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Quantum Field-Theory Models on Fractal Spacetime

I. Introduction and Overview

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Abstract. The present work explores the possibility of giving a non-perturbative definition of the quantum field-theory models in non-integer dimensions, which have been previously studied by Wilson and others using analytic continuation of dimension in perturbation integrals. The method employed here is to base the models on fractal point-sets of non-integer Hausdorff-Besicovitch dimension. Two types of scalar-field models are considered: the one has its propagator (= covariance operator kernel) given by a proper-time or heat-kernel representation and the other has a hierarchical propagator. The fractal lattice version of the proper-time propagator is shown to be reflection-positive. The hierarchical models are introduced and their properties discussed on an informal basis.

1. Introduction

In a classic 1973 paper, "Quantum Field-Theory Models in Less Than 4 Dimensions," Wilson studied the scalar interaction ϕ^4 and Fermi-type $(G\bar{\psi}\psi)^2$ interaction for spacetime dimension between 2 and 4 [45]. His method was perturbative (although in some cases infinite classes of diagrams were summed within a $1/N$ expansion) and the integrals associated to the Feynman-graphs were extended to noninteger d by the analytical continuation procedure introduced earlier as a regularization method for gauge theories [10, 33]. Since that time the question of what non-perturbative significance might be given to these models – if any – has remained open. However, more recently Gefen, Aharony, Mandelbrot and collaborators have made a relevant investigation of the possibility of achieving statistical-mechanical spin models, with the critical properties predicted by the ε-expansion method, by employing fractal lattices [25–30]. In this paper the same method is exploited to give a non-perturbative definition of quantum field-

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theory models in non-integer dimensions. These models are interesting as theoretical toys to illuminate the possibilities of quantum field-theory, as argued by Wilson in the conclusion of his paper:

"... it seems likely that a thorough study of these models in less than four dimensions will generate new ideas about the nature of field theory that do not depend on dimensionality and may apply to four-dimensional theories as well. It should be instructive to study the behavior of high-energy scattering, deep-inelastic scattering, bound states, etc. in these models."

Wilson himself discovered a number of interesting phenomena in these models. For example, he exhibited the possibility of anomalous scaling at high energies due to a non-Gaussian ultraviolet renormalization-group fixed point. Also, he argued for the equivalence – in dimension d just less than 4 – of four-fermion $(\bar{\psi}\psi)^2$ theories and Yukawa theories when these are defined by unconventional renormalizations. It is indeed possible that these models in $4-\varepsilon$ dimensions are actually physically relevant. In [14] Crane and Smolin motivate the consideration of fractal spacetimes, or "fractal spacetime foam," by quantum gravity considerations. Elsewhere the relevance of such models to problems of particle physics, particularly the Higgs sector of GUT's models, is considered [17]. However, in the present paper and its companion [19] we shall simply introduce and study two scalar-field "Euclidean" theories on fractal point-sets. We hope by this investigation to have demonstrated that fractal sets provide an effective and feasible method of realizing quantum field-theory in non-integer dimensions, at least for the scalar theories. The case of higher-spin fields, e.g. gauge fields and fermions, pose a vastly more difficult problem. Although some progress can be made on this problem by purely formal considerations [18], at this time there is really no rigorous framework for introducing spin structure into fractal sets. It is not clear therefore whether the fractal continuation applies to all the physical theories for which the usual analytical continuation is effective. The plan of this paper is as follows: in the following Sect. 2 the fractal sets employed in this work – which are amenable to a rigorous mathematical treatment – are introduced and discussed. In Sect. 3, scalar field theory models on fractal sets with propagator given by a proper-time or heat-kernel representation are defined. It is shown that this approach leads to actual quantum-mechanical models with positive-norm Hilbert space. The scaling and renormalizability properties of these models are then discussed. In the final Sect. 4, hierarchical-type scalar field-theory models on fractal sets are introduced, for which the renormalization problem can be solved in a rigorous fashion, employing the large-field and analyticity techniques of Gawedzki and Kupiainen [22–24]. We content ourselves here with an informal discussion of the model and its analysis. Precise statements of results and complete, rigorous proofs for the hierarchical model approximation are contained in the companion paper [19].

2. Fractal Spacetimes

The purpose of this section is to introduce and briefly discuss the fractal point-sets employed in this paper and also to set the notations.

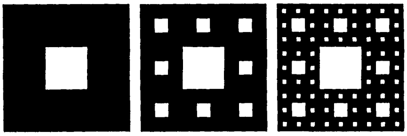

Fig. 1. Three stages in the construction of a Sierpinski carpet $\mathbb{F}^0(d, L, \mathscr{C})$, with $d=2$, $L=3$ and $\mathscr{C}=\{(1,1)\}$. The set has fractal dimension $\bar{d}=\log 8/\log 3$. The origin is labelled as 0

The most important class of fractal sets for our purposes are the *Sierpinski carpets* [42], which are two-dimensional generalizations of the well-known Cantor middle-thirds set. As a simple case, consider the set obtained from the unit square $[0,1]^2$ by partition into 9 subsquares and removal of the central open subsquare, repetition of the same operation on the 8 remaining closed subsquares, etc.

Generalization is possible both with respect to the Euclidean dimension d, the (integral) scale factor L, the edge-length L^N of the initial hypercubes, and the set $\mathscr{C}$ of sub-hypercubes removed at each stage. The various sub-hypercubes are labelled by d-component vectors with components ranging from 0 to $L-1$. We denote the corresponding fractal set by $\mathbb{F}^N(d, L, \mathscr{C})$. It has Hausdorff dimension

$$\bar{d}=\log(L^d-C)/\log L, \tag{2.1}$$

where $C=|\mathscr{C}|$. We allow the possibility that N is infinite.

The Sierpinski "hypercarpet" $\mathbb{F}^N(d, L, \mathscr{C})$ has an analytic description which is extremely useful. A point $x\in\mathbb{R}^d$ belongs to $\mathbb{F}^N(d, L, \mathscr{C})$ if and only if *for at least one* base-L expansion of x,

$$x=\sum_{k=-\infty}^{N-1} x_k L^k, \qquad x_k=\mathbb{Z}_L^d\equiv\{0,\dots,L-1\}^d, \tag{2.2}$$

none of the coefficients x_k belong to the distinguished subset $\mathscr{C}\subset\mathbb{Z}_L^d$. From the previous geometric description it is obvious that $\mathbb{F}^N(d, L, \mathscr{C})$ is closed, since it is a countable intersection of closed sets. From the analytic description it is obvious that it is perfect as well. Since $\mathbb{F}^N(d, L, \mathscr{C})$ is a closed, perfect and, therefore, uncountable set, it may plausibly be described as a continuum. For $d\geqq 2$ and reasonable choices of $\mathscr{C}$, it is possible also to see that the hypercarpet is at least path-connected. There are also discrete approximations to these sets, so-called *lattice Sierpinski hypercarpets*. The simplest of these is defined by

$$\mathbb{F}_M^N(d, L, \mathscr{C})\equiv\mathbb{F}^N(d, l, \mathscr{C})\cap L^{-M}\mathbb{Z}^d. \tag{2.3}$$

An example is illustrated in Fig. 2.

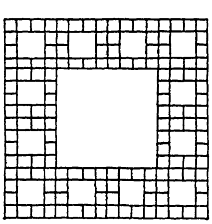

Fig. 2. The lattice Sierpinski carpet $\mathbb{F}_M^N(d, L, \mathscr{C})$ for $N=2$, $M=0$, and $d=2$, $L=4$ and $\mathscr{C}=\{(1,1), (1,2), (2,1), (2,2)\}$

An analytic description of this lattice is that it consists of those points $n \in L^{-M} \cdot \mathbb{Z}^d$ such that *for at least one base-L* expansion of n,

$$n = \sum_{k=-M}^{N} n_k L^k, \qquad n_k \in \mathbb{Z}_L^d, \tag{2.4}$$

none of the coefficients n_k belong to $\mathscr{C}$ and the only possible components of n_N are 0's and 1's. This lattice description associates lattice sites to the *vertices* of the elementary hypercubes of edge-length L^{-M}. An alternate lattice description is to associate lattice sites to the center of each elementary hypercube of edge-length L^{-M}. This lattice is dual to the previous one and is denoted $^*\mathbb{F}_M^N(d, L, \mathscr{C})$. An analytical description of this set is that it consists of those points $n \in L^{-M}(^*\mathbb{Z}^d) \equiv L^{-M}(\mathbb{Z}^d + (1/2, \ldots, 1/2))$ such that in its representation of the form

$$n = \sum_{k=-M}^{N-1} n_k L^k + L^{-M}(1/2, \ldots, 1/2) \tag{2.5}$$

none of the coefficients n_k belong to $\mathscr{C}$. An example is illustrated in Fig. 3.

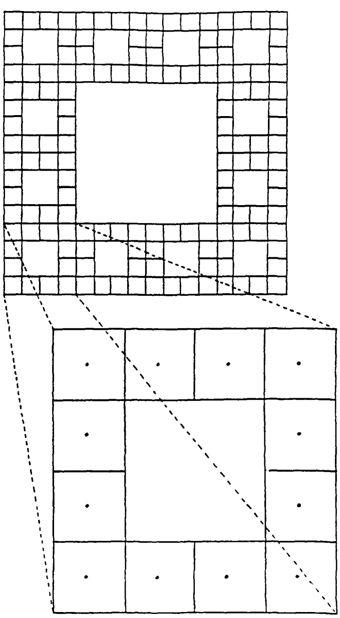

Fig. 3. The fractal lattice $^*\mathbb{F}_M^N(d, L, \mathscr{C})$ for $N=2$, $M=0$, and $d=2$, $L=4$, $\mathscr{C} = \{(1,1), (1,2), (2,1), (2,2)\}$. A block of sites is distinguished in outset

Another class of fractals which is useful for examples if not for serious work is the *Sierpinski hypergaskets* in Euclidean d-space, $d \geqq 2$, generalizations of the original Sierpinski gasket in $d=2$ [41]. The construction of the members of this class is similar to that of the Sierpinski carpet except that it is based upon the unit d-simplex rather than upon the unit d-hypercube. Partitioning the unit d-simplex in $\mathbb{R}^d$ into $(d+2)$ sub-simplices of edge-length 1/2, one removes the open central sub-simplex, repeats this same operation on the $(d+1)$ remaining closed sub-simplices, etc. Clearly, the d-dimensional hypergasket has

$$\bar{d} = \log(d+1)/\log 2\,. \tag{2.6}$$

An example is illustrated in Fig. 4.

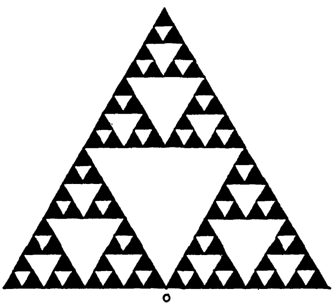

Fig. 4. Fourth stage in the construction of the Sierpinski gasket $\mathbb{G}^N(d)$, $d=2$, $N=0$. $\mathbb{G}^N(d)$ has finite volume but an infinite volume version $\mathbb{G}^d$ can be obtained by dilation through the origin 0. $\bar{d}=\log 3/\log 2$

The (hyper)gaskets are also clearly closed, perfect, connected, uncountable pointsets. The gasket in $\mathbb{R}^d$ with largest edge-length 2^N will here be denoted by $\mathbb{G}^N(d)$. The gaskets have also discrete approximations, the *lattice Sierpinski hypergaskets* $\mathbb{G}^N_M$ with largest edge-length 2^N and shortest edge-length 2^{-M}, consisting of all the vertices of the elementary d-simplices of edge-length 2^{-M} in the $(M+N)^{\text{th}}$ stage of construction of the gasket from an initial d-simplex of edge-length 2^N. An example is illustrated in Fig. 5.

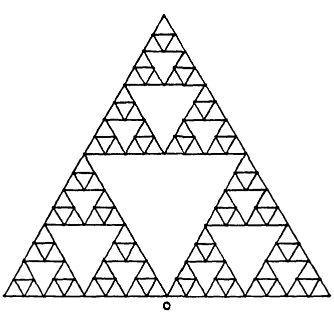

Fig. 5. Lattice version $\mathbb{G}^N_M(d)$ of the Sierpinski gasket: $d=2$, $M=4$, $N=0$

It is already apparent from these few examples that the fractal continuation of dimension is highly non-unique: there are many fractal sets in $\mathbb{R}^d$ with the same Hausdorff dimension $\bar{d}$. In the studies of Gefen, Aharony and Mandelbrot (GAM) it was shown that the critical properties of statistical systems on fractal lattices depend, in general, on these additional geometric properties. Of particular importance is the *order of ramification* at P, defined by Urysohn and Menger [36], or [35, Chap. 14], as the minimum number of points which must be removed from a set in order to disconnect an arbitrarily large or small, bounded set of points connected to P. In the case of the above examples, it is not difficult to see that the gaskets have finite order of ramification at every point while the carpets have infinite order of ramification at every point. Fractals with finite order of ramification at every point are essentially quasi-linear. It was found by GAM that phase transitions occur on finitely-ramified lattices only at $T=0$, as one would expect from a Peierls droplet argument. It is for this reason essentially that the carpets are interesting for field-theoretical construction, whereas the gaskets are only a toy example.

The infinite-volume versions of both the carpets and the gaskets, $\mathbb{F}(d, L, \mathscr{C})$ and $\mathbb{G}(d)$, are invariant under a group of *discrete scale transformations* through the origin O:

$$L^k \cdot \mathbb{F}(d, L, \mathscr{C}) = \mathbb{F}(d, L, \mathscr{C}), \tag{2.7}$$

$$2^k \cdot \mathbb{G}(d) = \mathbb{G}(d). \tag{2.8}$$

This makes the models based on these sets particularly convenient to study by renormalization group methods. However, the price is lack of translation invariance. Clearly, any set $\mathscr{S} \subseteqq \mathbb{R}^d$ which is simultaneously (1) closed, (2) invariant under a scale transformation $L>1$, and (3) invariant under an n-dimensional lattice group of translations generated by a set $\{v_1, \ldots, v_n\}$ of linearly independent vectors in $\mathbb{R}^d (n \leqq d)$ can be written as a Cartesian product $\mathscr{S}' \times \langle v_1, \ldots, v_n \rangle$, where $\langle v_1, \ldots, v_n \rangle$, the linear span of $\{v_1, \ldots, v_n\}$, is an n-dimensional hyperplane isomorphic to $\mathbb{R}^n$, and $\mathscr{S}'$ is some subset of the orthogonal complement $\mathbb{R}^d \ominus \langle v_1, \ldots, v_n \rangle$. In the above cases there is no remnant of translation-invariance. For a physical interpretation it is sometimes convenient in the case of the carpets to keep translation invariance in one direction – the "time-direction" – so that one may define a Hamiltonian or transfer-matrix. One can therefore introduce "*Hamiltonian*" *carpets*, distinct from the "*Euclidean*" *carpets* previously defined, as

$$\mathbb{F}_M(D, L, \mathscr{C}) \times L^{-M}\mathbb{Z} \tag{2.9}$$

in the lattice case, or

$$\mathbb{F}(D, L, \mathscr{C}) \times \mathbb{R}, \tag{2.10}$$

in the continuum case. The latter set has Hausdorff dimension

$$\bar{d} = \bar{D} + 1, \qquad \bar{D} = \log(L^D - C)/\log L. \tag{2.11}$$

Another symmetry which it is desirable to restore is reflection-symmetry, i.e. invariance under reflection through a coordinate hyperplane, the "time-zero" hyperplane in the case of the Hamiltonian lattice. For this purpose, one introduces the fractal sets obtained from $\mathbb{F}^N_M(d, L, \mathscr{C})$ by reflection through the d coordinate hyperplanes, denoted simply by $\mathbb{F}^N_M(\bar{d})$, where d is given by (2.1). In the Hamiltonian case, one considers $\mathbb{F}^N_M(\bar{D}, 1) \equiv \mathbb{F}^N_M(\bar{D}) \times (L^{-M}\mathbb{Z} \cap [-L^N, +L^N])$ for the lattice version or $\mathbb{F}^N(\bar{D}, 1) \equiv \mathbb{F}^N(\bar{D}) \times [-L^N, +L^N]$ for the continuum. See Figs. 6 and 7 below.

One should observe from Eqs. (2.1) and (2.11) that the carpets allow one to discuss the approach back to integer dimension $d=4$. Indeed, if one takes $L \nearrow +\infty$ but holds C fixed in (2.1), then

$$\bar{d} = d + \log(1 - C/L^d)/\log L \nearrow d. \tag{2.12}$$

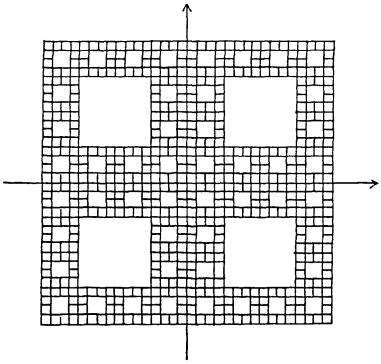

Fig. 6. A portion of the "Euclidean" lattice Sierpinski carpet $\mathbb{F}_0^{\bar{d}}$, $\bar{d} = \log 12/\log 4$, constructed from $\mathbb{F}_0(d, L, \mathscr{C})$ with $d=2$, $L=4$, $\mathscr{C}=\{(1,1), (1,2), (2,1), (2,2)\}$. The two coordinate hyperplanes (axes) are distinguished by heavy lining. The portion of the lattice shown is just $\mathbb{F}_0^{\bar{d}} \cap \Lambda^{(2)}$, $\Lambda^{(2)} \equiv [-L^2, +L^2]^d = [-4^2, +4^2]^2$

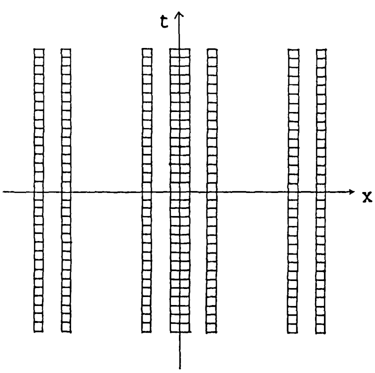

Fig. 7. A portion of the "Hamiltonian" lattice Sierpinski carpet $\mathbb{F}_0^{\bar{D}} \times \mathbb{Z}^1$, $\bar{D} = \frac{1}{2}$, constructed from $\mathbb{F}_0(D, L, \mathscr{C})$ with $D=1$, $L=4$, $\mathscr{C}=\{1,2\}$. The time axis is labelled "t" and the coordinate hyperplane orthogonal to the time axis (here, the spatial axis) is labelled "x". The portion of the lattice shown is just the intersection with $\Lambda^{(2)} \equiv [-L^2, +L^2]^d = [-4^2, +4^2]^2$

It is also easy to arrange that $\mathbb{F}^{\bar{d}} \nearrow \mathbb{R}^4$: e.g. choose $L_n = R^{2^n}$, $R > 1$ and the set $\mathscr{C}_n \subseteq \mathbb{Z}_{L_n}^{\bar{d}}$, so that at each stage of construction the central hypercube of B^d elementary hypercubes is removed ($B + D$ even). Since GAM found that the results of the ε-expansion are reproduced when the lacunarity $\mathscr{L}$ (a measure of the spatial inhomogeneity of the fractal) is taken to zero, it may be desirable to arrange this. $\mathscr{L} \nearrow 0$ can be accomplished as $\bar{d} \nearrow d$ by spreading the removed hypercubes evenly throughout the initial hypercube. For details see [27].

A final feature of each of the above continuum fractal sets is that they admit a *d-dimensional Hausdorff measure* $H^{\bar{d}}$ on a σ-algebra of measurable sets $\mathscr{M}$ (which includes the Borel sets in the subspace topology), σ-finite in the infinite-volume cases [20, 40]. This measure has the simple scaling property

$$H^{\bar{d}}(sE) = s^{\bar{d}} H^{\bar{d}}(E), \qquad s > 0, \qquad E \in \mathscr{M}. \tag{2.13}$$

The integral of an $H^{\bar{d}}$-measurable function f with respect to $H^{\bar{d}}$ over a fractal set $\mathbb{F}$ will be denoted simply by

$$\int_{\mathbb{F}} f(x) d^{\bar{d}}x. \tag{2.14}$$

We observe that where the set $\mathbb{F}$ has scale-invariance, $L \cdot \mathbb{F} = \mathbb{F}$, for some $L \in \mathbb{Z}^+$, the integral obeys a simple scaling law

$$\int_{\mathbb{F}} f(Lx) d^{\bar{d}}x = L^{-\bar{d}} \int_{\mathbb{F}} f(x) d^{\bar{d}}x. \tag{2.15}$$

Although we do not actually employ this fact, we remark that fractal lattice sums

$$\sum_{n \in \mathbb{F}_M^N(\bar{d})} L^{-M\bar{d}} f(n) \tag{2.16}$$

converge in the limit $M \to +\infty$ to the integrals with respect to $H^{\bar{d}}$ in (2.14), when the function is continuous with compact support: see [5, 6].

We may now compare the usual analytic continuation of dimension with the fractal continuation of dimension considered here, particularly as accomplished with the class of Sierpinski hypergaskets. In the first place, it is clear that a field-theory on a fractal continuum will have ultraviolet divergences, since no points of the set are isolated, whereas the analytic continuation is a perturbative regularization. As we shall see later, the expected divergences on the fractal set do indeed occur. Secondly, the fractal continuation in general destroys translation invariance and, generally, Euclidean symmetry, except the time-translation invariance restored in the "Hamiltonian" case. Of course, the analytic continuation formally preserved these spacetime symmetries. On the other hand, the behavior under scale transformations is similar. The transformation law like (2.15) is one of the properties postulated by Wilson in his axiomatic definition of the analytically-continued integration [45]. The chief difference here is that the scale transformations are restricted to those in a discrete group. With respect to positivity properties the fractal continuation is superior: the Lebesgue-Hausdorff integral (2.14) is a positive operation, whereas, as noted by Wilson, the formally defined d-dimensional integration need not be. Finally, both methods allow one to take the limit as $d \nearrow 4$ and, furthermore, one can expect some restoration of the effects of Euclidean symmetry in the case of the fractal continuation in that limit.

To conclude this section we wish just in passing to point out the possibility of using *random fractals* to achieve quantum field-theory models in non-integer dimensions, similar in spirit to the random-lattice technique of Christ, Friedberg, and Lee [14]. Such an approach would maintain more spacetime symmetry than a fixed lattice and allow a continuous adjustment of dimension as well.

3. The Standard Models (The Proper-Time Propagator)

With the preliminaries completed, we can now begin to consider certain quantum field-theory models defined on the fractal sets introduced above. The class of models we consider in this section are a natural analogue of the standard scalar quantum field-theories in ordinary Euclidean space. The propagator (covariance operator kernel) of these models is given by a proper-time or heat-kernel representation. We here recall this representation of the ordinary Euclidean scalar propagators both for the continuum and lattice cases [43, 31, 11]

Continuum

$$T = e^{\Delta/2} \tag{3.1a}$$

$$e^{m^2/2} \tag{3.2a}$$

$$(-\Delta + m^2)^{-1} = \int_0^\infty \frac{d\tau}{2} e^{-m^2\tau/2} T^\tau \tag{3.3a}$$

$$G^d(x, y; m) = \int_0^\infty \frac{d\tau}{2} e^{-m^2\tau/2} P_\tau^d(x, y) \tag{3.4a}$$

Lattice

$$T = 1 + \Delta/2d \tag{3.1b}$$

$$z^{-1} = 1 + m^2/2d \tag{3.2b}$$

$$(-\Delta + m^2)^{-1} = \frac{1}{2d} \sum_{N=0}^{\infty} z^{N+1} T^N \tag{3.3b}$$

$$G_0^d(n,m;z) = \frac{1}{2d} \sum_{N=0}^{\infty} z^{N+1} P_N^d(n,m) \tag{3.4b}$$

In the continuum column, Δ represents the usual Laplacian $\sum_{i=1}^{d} \partial_i^2$, $G^d(x,y;m)$ the propagator $\langle x|(-\Delta+m^2)^{-1}|y\rangle$, and $P_\tau^d(x,y)$ the heat-diffusion kernel $\langle x| e^{\tau\Delta/2}|y\rangle$. The adjacent column gives the lattice analogues. T is the Markov transition probability (or stepping) matrix of the symmetric random walk on $\mathbb{Z}^d$: for all $n, m \in \mathbb{Z}^d$,

$$T_{nm} = \begin{cases} 1/2d & \text{if} \quad |n-m| = 1 \\ 0 & \text{otherwise} \end{cases} . \tag{3.5}$$

Now Δ is the lattice Laplacian operator: for $f \in L^2(\mathbb{Z}^d)$,

$$(\Delta f)(n) = \sum_{m:\,|m-n|=1} f(m) - 2df(n). \tag{3.6}$$

Finally, $G_0^d(n,m;z)$ is the lattice propagator $\langle n|(-\Delta+m^2)^{-1}|m\rangle$ and $P_N^d(n,m) = \langle n|\, T^N |m\rangle$ the matrix of N-step transition probabilities.

There is no difficulty to introduce fractal lattice analogues of the proper-time representations and we shall do this first, establishing certain relevant properties, such as the reflection-positivity. To introduce propagators on fractal continua by means of a proper-time representation requires the proper mathematical definition of "Brownian-motion processes" on these objects, a problem now much investigated [5, 6, 32, 34]. We have fewer hard results in this case, but we shall discuss it briefly with a view of explaining the significance for quantum field-theories of known and conjectured results.

If T is a transition probability matrix on any of the unit-spacing fractal lattices $\mathbb{F}_0^d$ earlier introduced, then the bounded operator $T: l^2(\mathbb{F}_0^{\bar{d}}) \to l^2(\mathbb{F}_0^{\bar{d}})$ defined by

$$(Tf)(n) = \sum_{m \in \mathbb{F}_0^{\bar{d}}} T_{nm} f(m), \qquad f \in l^2(\mathbb{F}_0^{\bar{d}}) \tag{3.7}$$

is easily seen to be a contraction. If one further requires that T be *symmetric*,

$$T_{nm} = T_{mn}, \qquad m, n \in \mathbb{F}_0^{\bar{d}}, \tag{3.8}$$

then T is self-adjoint, so that, if (3.1b) is made a definition,

$$-\Delta \equiv 2d(1-T), \tag{3.9}$$

it follows that the operator $-\Delta$ is positive, self-adjoint on $l^2(\mathbb{F}_0^{\bar{d}})$. Therefore the definition is a reasonable one. Also, if T is a nearest-neighbor RW,

$$|n-m| > 1 \Rightarrow T_{nm} = 0, \tag{3.10}$$

then $-\Delta$ has the same property, i.e. it is *local.* The set of symmetric, nearest-neighbor RWs on the fractal lattices introduced above, $\mathbb{F}_0(\bar{d})$ and $\mathbb{G}_0(d)$, is not empty. In the case of $\mathbb{G}_0(d)$, every site of the lattice has precisely $2d$ nearest neighbors (see Fig. 5). Therefore, one may define

$$T_{nm}=\begin{cases}1/2d & |n-m|=1\\ 0 & \text{otherwise}\ .\end{cases} \tag{3.11}$$

In the case of $\mathbb{F}_0(\bar{d})$, points not on the boundary of a cut-out section (such as the point I indicated in Figs. 6 or 7) have $2d$ nearest neighbors, whereas points on a boundary (such as point B in Fig. 6 or 7) have $2d-1$ nearest neighbors. We therefore distinguish classes $\mathscr{I}$ of internal points and $\mathscr{B}$ of boundary points and set

$$T_{nm}=\begin{cases}1/2d & |n-m|=1\\ 1/2d & n=m\in\mathscr{B}\\ 0 & \text{otherwise}\ ,\end{cases} \tag{3.12}$$

so that a random walker at a boundary site has probability $1/2d$ of remaining at that site. This defines a symmetric, nearest-neighbor (in the sense of (3.10)) RW on $\mathbb{F}_0(\bar{d})$. Each of these RW's is obviously invariant under a large class of local isometries of the fractal lattices $\mathbb{G}_0(d)$ and $\mathbb{F}_0(\bar{d})$. We further observe that the RW on $\mathbb{F}_0(\bar{d})$ is invariant under the d reflections in the coordinate hyperplanes, in particular, in the Hamiltonian case, under reflection through the "time-zero" hyperplane. That is, if θ is reflection through the coordinate hyperplane H, then

$$T_{\theta n,\theta m}=T_{n,m}\,. \tag{3.13}$$

We refer to each of these as the *standard* RW on $\mathbb{G}_0(d)$ or $\mathbb{F}_0(\bar{d})$.

Now, in terms of any of these RW's we may introduce a lattice propagator $G_0^{\bar{d}}$ by means of the analogue of (3.3b):

$$G_0^{\bar{d}}\equiv\frac{1}{2d}\sum_{N=0}^{\infty} z^{N+1}T^N \tag{3.14}$$

or (3.4b)

$$G_0^{\bar{d}}(n,m;z)\equiv\frac{1}{2d}\sum_{N=0}^{\infty} z^{N+1}P_N^{\bar{d}}(n,m)\,, \tag{3.15}$$

with $P_N^d(n,m)\equiv\langle n|\,T^N\,|m\rangle$. In each case, the series converges absolutely for $z<0$ (or $m^2>0$) and $G_0^{\bar{d}}=(-\Delta+m^2)^{-1}$. Simple exponential decay properties and pointwise positivity of $G_0^{\bar{d}}$ are immediate. It is important also that, for the carpet case, $\mathbb{F}_0(\bar{d})$, this definition yields a *reflection-positive* $G_0^{\bar{d}}$, since this permits the quantum-mechanical models to be defined with this propagator, with positive-norm Hilbert-space and self-adjoint transfer matrix. The proof of this property proceeds along rather standard lines [31]. We first consider *operator monotonicity* of $G_0^{\bar{d}}$ with respect to the introduction of Dirichlet boundary conditions. Let Λ be the union of an arbitrary finite set of elementary hypercubes with vertices in $\mathbb{Z}^d$ and define

$$\mathbb{F}_0^\Lambda(\bar{d})\equiv\mathbb{F}_0(\bar{d})\cap\Lambda\,, \tag{3.16}$$

$$\mathrm{Int}\,\mathbb{F}_0^\Lambda(\bar{d})\equiv\mathbb{F}_0(\bar{d})\cap\mathrm{Int}\,\Lambda\,, \tag{3.17}$$

$$\partial\mathbb{F}_0^\Lambda(\bar{d})\equiv\mathbb{F}_0(\bar{d})\cap\partial\Lambda\,. \tag{3.18}$$

Note in particular that with $\Lambda^N \equiv [-L^N, +L^N]$,

$$\mathbb{F}_0^{\Lambda^N}(\bar{d}) = \mathbb{F}_0^N(\bar{d}). \tag{3.19}$$

For any Λ, the Laplacian on $\mathbb{F}_0^\Lambda(\bar{d})$ with Dirichlet data on $\partial\mathbb{F}_0^\Lambda(\bar{d})$, or the *Dirichlet Laplacian*, Δ_Λ, is defined by

$$\Delta_\Lambda \equiv \Pi_{\mathrm{Int}\,\mathbb{F}_0^\Lambda(\bar{d})} \Delta \Pi_{\mathrm{Int}\,\mathbb{F}_0^\Lambda(\bar{d})}, \tag{3.20}$$

where $\Pi_{\mathrm{Int}\,\mathbb{F}^\Lambda}$ is the projection onto $l^2(\mathrm{Int}\,\mathbb{F}_0^\Lambda(\bar{d}))$ as a subspace of $l^2(\mathbb{F}_0(\bar{d}))$. Clearly, $-\Delta_\Lambda \geqq 0$ for every Λ. In the same way, one introduces the covariance operator on $\mathbb{F}_0^{\bar{d}}$ with Dirichlet data on $\partial\mathbb{F}_0^\Lambda(\bar{d})$, or the *Dirichlet covariance* $G_{0,\Lambda}^{\bar{d}}$ as

$$G_{0,\Lambda}^{\bar{d}} \equiv \Pi_{\mathrm{Int}\,\mathbb{F}_0^\Lambda(\bar{d})} G_0^{\bar{d}} \Pi_{\mathrm{Int}\,\mathbb{F}_0^\Lambda(\bar{d})}. \tag{3.21}$$

Then we observe the following

Proposition 1. *For* $\Lambda_1 \subseteqq \Lambda_2$, $G_{0,\Lambda_1}^{\bar{d}} \leqq G_{0,\Lambda_2}^{\bar{d}}$.

The proof is standard [31] and proceeds by interpolating between boundary conditions on $\partial\Lambda_1$ and $\partial\Lambda_2$ by introducing a local mass perturbation $M\bar{\Pi}_{\mathrm{Int}\,\mathbb{F}_0^{\Lambda_1}(\bar{d})}$ of $(G_{0,\Lambda_2}^{\bar{d}})^{-1}$ and taking $M \to +\infty$. With this result on operator monotonicity we can now establish the reflection-positivity. If θ is the reflection through H earlier discussed and Λ is a union of elementary hypercubes with $\theta\Lambda = \Lambda$, then we define $\Theta : l^2(\mathbb{F}_0(\bar{d}) \to l^2(\mathbb{F}_0(\bar{d}))$ by

$$(\Theta f)(n) = f(\theta n) \tag{3.22}$$

and use the same symbol to denote its restriction to $l^2(\mathbb{F}_0^\Lambda(\bar{d}))$. It follows from (3.13) that

$$T\Theta = \Theta T \tag{3.23}$$

and, therefore, similar statements hold with T replaced by Δ, $G_0^{\bar{d}}$, Δ_Λ or $G_{0,\Lambda}^{\bar{d}}$. We have

Proposition 2. *Let T be a symmetric, nearest-neighbor, θ-invariant* RW *on* $\mathbb{F}_0(\bar{d})$ *(e.g. the standard* RW*). Then for $\Lambda = \Lambda^N$ or for any other θ-invariant Λ, $\theta\Lambda = \Lambda$, $G_{0,\Lambda}^{\bar{d}}$ is reflection-positive, i.e.*

$$\Pi_\pm \Theta G_{0,\Lambda}^{\bar{d}} \Pi_\pm \geqq 0, \tag{3.24}$$

where $\Pi_\pm$ are the projections onto the subspaces $l^2(\mathbb{F}_0^\Lambda(\bar{d}))$ of $l^2(\mathrm{Int}\,F_0^\Lambda(\bar{d}))$ supported on or above (respectively on or below) the hyperplane H.

Proof. Before we begin the proof proper, we note that we are proving here a version of reflection-positivity for reflection through a hyperplane H containing sites. If desired, it is not difficult to modify the definition of $\mathbb{F}(\bar{d})$ so that it is invariant under reflections in d hyperplanes, each parallel to and midway between two lattice hyperplanes, and similar proofs carry through in that case. We define, as implicitly above,

$$\Lambda = \{\text{points of } \mathrm{Int}\Lambda \text{ on or above (respectively on or below) } H\}, \tag{3.25}$$

$$\Lambda_\pm^0 = \{\text{points of } \mathrm{Int}\,\Lambda \text{ above (respectively below) } H = \Lambda_\pm / H\}, \tag{3.26}$$

and let $\Pi_{\pm}$, $\Pi_{\pm}^{0}$, Π_H be corresponding projection operators in $l^2(\mathrm{Int}\,\mathbb{F}^{\Lambda}(\vec{d}))$. Since

$$\Pi_{\pm}\Theta G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}=\Pi_{\pm}[G_{0,\Lambda}-(1-\Theta)G_{0,\Lambda}^{\vec{d}}]\Pi_{\pm}\,, \tag{3.27}$$

it suffices to show that

$$\Pi_{\pm}(1-\Theta)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}=\Pi_{\pm}G_{0,\Lambda/H}^{\vec{d}}\Pi_{\pm}\,, \tag{3.28}$$

for then (3.24) follows by the operator monotonicity, $G_{0,\Lambda}^{\vec{d}}-G_{0,\Lambda/H}^{\vec{d}}\geqq 0$. In fact, both the left-hand side and right-hand side of (3.28) are equal to $G_{0,\Lambda_{\pm}^{0}}^{\vec{d}}$.

For the right-hand side this is obvious since $G_{0,\Lambda/H}^{\vec{d}}=\bar{\Pi}_H G_{0,\Lambda}^{\vec{d}}\bar{\Pi}_H$, and thus

$$\Pi_{\pm}G_{0,\Lambda/H}^{\vec{d}}\Pi_{\pm}=\Pi_{\pm}^{0}G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}^{0}=G_{0,\Lambda_{\pm}^{0}}^{\vec{d}}\,. \tag{3.29}$$

For the left-hand side, note first that

$$\Pi_{\pm}(1-\Theta)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}=\Pi_{\pm}^{0}(1-\Theta)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}^{0}\,, \tag{3.30}$$

since

$$\Pi_H(1-\Theta)=(1-\Theta)\Pi_H=0\,. \tag{3.31}$$

Therefore, to show that

$$\Pi_{\pm}(1-\Theta)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}=G_{0,\Lambda_{\pm}^{0}}^{\vec{d}}\,, \tag{3.32}$$

it only remains to establish that

$$\Pi_{\pm}^{0}(-\Delta+m^2)\Pi_{\pm}^{0}(1-\Theta)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}^{0}=\Pi_{\pm}^{0}\,. \tag{3.33}$$

We now invoke the *locality* of Δ_{Λ}, which implies

$$\Pi_{\pm}^{0}\Delta_{\Lambda}\Pi_{\mp}^{0}=0\,. \tag{3.34}$$

Also, we again use (3.31). Thus,

$$\begin{aligned}&\Pi_{\pm}^{0}(-\Delta_{\Lambda}+m^2)\Pi_{\pm}^{0}(1-\Theta)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}^{0}\\&=\Pi_{\pm}^{0}(-\Delta_{\Lambda}+m^2)(\Pi_{+}^{0}+\Pi_{-}^{0}+\Pi_H)(1-\Theta)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}^{0}\\&=\Pi_{\pm}^{0}(1-\Theta)(-\Delta_{\Lambda}+m^2)G_{0,\Lambda}^{\vec{d}}\Pi_{\pm}^{0}\quad\text{since}\quad[\Theta,\Delta_{\Lambda}]=0\\&=\Pi_{\pm}^{0}(1-\Theta)\Pi_{\pm}^{0}=\Pi_{\pm}^{0}\,,\end{aligned} \tag{3.35}$$

which is (3.33). This gives (3.32), completes the identification (3.28), and ends the proof. □

Before turning to the continuum models, let us just briefly comment on the fractal lattice field theories associated to these propagators. In the finite volume case, these have Gibbs measures of the form

$$d\mu_{v,G_{0,\Lambda^N}^{\vec{d}}}(\phi)=Z^{-1}\exp\Big[-\sum_{n\in\mathrm{Int}\,\mathbb{F}_0^N(\vec{d})}v(\phi(n))\Big]\,d\mu_{G_{0,\Lambda^N}^{\vec{d}}}(\phi)\,, \tag{3.36}$$

where $\mu_{G_{0,\Lambda^N}^{\vec{d}}}$ is the Gaussian measure on $\mathbb{R}^{|\mathrm{Int}\,\mathbb{F}_0^N(\vec{d})|}$ with covariance $G_{0,\Lambda^N}^{\vec{d}}$ and Z normalizes the measure to unity:

$$Z^{-1}=\int_{\mathbb{R}^{|\mathrm{Int}\,\mathbb{F}_0^N(\vec{d})|}}\exp\Big[-\sum_{n\in\mathrm{Int}\,\mathbb{F}_0^N(\vec{d})}v(\phi(n))\Big]\,d\mu_{G_{0,\Lambda^N}^{\vec{d}}}(\phi)\,. \tag{3.37}$$

On the other hand, this is obviously the same as

$$d\mu_{v, G_{0,\Lambda^N}^{\bar{d}}}(\phi) = Z^{-1}\exp\left[+\sum_{n,m\in \mathrm{Int}\,\mathbb{F}_0^N(\bar{d})} J_{nm}\phi(n)\phi(m)\right]\prod_n dv(\phi(n)), \tag{3.38}$$

with v the spin-weight measure

$$dv(\phi) = e^{(2d+m^2)\phi^2 - v(\phi)}d\phi, \tag{3.39}$$

and

$$J_{nm} = d\cdot T_{nm}. \tag{3.40}$$

We see therefore that these are nearest-neighbor, ferromagnetic continuous-spin models, similar to the Ising spin models on fractal lattices investigated by GAM. It is expected from simple renormalization-group considerations if $\bar{d} = 4-\varepsilon$, $\varepsilon > 0$, and if the lacunarity $\mathscr{L}$ goes to zero as $\varepsilon \searrow 0$, then the critical properties of these models should be governed by a Wilson-Fisher non-Gaussian fixed point as predicted by the ε-expansion [44]. It would be interesting to test this conjecture numerically by determining critical exponents of fractal lattice Ising models with $\bar{d} = 4-\varepsilon$ using the Monte Carlo renormalization-group method, such as was already done some time ago by Blote and Swendson for the $d=4$ Ising model [11]. This would not only test our theoretical understanding of these models, but also it would be a check whether non-integer dimensions can be effectively realized for computer simulation by the fractal continuation method.

There is much less that can be said with certainty about the continuum models, because the diffusion processes on fractal spaces, in terms of which the continuum propagator would be defined by the analogue of (3.4a):

$$G^{\bar{d}}(x, y; m) = \int_0^\infty \frac{d\tau}{2} e^{-m^2\tau/2} P_\tau^{\bar{d}}(x, y), \tag{3.41}$$

are only now beginning to be rigorously investigated [5, 6, 32, 34] (for earlier non-rigorous studies see [4, 38, 39]). We shall not have much to contribute to these efforts, only outlining an approach to the construction of such processes and illustrating it with a simple existence result for the Sierpinski gasket. We then go on to consider the field-theoretic implications of the known and conjectured results.

On a scaling fractal $\mathbb{F}^{\bar{d}}$ with a discrete group of scale invariances, $s = L^k$, $k\in\mathbb{Z}$, where $L\cdot\mathbb{F}^{\bar{d}} = \mathbb{F}^{\bar{d}}$, it is natural to conjecture that the propagator $G^{\bar{d}}$ obeys a scaling law

$$G^{\bar{d}}(\sigma^\theta x, \sigma^\theta y; \sigma^{-1}m) = \sigma^{-\tilde{d}+2}G^{\bar{d}}(x, y; m), \tag{3.42}$$

analogous to the corresponding scaling law for the standard Euclidean propagator G^d, with $\tilde{d} = d$, $\theta = 1$. For our purposes, the exponent $\tilde{d}-2$ in this scaling law defines the *spectral dimensionality* associated to the heat-diffusion process on the fractal. This dimensionality, originally introduced in [3, 38], characterizes a variety of properties of physical systems on fractal sets. We note that generally $\tilde{d} \neq \bar{d}$. The exponent θ, however, is not an independent parameter, if one assumes for the heat-diffusion kernel a scaling law also, which leads via (3.41) to (3.42):

$$P_{\sigma^2\tau}^{\bar{d}}(\sigma^\theta x, \sigma^\theta y) = \sigma^{-\tilde{d}}P_\tau^{\bar{d}}(x, y). \tag{3.43}$$

Since, however, one has the normalization condition

$$\int_{\mathbb{F}^{\bar{d}}} d^{\bar{d}}x P_\tau^{\bar{d}}(x,y) = 1\,, \qquad \tau > 0\,, \tag{3.44}$$

it follows that

$$\theta = \tilde{d}/\bar{d}\,. \tag{3.45}$$

A more constructive way of stating these scaling conjectures is as existence of the *scaling limit* of the lattice Green's function pointwise for

$$G^{\bar{d}}(x,y;m) = \lim_{M\to\infty} L^{M(\bar{d}-2/\theta)} G_0^{\bar{d}}([\![L^M x]\!], [\![L^M y]\!]; z^{(M)})\,, \tag{3.46}$$

for some choices of $\tilde{d}$ and θ, where also

$$(z^{(M)})^{-1} \equiv 1 + \frac{1}{2d} m^2 L^{-2M/\theta}\,, \qquad m^2 > 0; \tag{3.47}$$

or, likewise, existence of the limit

$$P_\tau^{\bar{d}}(x,y) = \lim_{M\to\infty} L^{M\bar{d}} P_{[\![d\cdot L^{2M/\theta}\tau]\!]}([\![L^M x]\!], [\![L^M y]\!])\,, \tag{3.48}$$

for $\tau > 0$. In these formulas, $[\![x]\!]$ is the element of $\mathbb{F}_0^{\bar{d}}$ nearest to $x \in \mathbb{F}_0^{\bar{d}}$ according to some conventional assignment. Obviously, these limits, if they exist and are non-trivial for some $\tilde{d}$, θ, imply (3.42), (3.43).

In the case of the Sierpinski gasket we can carry out a version of this scaling limit and establish the scaling law (3.42). This relation as well as the existence of the heat-diffusion process and many fine sample-path properties have already been established in [5]. However, since the Sierpinski carpet case appears more difficult, it may be useful to record an alternative approach. The present proof is based upon an exact renormalization-group equation for the lattice Green's function, $G_0(n,m;z)$, due to Rammal [39]. We simply quote here the result (for $d=2$):

$$G_0(n,m;z') = \zeta^{-1}(z) G_0(2n,2m;z) \tag{3.49}$$

with

$$z' = z^2/(4-3z) \tag{3.50}$$

and

$$\zeta^{-1}(z) = z(2+z)/(4+z)(2-z)\,. \tag{3.51}$$

The RG transformation (3.50) has two fixed points on the unit interval $[0,1]$, a stable fixed point $z^* = 1$ and an unstable fixed point $z^* = 0$. For $z = 1-\varepsilon$, $0 < \varepsilon \ll 1$, one obtains

$$\varepsilon' = 5\varepsilon + O(\varepsilon^2)\,. \tag{3.52}$$

Let us make a conventional assignment $[\![x]\!]$ as follows: every point $x \in \mathbb{G}_0^2$ belongs to a unique elementary triangle T_x of $\mathbb{G}_0^2$ if $x \notin \mathbb{G}_0^2$ and to two adjacent triangles if $x \in \mathbb{G}_0^2$. We define

$$[\![x]\!] = \begin{cases} x & \text{if} \quad x \in \mathbb{G}_0^2 \\ \text{highest point of } T_x & \text{if} \quad x \notin \mathbb{G}_0^2\,. \end{cases} \tag{3.53}$$

Now, on the basis of the RG equation (3.49), (3.50) one expects the limit

$$\begin{aligned} G(x, y; m) &= \lim_{M\to\infty} 2^{M\left(\hat{a}-\frac{2}{\theta}\right)} G_0(\llbracket 2^M x\rrbracket, \llbracket 2^M y\rrbracket; z^{(M)}) \\ &= \lim_{M\to\infty} \left(\frac{3}{5}\right)^M G_0(\llbracket 2^M x\rrbracket, \llbracket 2^M y\rrbracket; z^{(M)}) \end{aligned} \tag{3.54}$$

with

$$z^{(M)} \equiv \left(1 + \frac{m^2}{2d} 5^{-M}\right)^{-1}, \tag{3.55}$$

to exist and be nontrivial. In fact, the proof of the limit requires a slight modification of (3.54), (3.55) to

$$G(x, y; m) = \lim_{M\to+\infty} \zeta_M^{(M)} \cdot G_0(\llbracket 2^M\rrbracket, \llbracket 2^M\rrbracket; z_M^{(M)}) \tag{3.56}$$

with

$$z_M^{(M)} = \left(1 + \frac{m^2}{2d} 5^{-M} + \tilde{\varepsilon}_M^{(M)}\right)^{-1}, \qquad \tilde{\varepsilon}^{(M)} = O(m^4 5^{-2M}) \tag{3.57}$$

and

$$\zeta_M^{(M)} = (\tfrac{3}{5})^M (1 + \tilde{\eta}_M^{(M)}), \qquad \tilde{\eta}^{(M)} = O(m^2 5^{-M}). \tag{3.58}$$

Iteration of (3.49) gives, for $k \leqq M$,

$$\zeta_M^{(M)} \cdot G_0(\llbracket 2^M x\rrbracket, \llbracket 2^M y\rrbracket; z_M^{(M)}) = \zeta_k^{(M)} \cdot G_0(\llbracket 2^k x\rrbracket, \llbracket 2^k y\rrbracket; z_k^{(M)}), \tag{3.59}$$

where

$$\begin{aligned} z_{l-1}^{(M)} &= (z_l^{(M)})^2/(4 - 3z_l^{(M)}), \\ \zeta_{l-1}^{(M)} &= \zeta_l^{(M)} \cdot \zeta(z_l^{(M)}) = \zeta_l^{(M)} \cdot (4 + z_l^{(M)})(2 - z_l^{(M)})/z_l^{(M)}(2 + z_l^{(M)}). \end{aligned} \tag{3.60a, b}$$

If M, k are sufficiently large, then $z_l^{(M)} = (1 + z_l^{(M)})^{-1}$, $0 < \varepsilon_l^{(M)} \ll 1$, for all $1, k \leqq l \leqq M$, and the iteration equations simplify to

$$\varepsilon_{l-1}^{(M)} = 5\varepsilon_l^{(M)} + O((\varepsilon_l^{(M)})^2) \tag{3.61}$$

and

$$\zeta_{l-1}^{(M)} = \zeta_l^{(M)} \cdot \tfrac{5}{3}(1 + O(\varepsilon_l^{(M)})). \tag{3.62}$$

Defining $\tilde{\varepsilon}_l^{(M)}$, $\tilde{\eta}_l^{(M)}$, generally for $k \leqq l \leqq M$,

$$\varepsilon_l^{(M)} = \frac{m^2}{2d} 5^{-l} + \tilde{\varepsilon}_l^{(M)}, \tag{3.63}$$

$$\zeta_l^{(M)} = (\tfrac{3}{5})^l (1 + \tilde{\eta}_l^{(M)}), \tag{3.64}$$

it is now possible by a "fine-tuning" of $\tilde{\varepsilon}_M^{(M)}$, $\tilde{\eta}_M^{(M)}$ to achieve the bounds

$$|\tilde{\varepsilon}_l^{(M)}| \leqq B m^4 5^{-2l} \tag{3.65}$$

and

$$|\tilde{\eta}_l^{(M)}| \leqq C m^2 5^{-l}, \tag{3.66}$$

as well as existence of the limits

$$z_k = \lim_{M \to +\infty} z_k^{(M)} \tag{3.67}$$

and

$$\zeta_k = \lim_{M \to +\infty} \zeta_k^{(M)}. \tag{3.68}$$

This strategy is exhaustively discussed in the companion paper and we refer the reader to that discussion to avoid needless reduplication of arguments. Therefore, with the analyticity of the resolvent kernel for $|z| < 1$, the existence of the limit (3.56) follows:

$$\begin{aligned} G(x, y; m) &= \lim_{M \to +\infty} \zeta_k^{(M)} \cdot G_0(\llbracket 2^k x \rrbracket, \llbracket 2^k y \rrbracket; z_k^{(M)}) \\ &= \zeta_k \cdot G_0(\llbracket 2^k x \rrbracket, \llbracket 2^k y \rrbracket; z_k). \end{aligned} \tag{3.69}$$

One can then also use the lattice resolvent identity to argue that the limit obeys the appropriate scaling law:

$$\begin{aligned} &G(2x, 2y; 5^{1/2} m) - \frac{5}{3} G(x, y; m) \\ &= \lim_{k \to +\infty} \left(\frac{3}{5}\right)^k \left[\begin{aligned} &G_0\left(\llbracket 2^{k+1} x \rrbracket, \llbracket 2^{k+1} y \rrbracket; \left(1 + \frac{m^2}{4} 5^{-(k+1)} + \tilde{\varepsilon}_k \right)^{-1} \right) \\ &- G_0\left(\llbracket 2^{k+1} x \rrbracket, \llbracket 2^{k+1} \rrbracket; \left(1 + \frac{m^2}{4} 5^{-(k+1)} + \tilde{\varepsilon}_{k+1} \right)^{-1} \right) \end{aligned} \right] \\ &= \lim_{k \to +\infty} \left(\frac{3}{5}\right)^k (\tilde{\varepsilon}_{k+1} - \tilde{\varepsilon}_k) \times \sum_{p \in \mathbb{G}_0^2} G_0(\llbracket 2^{k+1} x \rrbracket, p; z_{k+1}) G_0(p; \llbracket 2^{k+1} y \rrbracket; z_k) \\ &= \lim_{k \to +\infty} \left(\frac{3}{5}\right)^k O(m^4 5^{-2k}) 3^{k+1} \sum_{z \in \mathbb{G}_{k+1}^2} 3^{-k+1} G_0(\llbracket 2^{k+1} \rrbracket, \llbracket 2^{k+1} \rrbracket; z_{k+1}) \\ &\quad \times G_0(\llbracket 2^{k+1} z \rrbracket, \llbracket 2^{k+1} y \rrbracket; z_k) \\ &= \lim_{k \to +\infty} O(m^4 5^{-k}) \int_{\mathbb{G}^2} d^{\bar{d}} z G(x, z; m) G(2z, 2y; 5^{-1/2} m) = 0, \end{aligned} \tag{3.70–73}$$

assuming that the Green's function is such that the lattice sum is at least majorized by the continuum integral. Therefore,

$$G(2x, 2y; 5^{1/2} m) = \tfrac{5}{3} G(x, y; m), \tag{3.74}$$

as expected.

To establish similar results in the case of the Sierpinski carpet, which is infinitely ramified, appears, however, to be much harder. At the time of writing, verification of the scaling law of the Green's function on the carpets is an apparently open question.

Let us therefore simply discuss the consequences of the scaling law (3.42) in those cases where it can be verified. We observe that (3.42) implies for the free scalar field with covariance $G^{\bar{d}}$ a scaling dimension

$$d_\phi = (\bar{d} - 2)/2\theta = (\bar{d}/2) - \theta^{-1}. \tag{3.75}$$

There is no difficulty at a formal level in setting up a perturbation theory for theories with polynomial interactions $v(\phi)=\sum_N (\lambda_N/N!)\phi^N$ and applying the standard power-counting arguments. One finds as the condition for superrenormalizability of the ϕ^N interaction that

$$Nd_\phi<\bar{d}, \tag{3.76}$$

or

$$N(\tilde{d}-2)/2\theta<\tilde{d}/\theta, \tag{3.77}$$

i.e.,

$$N(\tilde{d}-2)/2<\tilde{d}. \tag{3.78}$$

This is precisely the familiar superrenormalizability criterion, but with the Euclidean dimension replaced by the spectral dimension: rather surprisingly, the renormalizability is determined by the spectral dimension $\tilde{d}$ rather than the fractal dimension $\bar{d}$.

One point which should be emphasized is that there are ultraviolet divergences in these fractal theories. For example, by power-counting one finds that the integral corresponding to the self-energy graph in the ϕ^4 theory has superficial degree of divergence $\omega=2(\tilde{d}-3)/\theta$ and is therefore divergent for $\tilde{d}\geqq 3$. In contrast, the analytically continued integral corresponding to this graph is a meromorphic function of the complex dimension parameter d with poles only at $d=4,5,6,\ldots$: in particular, it is finite for $3<d<4$. Therefore, the fractal continuation is not a regularization as is the analytic continuation. On the other hand, one expects that the model on a fractal space is perturbatively renormalizable for $\tilde{d}<4$. At present, this appears difficult to demonstrate. Nonperturbatively, one expects that the ϕ^4 theories have only trivial continuum (scaling) limits when $\tilde{d}>4$ (and the coupling is positive) [1, 2, 21], whereas for $\tilde{d}<4$ there should be two families of solutions: a two-parameter family of superrenormalizable, asymptotically-free solutions and a one-parameter family of solutions with a non-Gaussian UV fixed point [45]. The explicit construction of the theories with non-Gaussian fixed points for $\tilde{d}<4$ is outlined in the following section for a simplified choice of the propagator.

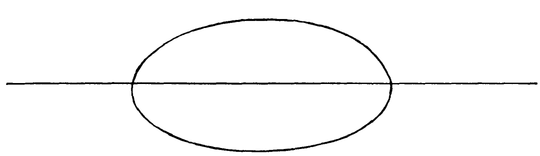

Fig. 8. See text

4. The Hierarchical Models: Informal Discussion

The model we deal with here is a hierarchical one, such as was first introduced by Dyson [15, 16], in a version which has been much studied in the last few years by Gawedzki and Kupiainen [22–24]. The model is based in our case, however, on a unit lattice Sierpinski carpet of the dual form $*\mathbb{F}_0^N(d,L,\mathscr{C})$ (see (2.5) and Fig. 3), which is particularly adapted for block-spin [46] renormalization-group (RG) transformations. The Gibbs measures are therefore given by

$$d\mu_{w,G_0}(s) = Z^{-1} \exp\Big[- \sum_{n \in {}^*\mathbb{F}_0^N} w(s_n) \Big] d\mu_{G_0}(s) \tag{4.1}$$

and μ_{G_0} is the Gaussian measure on $\mathbb{R}^{|{}^*\mathbb{F}_0^N|}$ with mean O and covariance G_0:

$$G_0(n,m) = \sum_{k=0}^{\infty} L^{-\alpha k} \delta_{[\![L^{-(k+1)}n]\!],[\![L^{-(k+1)}m]\!]} A(n_k) A(m_k). \tag{4.2}$$

We choose $L^d - C = L^{\bar{d}}$ to be even. In (4.2), n_k, m_k denote k^{th} place coefficients in base-L expansions of n, m as in (2.5). $[\![\cdot]\!]$ denotes the d-integer part of an element of $\mathbb{R}^d$, and A is a function on $\mathbb{Z}_L^d/\mathscr{C}$ with values in $\{\pm 1\}$, each value taken $(L^d - C)/2$ number of times. If we introduce the usual BS-field,

$$s'_m = (Bs)_m = L^{(\alpha/2) - \bar{d}} \sum_{n : [\![L^{-1}n]\!] = m} s_n, \tag{4.3}$$

and note the decompositions

$$d\mu_{G_0}(s) = d\mu_{G_0}(s') \prod_{m \in {}^*\mathbb{F}_0^{N-1}} d\mu_1(Z_m^0) \tag{4.4}$$

with

$$s_n = L^{-\alpha/2} s'_{[\![L^{-1}n]\!]} + A(n_0) Z^0_{[\![L^{-1}n]\!]}, \tag{4.5}$$

then integration of fluctuation fields Z^0 leads to the following recursion formula for the spin weight-exponent $w(s)$:

$$e^{-w'(s)} = \int \exp[-\tfrac{1}{2} L^{\bar{d}} (w(L^{-\alpha/2}s + z) + w(L^{-\alpha/2}s - z))] d\mu_1(z) / \int \exp[-L^{\bar{d}} w(z)] d\mu_1(z). \tag{4.6}$$

This is just the celebrated *approximate recursion formula* of Wilson [46], as it arises in the usual Euclidean case, except that here the Hausdorff-Besicovitch or fractal dimension $\bar{d}$ appears rather than the Euclidean dimension d. This is the crucial point for the analysis of these models. In addition to the trivial Gaussian fixed point $w(s) \equiv 0$, it was discovered by Wilson and Fisher [44] and subsequently rigorously verified [7, 8, 13, 22] that the recursion formula has for $d = 4 - \varepsilon$ a second non-Gaussian fixed point

$$w^*(s) = a^* s^2 + \lambda^* s^4 + \tilde{w}^*(s) \tag{4.7}$$

with $\lambda^* = O(\varepsilon)$. This fixed point governs the critical behavior of the model for $\bar{d} < 4$. As an aside, we remark that formula (3.6) already shows that the critical properties of hierarchical models on lattice Sierpinski carpets depend only upon the Hausdorff-dimension. This is in contrast to the Ising models on fractal lattices studied by GAM [25, 27–30], whose critical exponents depend strongly on a host of geometrical properties of the lattice but in agreement with a recent conjecture of Penrose [37], that a simple form of universality might hold for statistical models with long-range interactions on a fractal.

However, our interest here is in the renormalization problem of quantum field theory, which Wilson has related to the RG transformation of unit lattice statistical theories [46]. As preliminary to the rigorous discussion of this problem in the follow-up paper and to set our notation, let us consider the RG flows of the model in a formal, perturbative approach. It is convenient to represent the RG transformation (4.6) in the form:

$$w'(s)=y'(s)-y'(0)\,, \tag{4.8}$$

$$y'(s)=-\log\int d\mu(z)\exp[-L^{\bar{d}}q(s,z)]\,, \tag{4.9}$$

$$q(s,z)=\tfrac{1}{2}[w(L^{-\alpha/2}s+z)+w(L^{-\alpha/2}s-z)]\,. \tag{4.10}$$

Then, y' may be written as a cluster expansion

$$y'(s)=L^{\bar{d}}\langle q(s,\cdot)\rangle-\frac{1}{2!}L^{2\bar{d}}\langle q^2(s,\cdot)\rangle^T+\frac{1}{3!}L^{3\bar{d}}\langle q^3(s,\cdot)\rangle^T+\dots\,, \tag{4.11}$$

where the expectations are with respect to $d\mu(z)=e^{-z^2/2}dz/(2\pi)^{1/2}$. We take our initial spin weight exponent w in the form

$$w(s)=\frac{1}{2!}r:s^2:_{G_0}+\frac{1}{4!}u:s^4:_{G_0}+\frac{1}{6!}t:s^6:_{G_0}\,. \tag{4.12}$$

Here, $:F(s):_{G_0}$ denotes the normal-ordering with respect to covariance G_0:

$$:F[s]:_{G_0}\equiv\exp\left[-\frac{1}{2}\sum_{n,m\in{}^*\mathbb{F}_0^N}G_0(n,m)\partial^2/\partial s_n\partial s_m\right]F[s]\,. \tag{4.13}$$

We drop the subscript G_0 as convenient.

In this form, it is easy to see that the RG transformations diagonalizes to first-order:

$$L^{\bar{d}}\langle q\rangle=\frac{1}{2}L^2:s^2:+\frac{1}{4!}L^{\varepsilon}u:s^4:+\frac{1}{6!}L^{-2(1-\varepsilon)}t:s^6:. \tag{4.14}$$

The second-order contribution is also easily calculated from (3.9–12) and yields

$$y'(s)=y'(0)+\frac{1}{2!}r':s^2:+\frac{1}{4!}u':s^4:+\frac{1}{6!}t':s^6:+\dots \tag{4.15}$$

with

$$r'=L^2[r-(\alpha_2u^2+\alpha_1ru+\alpha_3r^2+R_2)]\,, \tag{4.16}$$

$$u'=L^{\varepsilon}[u-(\beta_2u^2+\beta_1ru+U_2)]\,, \tag{4.17}$$

$$t'=L^{-2(1-\varepsilon)}[t-(\delta_2u^2+T_2)]\,. \tag{4.18}$$

The ... in (4.15) represents higher-order induced terms, like $\frac{1}{8!}h':s^8:$, which, however, on assumptions on the initial weight (4.12), will be seen to be negligible. R_2, U_2, T_2 are homogeneous, second-order multinomials in r, u, t representing quadratic contributions other than those distinguished. The coefficients α_1, α_2, α_3, β_1, β_2, δ_2, etc. are all $O(L^{\bar{d}})$ as $L\nearrow+\infty$ and, in particular, $\alpha_1>0$, $\beta_2>0$.

Let us assume initially that

$$u=O(\varepsilon L^{-\bar{d}}\log L)\,, \tag{4.19}$$

$$r=O(\varepsilon L^{-\bar{d}}\log L)\,, \tag{4.20}$$

$$t=O(\varepsilon^2L^{-\bar{d}}\log L)\,. \tag{4.21}$$

It is clear from (4.17), (4.18) that (4.19), (4.21) are preserved under iteration. Also, we note that higher power couplings, like h', are $O(\varepsilon^3 L^{-\bar{d}} \log^3 L)$ at least. Because of the relevancy of the variable r, we cannot automatically conclude that $r' = O(\varepsilon L^{-\bar{d}} \log L)$. However, this is true for r located initially on the critical hypersurface, $r = r_c(u,t)$. The critical surface must pass through $(r,u,t) = (0,0,0)$ and, since $t = O(L^{\bar{d}} u^2)$, we conclude that $r_c(u,t) = O(u)$. By this observation the RG equations for r, u decouple from those of the other variables to $O(L^{2\bar{d}} u^3)$: in particular, terms like $\alpha_4 rt = O(L^{2\bar{d}} u^3)$ may be neglected. If we parameterize the critical hypersurface by:

$$r_c(u) = r_{1c} u + r_{2c} u^2 + O(L^{2d} u^3) \tag{4.22}$$

and use the stability condition

$$r_c(u)' = r_c(u) + O(L^{3\bar{d}} u^3), \tag{4.23}$$

with (4.22) and (4.16–17), we find that

$$r_{1c} \equiv 0 \tag{4.24}$$

and

$$L^2(r_{2c} - \alpha_2) = r_{2c} L^{2\varepsilon} \quad \text{or} \quad r_{2c} = (1 - L^{-2(1-\varepsilon)})^{-1} \alpha_2 . \tag{4.25}$$

Only a single interaction equation is now of concern:

$$u' = L^{\varepsilon}[u - \beta_2 u^2 + O(L^{2\bar{d}} u^3)] . \tag{4.26}$$

The fixed points of (4.26), (4.22) in the chosen domain (4.19–21) are then seen to be $(r,u) = (0,0) = O$ and $(r^*, u^*) = F$ with

$$u^* = \beta_2^{-1}(1 - L^{-\varepsilon}) + O(\varepsilon^2 L^{-\bar{d}} \log^2 L) = O(\varepsilon L^{-\bar{d}} \log L), \tag{4.27}$$

$$r^* = r_{2c}(u^*)^2 + O(\varepsilon^3 L^{-\bar{d}} \log^3 L) = O(\varepsilon^2 L^{-\bar{d}} \log L). \tag{4.28}$$

The linearized RG flow diagram for the model in the small-coupling domain is sketched in Fig. 9.

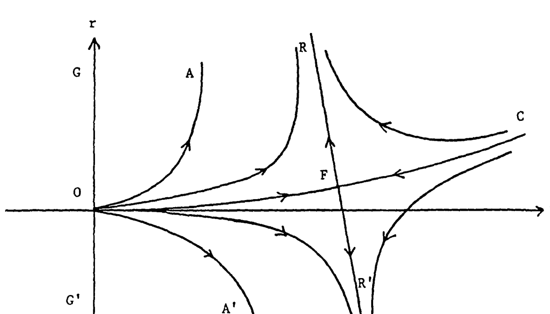

Fig. 9. The RG flow of the hierarchical model for r, $u = O(\varepsilon L^{-\bar{d}} \log L)$. The Gaussian fixed point, 0, the $W-F$ fixed point, F, and the critical hypersurface, OFC, are indicated

Now, as usual, if we consider the "Euclidean" hierarchical scalar field-theory on $\mathbb{F}_M^N$ with path measure

$$dy_{v,G_M}(\phi)=\exp\left[-\sum_{n\in{}^*\mathbb{F}_M^N} L^{-M\bar{d}}v(\phi(n))\right]d\mu_{G_M}(\phi)$$

$$\Big/\int_{\mathbb{R}^{|{}^*\mathbb{F}_M^N|}}\times\exp\left[-\sum_{n\in{}^*\mathbb{F}_M^N} L^{-M\bar{d}}v(\phi(n))\right]d\mu_{G_M}(\phi), \tag{4.29}$$

where

$$G_M(n,m)=\sum_{k=-M}^{\infty} L^{-\alpha k}\delta_{[\![L^{-(k+1)}n]\!],[\![L^{-(k+1)}m]\!]}A(n_k)A(m_k), \tag{4.30}$$

one can relate the Green functions of (4.29) with the statistical correlation functions of (4.1) by a simple rescaling

$$\phi(n)=L^{M\alpha/2}s_{L^Mn}, \tag{4.31}$$

with the definition

$$w(s)\equiv L^{-M\bar{d}}v(L^{M\alpha/2}s). \tag{4.32}$$

One finds

$$G^{(M)}(x_1,\dots,x_p;v,N)=L^{M\alpha p/2}C([\![L^Mx_1]\!],\dots,[\![L^Mx_p]\!];w,N+M). \tag{4.33}$$

Clearly, the renormalization problem, to find a sequence of cutoff dependent interactions $v^{(M)}$ so that the limit of the left-hand side of (4.33) exists as $M\nearrow+\infty$, is equivalent to the problem of finding a sequence of spin-weights $w^{(M)}$, so that the *scaling limit*, $M\nearrow+\infty$, of the right-hand side exists. It is also useful to reformulate the renormalization problem in terms of the Green's functions $G_k^{(M)}(n_1,\dots,n_p;v^{(M)},N)$ of block-averaged fields

$$\Phi_k^{(M)}(n)=\frac{1}{L^{-k\bar{d}}}\sum_{m\in\square_k^{(M)}(n)} L^{-M\bar{d}}\phi(m)=L^{-(M-k)\bar{d}}\sum_{m:[\![L^km]\!]=n}\phi(m), \tag{4.34}$$

defined as

$$G_k^{(M)}(n_1,\dots,n_p;v^{(M)},N)\equiv\int_{\mathbb{R}^{|{}^*\mathbb{F}_M^N|}}\Phi_k^{(M)}(n_1)\dots\Phi_k^{(M)}(n_p)d\mu_{v,G_M}(\phi). \tag{4.35}$$

Again, the limit as $M\nearrow+\infty$ is required to exist [24].

As realized by Wilson, the existence of the limits in (4.33) or (4.35) is guaranteed by the appropriate approach of the sequence $w^{(M)}$ to the critical hypersurface and by certain scaling relations for the correlation functions, which are a consequence of the RG having a fixed point and a simple form near the fixed point like (4.16–17) (see [46, Sect. 7.2]). In the hierarchical models discussed here the situation is simplified because one can easily derive such a scaling relation without any assumption about fixed points. If $[\![L^{-1}m_i]\!]\neq[\![L^{-1}m_j]\!]$ for $i\neq j$, one finds that

$$C(m_1,\dots,m_p;w,N)=L^{-\alpha p/2}C([\![L^{-1}m_1]\!],\dots,[\![L^{-1}m_p]\!];w',N-1) \tag{4.36}$$

and, by iteration, for $-N\leqq k\leqq 0$,

$$C(m_1,\dots,m_p;w,N)=L^{k\alpha p/2}C([\![L^km_1]\!],\dots,[\![L^km_p]\!];w_k,N+k) \tag{4.37}$$

so long as $[\![L^k m_i]\!] \neq [\![L^k m_i]\!]$ for $i \neq j$, where w_k is the result of $(-k)$ RG transformations of w. Let $Q = Q(x_1, \ldots, x_p)$ be the least integer such that $[\![L^Q x_i]\!] \neq [\![L^Q x_j]\!]$ for all. Then

$$L^{M\alpha p/2} C([\![L^M x_1]\!], \ldots, [\![L^M x_p]\!]; w^{(M)}, N+M) = L^{k\alpha p/2} C([\![L^k x_1]\!], \ldots, [\![L^k x_p]\!]; w_k^{(M)}, N+k), \tag{4.38}$$

for all $k, M \geqq k \geqq Q$. Clearly, the scaling limit of the right-hand side of (4.33) exists if it is possible to choose $w^{(M)} \equiv w_M^{(M)}$ in such a fashion that, for all $k \geqq 0$, the limits $w_k = \lim_{M \to +\infty} w_k^{(M)}$ exist, and, in that case,

$$\begin{aligned} G(x_1, \ldots, x_p; N) &\equiv \lim_{M \to +\infty} G^{(M)}([\![L^M x_1]\!], \ldots, [\![L^M x_p]\!]; v^{(M)}, N) \\ &= L^{k\alpha p/2} C([\![L^k x_1]\!], \ldots, [\![L^k x_p]\!]; w_k, N+k). \end{aligned} \tag{4.39}$$

The same condition guarantees the convergence as $M \to +\infty$ of (4.35) since

$$G_k^{(M)}(n_1, \ldots, n_p; v^{(M)}, N) = L^{k\alpha p/2} C(L^k n_1, \ldots, L^k n_p; w_k^{(M)}, N+k). \tag{4.40}$$

There is considerable latitude in choosing the sequence $w^{(M)}$. In the first place, according to Wilson's ideas, each of the RG trajectories emanating from a fixed point corresponds to a quantum field-theory. In Fig. 9, there are several such renormalized trajectories: O, OG, OA, OF, OG', OA' and also F, FR, FR'. The flowlines OG, OG' correspond to the trajectory of Gaussian theories and the flowlines like OA, OF, OA' are asymptotically-free, superrenormalizable theories. In our work, we are concerned with the theories corresponding to the trajectories FR, FR', with the non-Gaussian UV fixed point, and the massless, scale-invariant theory corresponding to the fixed point F itself. Furthermore, for each of those renormalized theories, there are a variety of renormalization cut-off prescriptions $w^{(M)}$ which suffice to yield that theory. In this work, we adopt essentially the choice

$$w^{(M)} = w_{cr}^{(M)} + \theta_R L^{-M/\nu} w_d, \tag{4.41}$$

where $w_{cr}^{(M)}$ is a sequence on the critical hypersurface tending as $M \to +\infty$ to $w_F = w^*$, w_d is the eigenfunction of the linearized RGT at F for the leading eigenvalue $L^{M/\nu}$, and θ_R is a renormalized coupling corresponding to the strength of the interaction w_d in the renormalized theory at unit scale $k=0$.

The rigorous construction of the various hierarchical field-theory models, particularly the theories with the non-Gaussian UV fixed points, is the subject of the companion paper [19].

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