

Conjecture 6.2. *If q is not a root of unity, there are isomorphisms*

$$\begin{aligned} H_r^{lf}(X_r^\varepsilon, X_r^{\varepsilon-}; L_r) &\xrightarrow{\sim} (V_{n_1} \otimes \dots \otimes V_{n_s})_{\Sigma n_i - s - 2r}, \\ H_r^{lf}(X_r^\varepsilon; L_r) &\xrightarrow{\sim} \operatorname{Ker} E \,|\, (V_{n_1} \otimes \dots \otimes V_{n_s})_{\Sigma n_i - s - 2r}. \end{aligned} \tag{6.2}$$

If q is a root of unity, there are isomorphisms

$$\begin{aligned} H_r^{lf}(X_r^\varepsilon, X_r^{\varepsilon-}; L_r) &\xrightarrow{\sim} (V_{n_1}^L \otimes \dots \otimes V_{n_s}^L)_{\Sigma n_i - s - 2r}, \\ H_r^{lf}(X_r^\varepsilon; L_r) &\xrightarrow{\sim} \operatorname{Ker} E \,|\, (V_{n_1}^L \otimes \dots \otimes V_{n_s}^L)_{\Sigma n_i - s - 2r}. \end{aligned} \tag{6.3}$$

Finally, let $Y_r = \mathscr{C}_r(\mathbf{C} \setminus \{w_1, \dots, w_s\})$ and L_r be the local system over Y_r defined by $q, n_1, \dots, n_s$.

Appendix

We summarize some known facts about $U_q(sl_2)$, following essentially [15–17]. Fix a non-zero complex number q. Let $U_q(sl_2)$ be the algebra with unit over $\mathbf{C}$ with generators E, F, H and relations

$$[H,E]=2E,$$

$$[H,F]=-2F,$$

$$[E,F]=q^H-q^{-H}.$$

We often denote $K^2=q^H$. Of course, q^H is not well-defined in the algebra but its action on modules where H takes integer values is. A more precise definition is the following: let $U(sl_2)$ be the complex algebra with unit with generators E, F, H, K^2, K^{-2} and relations

$$[H,E]=2E, \quad [H,F]=-2F, \quad K^2K^{-2}=K^{-2}K^2=1,$$
$$[E,F]=K^2-K^{-2}, \quad K^2H=HK^2.$$

$U(sl_2)$ is a $\mathbf{Z}$-graded algebra, with the assignment $\deg(E)=-\deg(F)=1$, $\deg(H)=\deg(K^{\pm 2})=0$. Let G_q be the category of $\mathbf{Z}$-graded left $U(sl_2)$-modules $M=\bigoplus_{n\in\mathbf{Z}} M_n$ such that

(i) For all $\xi\in M$ there exists an N such that $E^N\xi=0$,
(ii) $HM_n=nM_n$ and $K^2M_n=q^nM_n$.

The degree of homogeneous elements of a module in G_q is called *weight.* Following common usage, we refer to objects in G_q as ($\mathbf{Z}$-graded) $U_q(sl_2)$-modules.

Let n be an integer, and $q\in\mathbf{C}\backslash 0$. The *Verma module* V_n is the quotient of $U(sl_2)$ by the left ideal generated by E, K^2-q^{n-1} and $H-(n-1)$, with left action of $U(sl_2)$. The module V_n is in G_q and is generated by a highest weight vector v_n (= image of 1) of weight $n-1$. A basis of V_n is given by the vectors F^jv_n, $j=0,1,\ldots$, and one has the explicit formulae

$$EF^jv_n=\frac{[j][n-j]}{q-q^{-1}}F^{j-1}v_n,$$

$$HF^jv_n=(n-1-2j)F^jv_n.$$

The notation we use for q-numbers are

$$[j]\equiv[j]_q=q^j-q^{-j}, \quad \begin{bmatrix} j \\ l \end{bmatrix}=\frac{[j][j-1]\ldots[j-l+1]}{[l][l-1]\ldots[1]},$$

$$[j]!=[j][j-1]\ldots[2][1], \quad [0]!=1.$$

If q is a root of unity we define a number p as the smallest positive integer such that

$$q=e^{\pi ip'/p}$$

for some integer $p'>0$. If q is not a root of unity we set $p=\infty$.

Proposition A1.
(i) *If q is not a root of unity, V_n is irreducible for $n \leqq 0$. It contains a proper submodule SV_n generated by the singular vector*[2] *$F^n v_n$, if $n \geqq 1$. The quotient V_n/SV_n is an irreducible n-dimensional representation.*

(ii) *If $q = e^{\pi i p'/p}$ is a root of unity, then V_n contains a proper submodule SV_n generated by the singular vector $F^{\bar{n}} v_n$, where $1 \leqq \bar{n} \leqq p$ and $\bar{n} \equiv n (\operatorname{mod} p)$. The quotient V_n/SV_n is irreducible, of dimension $\bar{n}$.*

The *Shapovalov form* on V_n is the symmetric bilinear form $(\,,\,): V_n \times V_n \to \mathbf{C}$, uniquely characterized by
(i) $(v_n, v_n) = 1$,
(ii) $(E\xi, \eta) = (\xi, F\eta)$, $(H\xi, \eta) = (\xi, H\eta)$, $\xi, \eta \in V_n$.
The null space of $(\,,\,)$ is SV_n.

The action of $U(sl_2)$ on tensor products of modules in G_q is defined by the coassociative coproduct $\Delta: U(sl_2) \to U(sl_2) \otimes U(sl_2)$ defined on generators as

$$\begin{aligned}\Delta(H) &= H \otimes 1 + 1 \otimes H\,,\\ \Delta(K^{\pm 2}) &= K^{\pm 2} \otimes K^{\pm 2}\,,\\ \Delta(E) &= E \otimes 1 + K^2 \otimes E\,,\\ \Delta(F) &= F \otimes K^{-2} + 1 \otimes F\,.\end{aligned}$$

The action on tensor products with s factors is given by $\Delta^{(s)}: U(sl_2) \to U(sl_2) \otimes \ldots \otimes U(sl_2)$ with $\Delta^{(s+1)} = (\Delta^{(s)} \otimes 1)\Delta$, $\Delta^{(2)} = \Delta$. The universal R-matrix of $U_q(sl_2)$ is the formal series

$$R = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{(q-q^{-1})^k}{[k]!} q^{\frac{1}{2} H \otimes H} E^k \otimes F^k\,.$$

This series is well defined on any tensor product module in G_q since only finitely many terms are non-vanishing when R acts on a vector. Also, singular denominators cancel.

Let $(\,,\,)$ denote the product of Shapovalov forms: $(\,,\,): V_{n_1} \otimes \ldots \otimes V_{n_s} \times V_{n_1} \otimes \ldots \otimes V_{n_s} \to \mathbf{C}$.

Proposition A2. *Let $R_{i,i+1} = 1 \otimes \ldots \otimes 1 \otimes R \otimes 1 \otimes \ldots \otimes 1$ be the R-matrix acting on the i^{th} and $(i+1)^{st}$ factor in $V_{n_1} \otimes \ldots \otimes V_{n_s}$ and $P_{i,i+1}$ the transposition $\xi_1 \otimes \ldots \otimes \xi_s \mapsto \xi_1 \otimes \ldots \otimes \xi_{i+1} \otimes \xi_i \otimes \ldots \otimes \xi_s$. Then*

$$(R_{i,i+1}\xi, \eta) = (\xi, P_{i,i+1} R_{i,i+1} P_{i,i+1} \eta)$$

for all $\xi, \eta \in V_{n_1} \otimes \ldots \otimes V_{n_s}$.

Let $W_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$ be the space of singular vectors of weight $n-1$ in $V_{n_1} \otimes \ldots \otimes V_{n_s}$. The family of vector spaces $W_n(V_{\alpha(n_1)} \otimes \ldots \otimes V_{\alpha(n_s)})$, $\alpha \in S_s$ carries an R-matrix representation of the colored braid groupoid $B_{1,\ldots,1}$. As a consequence of Proposition A2 we have

Proposition A3. *Let $F_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$ be the quotient of $W_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$ by the null space $\mathcal{N}$ of $(\,,\,)$ restricted to $W_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$. The representation of $B_{1,\ldots,1}$ on*

[2] A singular vector is a vector annihilated by E

$\{W_n(V_{\alpha(n_1)}\otimes\ldots\otimes V_{\alpha(n_s)})\}_{\alpha\in S}$ *reduces to a well-defined representation on* $\{F_n(V_{\alpha(n_1)}\otimes\ldots\otimes V_{\alpha(n_s)})\}_{\alpha\in S_s}$.

The subquotient $F_n(V_{n_1}\otimes\ldots\otimes V_{n_s})$ is called the *fusion rule subquotient* of $V_{n_1}\otimes\ldots\otimes V_{n_s}$ with weight $n-1$. It can be characterized more explicitly.

Proposition A4. *Let* $p-1\geqq n_1, n_2$, $n\geqq 1$. *Then*

$$N^n_{n_1n_2}=\dim F_n(V_{n_1}\otimes V_{n_2}) = \begin{cases} 1 & \text{if } |n_1-n_2|+1\leqq n\leqq \min(n_1+n_2-1, 2p-n_1-n_2-1) \\ 0 & \text{otherwise}. \end{cases}$$

Thus, if $N^n_{n_1n_2}=1$, there is a singular vector in $V_{n_1}\otimes V_{n_2}$ of weight $n-1$ which is not in the null space of (,). Correspondingly, we have a homomorphism

$$C^n_{n_1,n_2}: V_n \to V_{n_1}\otimes V_{n_2}.$$

Suppose in the following that $p-1\geqq n_1,\ldots,n_s$, $n\geqq 1$. Introduce the *path space* $P^n_{n_1},\ldots,n_s$ as the space of complex linear combinations of sequences $(m_1,\ldots,m_{s-2})$ of integers in $[1,p-1]$ such that $N^n_{n_1,m_2}=N^{m_{i-1}}_{n_i m_i}=N^{m_{s-2}}_{n_{s-1}n_s}=1$ $(2\leqq i\leqq s-2)$.

Proposition A5. *The homomorphism*

$$P^n_{n_1,\ldots,n_s}\to W_n(V_{n_1}\otimes\ldots\otimes V_{n_s}),$$

$$(m_1,\ldots,m_{s-2})\mapsto(1\otimes\ldots\otimes 1\otimes C^{m_{s-1}}_{n_{s-1}n_s})\ldots(1\otimes C^{m_1}_{n_2m_2})C^n_{n_1m_1}v_n$$

composed with the canonical projection $W_n\to F_n$, *gives an isomorphism*

$$P^n_{n_1,\ldots,n_s}\xrightarrow{\sim} F_n(V_{n_1}\otimes\ldots\otimes V_{n_s}).$$

The proofs of the last two propositions can be extracted from [17], noticing that since the vectors of the form $\xi_1\otimes\ldots\otimes\xi_s$ with some $\xi_j\in SV_{n_j}$ are in the null space of (,), we can replace everywhere V_n by the irreducible quotient V_n/SV_n.

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