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Einstein's Equations near Spatial Infinity*

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Abstract. A new class of space-times is introduced which, in a neighbourhood of spatial infinity, allows an expansion in negative powers of a radial coordinate. Einstein's vacuum equations give rise to a hierarchy of linear equations for the coefficients in this expansion. It is demonstrated that this hierarchy can be completely solved provided the initial data satisfy certain constraints.

1. Introduction and Motivation

Minkowski space, the arena of special relativity, has a much richer structure "at infinity" than flat Riemannian spaces. One can move to infinity along timelike, spacelike and null lines, which cannot be mapped into each other by Poincaré transformations. Correspondingly one can investigate the behaviour of fields on Minkowski space in different asymptotic regions.

Light cones, or more generally null hypersurfaces, are characteristic hypersurfaces for hyperbolic equations, constructed geometrically from the Minkowski metric. (The scalar-wave equation, Maxwell's equations and the Yang–Mills equations are important examples.) The radiation contained in such fields propagates to infinity along the bicharacteristics of such hypersurfaces, i.e. null lines. For this reason, Bondi et al. [1] in their studies of radiation considered expansions of the type

$$\Phi = \frac{{}^{0}\psi(u,\theta,\phi)}{r} + \frac{{}^{1}\psi(u,\theta,\phi)}{r^2} + \cdots, \tag{1.1}$$

where $u = t - r$ and (r, θ, ϕ) are standard polar coordinates. If Φ satisfies the scalar-wave equation $\Box\Phi = \eta^{\mu\nu}\Phi_{,\mu\nu} = 0(\eta^{\mu\nu} = \mathrm{diag}(-\,+\,+\,+), \mu,\nu = 0,1,2,3)$, one gets a recursion relation for the coefficients of the expansion (1.1)

$$-2n\frac{\partial^n\psi}{\partial u} = \{L^2 + n(n-1)\}^{n-1}\psi, \quad n \geqq 1, \tag{1.2}$$

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where L^2 is the Laplace operator on the unit sphere. Hence specifying ${}^0\psi$ arbitrarily, all ${}^n\psi$ are determined up to quadrature.

Bondi et al. tried very successfully to generalize from linear fields on Minkowski space to source-free solutions of Einstein's equations. They looked at space times, whose metric allows an expansion of the type (1.1) along null hypersurfaces of the metric! Further work by Penrose [2] and many others led to the concept of future— and past—null infinity ($\mathscr{I}^+, \mathscr{I}^-$) as a definition of asymptotic flatness. Quite recently Friedrich [3] showed that there exists a large class of solutions having the properties Bondi et al. assumed.

In Minkowski space $\mathscr{I}^+$ and $\mathscr{I}^-$ "meet at spacelike infinity," which can be described by a point I^0 in the conformal extension of Minkowski space. Static solutions like the Schwarzschild metric show that the "point I^0" loses some essential properties which it has in the flat case. The physical reason for this is the long range $1/r$ potential of the gravitational field. Ashtekar and Hansen [4] developed a framework which assigns a structure to spacelike infinity called AEFANSI. The essential idea is to represent $\mathscr{I}^+, \mathscr{I}^-$ and spacelike infinity in a unified picture. Up to now, however, it is not clear, whether this picture is compatible with the field equation when radiation is present and the ADM-mass is non-zero.

Other investigations [5], in which linear fields on a Schwarzschild background are considered, cast some doubt whether the assumptions one would like to make near I^0, such as the simultaneous existence of both $\mathscr{I}$'s, are compatible with the field equations. Hence there is some interest in the generic behaviour of solutions of the vacuum field equations near spacelike infinity.

In [12] Ashtekar proposes a definition of spacelike infinity, which contains essentially no assumptions about existence of and behaviour at $\mathscr{I}$. Technicalities aside, Ashtekar's approach is equivalent to our treatment for "the first order in $1/r$."

To explain our approach in more detail, let us go back to the wave equation on Minkowski space. Using 4-dimensional "polar coordinates"

$$\rho^2 = \eta_{\mu\nu} x^\mu x^\nu, \qquad \frac{t}{r} = \tan\chi, \tag{1.3}$$

(r,θ,ϕ) polar coordinates in $t = \text{constant}$, the metric is $(\phi^a) = (\chi,\theta,\phi)$,

$$\eta_{\mu\nu} dx^\mu dx^\nu = d\rho^2 + \rho^2 {}^0h_{ab} d\phi^a d\phi^b = d\rho^2 + \rho^2(-d\chi^2 + \cos^2 h\chi d\Omega^2). \tag{1.4}$$

The curves $\phi_a = \text{const}$, $\rho \to \infty$ are spacelike geodesics going to spacelike infinity. In analogy with (1.1) we look for solutions of $\Box \Phi = 0$, which admit an expansion of the form

$$\Phi = \frac{{}^1\Phi(\chi,\theta,\phi)}{\rho} + \frac{{}^2\Phi(\chi,\theta,\phi)}{\rho^2} + \cdots. \tag{1.5}$$

Calculating $\Box$ in "polar coordinates" and inserting (1.5), one finds

$$ {}^0h^{ab} D_a D_b {}^n\Phi + n(n-2) {}^n\Phi = 0, \tag{1.6}$$

where D_a is the covariant derivative associated with ${}^0h_{ab}$. The system decouples completely! One finds an invariant wave equation on the hyperboloid described by the metric ${}^0h_{ab}$. This is not surprising, because the invariant wave operator always factorizes in generalized polar coordinates! Consequently, one has solutions to the wave equation of the form

$$\Phi = \frac{{}^nA\left(\frac{t}{r},\theta,\phi\right)}{r^n} = \frac{{}^n\Phi\left(\frac{t}{r},\theta,\phi\right)}{\rho^n}. \tag{1.7}$$

because $\rho = r\sqrt{1-(t/r)^2}$. Here ${}^n\Phi(\chi,\theta,\phi)$ is determined by (1.6) on the hyperboloid.

Using standard spherical harmonics on S^2 as data at $t/r = 0$, one gets a unique collection of fields ${}^n\Phi$. Guided by analogy to the static case, one can consider those fields as "time dependent multipole moments near spacelike" infinity.

In contrast to the static case, however, one can choose arbitrary data for $\Box\Phi$ on $t = 0$. It is a matter of convenience or simplicity to consider, as we do, fields with data

$$\Phi(0,x^a) = \frac{Y_{lm}(\theta,\phi)}{r^n}, \quad \frac{\partial}{\partial t}\Phi(0,x^a) = \frac{Y_{lm}(\theta,\phi)}{r^{n+1}}. \tag{1.8}$$

It is straightforward to verify all this also for the Maxwell field or for general rest mass-zero, spin-s fields. For example, there exist solutions of Maxwell's equations of the form

$$F_{\mu\nu} = \frac{{}^nF_{\mu\nu}\left(\frac{t}{r},\theta,\phi\right)}{\rho^n}, \quad n \geqq 2, \tag{1.9}$$

where $F_{\mu\nu}$ are the components in a Cartesian frame. It is important to realize that the following limit describes the radiation field: If we substitute $t = u + r$, $\rho = r\sqrt{1-(t/r)^2}$, we get

$$F_{\mu\nu}(r,u,\theta,\phi) = \frac{{}^nF_{\mu\nu}\left(\frac{u+r}{r},\theta,\phi\right)}{r^n\sqrt{1-\left(\frac{u+r}{r}\right)^2}}. \tag{1.10}$$

One can establish that $\lim_{r\to\infty} rF_{\mu\nu}$ exists and describes the radiation field on $\mathscr{I}^+$. The key point is that a certain combined limit $t/r \to 1, \rho \to \infty$, extracts information at $\mathscr{I}^+$. All this can be done explicitly for each n, because one has an integral representation of solutions of spin-s fields in terms of the data. One convenient way would be to decompose the field equations in spherical harmonics and use the Euler–Poisson–Darboux equation for the (χ,ρ)-dependence.

We generalize to Einstein's theory as follows. We consider metrics of the form

$$d\rho^2\left(1 + \frac{{}^1\sigma}{\rho} + \frac{{}^2\sigma}{\rho^2}\cdots\right)^2 + \rho^2\left({}^0h_{ab} + \frac{1}{\rho}{}^1h_{ab} + \cdots\right)d\phi^a d\phi^b. \tag{1.11}$$

Here ${}^n\sigma$, ${}^nh_{ab}$ are functions on the unit hyperboloid with metric ${}^0h_{ab}$ which we shall denote by $\mathscr{H}$. The field equations $R_{\mu\nu}=0$ lead to a hierarchy of linear partial differential equations on $\mathscr{H}$ of the following structure.

The linear equations for ${}^1\sigma$, ${}^1h_{ab}$ turn out to be equivalent to the ones found by Geroch [6], and by Ashtekar and Hansen [4] and, in the linear approximation, by Sommers [7]. Note ${}^2\sigma$ and higher can always be transformed away by a coordinate change. The remaining system for ${}^nh_{ab}$ is: (L, M are certain linear differential operators on $\mathscr{H}$)

$$\begin{aligned} {}^nL({}^nh_{ab}) &= {}^nJ_{ab}({}^1\sigma, {}^1h_{cd}, \ldots {}^{n-1}h_{cd}), \\ -\frac{n}{2}\left(D_a{}^nh^a_b - \frac{1}{2}D_b{}^nh^a_a\right) &= {}^nJ_b({}^1\sigma, {}^1h_{cd}, \ldots {}^{n-1}h_{cd}), \\ {}^nM({}^nh^a_a) &= {}^nJ({}^1\sigma, {}^1h_{cd}, \ldots {}^{n-1}h_{cd}). \end{aligned} \tag{1.12}$$

This system is overdetermined, but as the main result of this paper shows, it can always be solved provided the constraints are satisfied. In this paper we do not analyse the invariant meaning of the ansatz (1.11). This can and will be done in a further paper, when we will have seen that useful information can be extracted from Eqs. (1.12).

The system (1.12) has solutions satisfying (1.11). All stationary, asymptotically flat space-times are in fact analytic in $1/\rho$ [8]. Nonstationary examples are the C-metric [9] and the analogues of the Einstein–Rosen [10] waves constructed near spacelike infinity.

It is our hope that investigations of the solutions of the linear equations (1.12) will lead to further insight into the relation between $\mathscr{I}^+$ and $\mathscr{I}^-$.

The plan of our paper is as follows. In Sect. 2 we define the space-times considered and simplify the metric by certain coordinate transformations. In Sect. 3 we derive the hierarchy of the field equations and prove the main theorem about the solvability. Section 4 contains some conclusions.

After completion of this work we learnt that space-times satisfying similar requirements but at timelike, rather than spacelike, infinity were considered in unpublished work by Eardley [11].

2. Space-Times Admitting a Radially Smooth Minkowskian Spacelike Infinity

In this section we define the class of space-times we are considering.

Definition (2.1). (M,g) is radially smooth of order m at spatial infinity, if the following holds:

(1) For a part of M, a chart (x^μ) exists which is defined for

$$\rho_0 < \rho < \infty, \qquad \rho^2 = \eta_{\mu\nu}x^\mu x^\nu.$$

(2) The components of the metric in this chart satisfy

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{m} \frac{1}{\rho^n}{}^nl_{\mu\nu}\left(\frac{x^\sigma}{\rho}\right) + {}^{m+1}f_{\mu\nu}(x^\sigma), \tag{2.1}$$

where

(3) ${}^{n}l_{\mu\nu}$ is C^∞ in x^σ/ρ and $|{}^{m}f_{\mu\nu}| \leqq \frac{\text{const}}{\rho^m}$, $|\partial^m f_{\mu\nu}| \leqq \frac{\text{const}}{\rho^{m+1}}, \ldots |\partial\partial\partial^m f_{\mu\nu}| \leqq \frac{\text{const}}{\rho^{m+3}}$, in short ${}^{m}f_{\mu\nu} = 0^3(1/\rho^m)$. Obvious generalizations of this definition are the cases where $m = \infty$ or $g_{\mu\nu}$ is C^∞ (analytic) in $1/\rho$.

This is essentially the same definition as in [12], however "$C^{>0}$" is replaced by the stronger assumption of radial smoothness.

Clearly given (2.1) there is a large freedom of finding coordinates $\bar{x}^\mu$ and $\bar{\rho}$ such that (2.1) holds again. For example, define $\bar{x}^\mu$ by

$$x^\mu = \bar{x}^\mu + \sum_{n=1}^{s} \frac{a^\mu(\bar{x}^\nu/\bar{\rho})}{\bar{\rho}^n}, \qquad \bar{\rho}^2 = \eta_{\mu\nu}\bar{x}^\mu\bar{x}^\nu, \tag{2.2}$$

$$\begin{aligned}\frac{\partial x^\mu}{\partial \bar{x}^\nu} &= \delta^\mu_\nu + \sum_{n=1}^{s}\left\{-\frac{n}{\bar{\rho}^{n+1}}a^\mu(\bar{x}^\sigma/\bar{\rho})\frac{\bar{x}^\nu}{\bar{\rho}} + \frac{1}{\bar{\rho}^{n+1}}a^\mu{}_{,\lambda}\left[\delta^\lambda_\nu - \eta_{\nu\sigma}\frac{\bar{x}^\sigma}{\bar{\rho}}\frac{\bar{x}^\lambda}{\bar{\rho}}\right]\right\}\\ &= \delta^\mu_\nu + \sum_{n=1}^{s} b^\mu_\nu\left(\frac{\bar{x}^\sigma}{\bar{\rho}}\right)\Big/\bar{\rho}^{n+1}.\end{aligned}$$

Choosing $s \geqq m-1$, (2.1) is preserved. There are, however, further transformations. Consider

$$\begin{aligned}x^\mu &= \bar{x}^\mu + \xi^\mu(\bar{x}^\nu/\bar{\rho}),\\ \frac{\partial x^\mu}{\partial \bar{x}^\nu} &= \delta^\mu_\nu + \xi^\mu{}_{,\lambda}\frac{1}{\bar{\rho}}\left(\delta^\lambda_\nu - \eta_{\lambda\sigma}\frac{\bar{x}^\sigma}{\bar{\rho}}\frac{\bar{x}^\lambda}{\bar{\rho}}\right).\end{aligned} \tag{2.3}$$

These are "supertranslations," i.e. direction dependent shifts of the origin. One is tempted to conjecture that those are all transformations preserving (2.1). However, if $g_{\mu\nu} = \eta_{\mu\nu}$, we consider

$$\begin{aligned}x^\mu &= \bar{x}^\mu + c^\mu \ln\bar{\rho}, \qquad c^\mu = \text{const},\\ \frac{\partial x^\mu}{\partial \bar{x}^\nu} &= \delta^\mu_\nu + c^\mu\frac{1}{\bar{\rho}}\eta_{\nu\sigma}\frac{\bar{x}^\sigma}{\bar{\rho}}.\end{aligned} \tag{2.4}$$

Because the metric components of $\eta_{\mu\nu}$ are constant, we get a metric of the type (2.1) by this transformation. Probably those transformations can only be applied to cases in which one has a Killing vector of $g_{\mu\nu}$, acting like a translation near infinity. Examples are given by static space-times.

The transformation (2.4) can also be used to produce a counter-example to the conjecture that the AEFANSI structure is unique [12]. (The possible occurrence of "logarithmic" transformations at spatial infinity was first pointed out by Bergmann [13].)

The class of space-times defined above can also be characterized in a coordinate-independent fashion. We postpone this, however, because we want first to establish some consequences of the vacuum field equations.

It is convenient to use ρ for a coordinate, together with coordinates on the manifold of directions.

Lemma (2.1). *Let (ϕ^a) $(a = 1, 2, 3)$ be a local chart on the manifold of directions x^μ/ρ.*

Then (2.1) transforms into

$$d\rho^2\left[1+\sum_{n=1}^{m}\frac{{}^n\tilde{\sigma}(\phi)}{\rho^n}+O^3\left(\frac{1}{\rho^{m+1}}\right)\right]$$
$$+2\rho d\rho d\phi^a\left[0+\sum_{n=1}^{m}\frac{{}^nA_a(\phi)}{\rho^n}+O^3\left(\frac{1}{\rho^{m+1}}\right)\right]$$
$$+\rho^2\left[{}^0h_{ab}(\phi)+\sum_{n=1}^{m}\frac{{}^nh_{ab}(\phi)}{\rho^n}+O^3\left(\frac{1}{\rho^{m+1}}\right)\right]d\phi^a d\phi^b. \tag{2.5}$$

Proof. There exist functions $w^\mu(\phi^a)$ such that

$$\frac{x^\mu}{\rho}=w^\mu(\phi^a),\quad dx^\mu=w^\mu d\rho+\rho w^\mu_{,a}d\phi^a \tag{2.6}$$

Substituting into (2.1) leads to (2.5) with

$$\eta_{\mu\nu}dx^\mu dx^\nu=d\rho^2+\rho^{2\,0}h_{ab}d\phi^a d\phi^b, \tag{2.7}$$

$$ {}^n\tilde{\sigma}={}^nl_{\mu\nu}w^\mu w^\nu,\quad {}^nh_{ab}={}^nl_{\mu\nu}w^\mu_{,a}w^\nu_{,b}, \tag{2.8}$$
$$ {}^nA_a={}^nl_{\mu\nu}w^\mu w^\nu_{,a}.$$

Later it will be useful to substitute

$$\left(1+\sum_{n=1}^{m}\frac{{}^n\tilde{\sigma}}{\rho^n}\right)=\left(1+\sum_{n=1}^{m}\frac{{}^n\sigma}{\rho^n}\right)^2+O^3\left(\frac{1}{\rho^{m+1}}\right), \tag{2.9}$$

which defines ${}^n\sigma$ uniquely.

The metric ${}^0h_{ab}d\phi^a d\phi^b$ is the metric of the unit hyperbolid $\mathscr{H}$, i.e. $\rho^2=1$ in Minkowski space. The topology of $\mathscr{H}$ is $S^2\times\mathbb{R}$, so we need at least two charts (ϕ^a) to cover $\mathscr{H}$. We now use part of the coordinate freedom to simplify (2.5).

Lemma (2.2). *There exists a coordinate transformation such that the metric has the form (2.5), (2.9) with*

$$ {}^2\sigma=3\sigma=\cdots={}^m\sigma=0,$$
$$ {}^1A_a={}^2A_a={}^3A_a=\cdots={}^mA_a=0. \tag{2.10}$$

Proof. We first show that (2.10) is true for ${}^2\sigma$ and ${}^1A_a$. Take

$$\phi^a=\bar{\phi}^a+\frac{1}{\rho}{}^1G^a(\bar{\phi}^b),\quad \bar{\rho}=\rho,$$
$$d\phi^a=d\bar{\phi}^a+\frac{1}{\rho}{}^1G^a_{,b}d\bar{\phi}^b-\frac{1}{\rho^2}d\rho\,{}^1G^a. \tag{2.11}$$

Substituting into (2.5) one gets a "mixed" term

$$\rho d\bar{\phi}^a d\rho\{{}^1A_a-{}^1G^{b0}h_{ab}\}. \tag{2.12}$$

Hence, choosing

$$ {}^1G^b = {}^1A_a {}^0h^{ab}, \tag{2.13} $$

the leading term of $d\bar{\phi}^a d\rho$ is removed. Denote this metric again as (2.5) with ${}^1A_a = 0$.

Now we transform

$$ \rho = \bar{\rho} + \frac{{}^2F(\phi^a)}{\bar{\rho}}, \tag{2.14} $$

$$ d\rho = d\bar{\rho} - \frac{{}^2F d\bar{\rho}}{\bar{\rho}^2} + \frac{1}{\bar{\rho}} {}^2F_{,a} d\phi^a. $$

The $1/\bar{\rho}^2$-term in $d\bar{\rho}^2$ becomes $(-{}^2F + {}^2G^a) \cdot 1/\bar{\rho}^2$. Hence

$$ {}^2F = {}^2\sigma \tag{2.15} $$

removes ${}^2\sigma$ in the metric and no $d\bar{\rho} d\phi^a$-term of order 1 is introduced. Now one can use induction to show that if ${}^1\sigma = {}^3\sigma = {}^{n-1}\sigma = 0$, ${}^1A_a = \ldots {}^{n-2}A_a = 0$, a transformation

$$ \phi^a = \bar{\phi}^a + \frac{1}{\bar{\rho}^{n-1}} {}^{n-1}G^a, \tag{2.16} $$

$$ \rho = \bar{\rho} + \frac{1}{\bar{\rho}^{n-1}} {}^nF, $$

removes the terms ${}^n\sigma$, ${}^{n-1}A_a$. In the last step ${}^nA_a$ is removed in a similar way ($n \leqq m$).

Thanks to this lemma we may confine ourselves to metrics of the form

$$ N^2 d\rho^2 + \rho^2 \left({}^0h_{ab} + \frac{1}{\rho} {}^1h_{ab} + \cdots + \frac{{}^nh_{ab}}{\rho^n} + \cdots \right) d\phi^a d\phi^b, \tag{2.17} $$

where $N = 1 + {}^1\sigma/\rho \cdot {}^1\sigma$ and ${}^nh_{ab}$ depend only on ϕ^a and the remainder-term in (2.17) was omitted.

The term ${}^1\sigma/\rho$ could in general only be removed by a transformation of the type $\rho = \bar{\rho} + a(\bar{\phi}) \ln \bar{\rho}$ which would introduce $\ln \bar{\rho}$-terms in the $d\bar{\phi}^a d\bar{\phi}^b$-coefficients.

The most natural question now is to find all coordinate transformations preserving the form (2.17). We postpone this for reasons mentioned above. We conjecture, however, that if g is not stationary, the supertranslations

$$ \rho = \bar{\rho} \left[1 + \omega(\bar{\phi}^a) + \frac{{}^2F(\bar{\phi}^a)}{\bar{\rho}} + \cdots, \right. \tag{2.18} $$

$$ \phi^a = \bar{\phi}^a + \frac{1}{\bar{\rho}} {}^0h^{ab} \omega_{,b} + \frac{{}^2G^a}{\bar{\rho}^2} + \cdots, $$

are all transformations preserving (2.17). Here $\omega(\phi^a)$ can be chosen arbitrarily and characterizes the supertranslation. The further coefficients ${}^nF$, ${}^nG$ in (2.18) are then determined by ω, ${}^1\sigma$, ${}^{n'}h (n' \leqq n)$ and their derivatives. Substituting (2.18) into (2.17) one finds for the change of the $1/\rho$-quantities:

$$ {}^1\bar{\sigma} = {}^1\sigma, \qquad {}^1\bar{h}_{ab} = {}^1h_{ab} + 2D_a D_b \omega + 2\omega {}^0h_{ab}. \tag{2.19} $$

Note that one has to expand ${}^{0}h_{ab}(\bar{\phi}^c + 1/\bar{\rho}\ldots)$ to get (2.19).

3. The Field Equations Up To First Order

We now want to study the restrictions on the line element (2.17) imposed by Einstein's vacuum equations. The 3 + 1-splitting which is inherent in (2.17) suggests a similar decomposition of the field equations $R_{\mu\nu} = G_{\mu\nu} = 0$. We write

$$H := -2\rho^2\eta^\mu\eta^\nu G_{\mu\nu} = -2\rho^2\eta^\mu\eta^\nu\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = 0, \tag{3.1}$$

$$F_a := \rho q_a^\mu \eta^\nu R_{\mu\nu} = 0, \tag{3.2}$$

$$F_{ab} := N q_a^\mu q_b^\nu R_{\mu\nu} = 0. \tag{3.3}$$

Here η_μ is the unit normal $N\nabla_\mu\rho$ and q_a^μ projects out directions parallel to η_μ. The occurrence of ρ and N in the definitions of H, F_a and F_{ab} is motivated only by convenience. Next we express the left sides of (3.1, 2, 3) in terms of N and h_{ab}. To this end we define the, again suitably rescaled, second fundamental form of $\rho = \text{const}$ by

$$\rho p_{ab} = q_a^\mu q_b^\nu \nabla_\mu \eta_\nu. \tag{3.4}$$

Let D_a be the covariant derivative associated with h_{ab} and let $\mathfrak{R}_{ab}$, $\mathfrak{R}$ denote its Ricci tensor and Ricci scalar, respectively. (Our conventions are $D_{[a}D_{b]}V_c = \frac{1}{2}\mathfrak{R}_{abc}{}^d V_d$, $\mathfrak{R}^b_{abc} = \mathfrak{R}_{ac}$). By contracting the Gauß equation (see Schouten [14]), we obtain the identity

$$H = \mathfrak{R} + p_{ab}p^{ab} - p^2, \tag{3.5}$$

and, from Codazzi's equation [14],

$$F_a = D_b(p_a^b - p\delta_a^b). \tag{3.6}$$

Furthermore we have the identity

$$F_{ab} = (\rho p_{ab})' - 2Np_{ac}p_b^c + D_aD_bN - N\mathfrak{R}_{ab} + Npp_{ab}, \tag{3.7}$$

where a prime means $\partial/\partial\rho$ (see e.g. Fischer and Marsden [15, p. 562]). Indices here are raised and lowered with h_{ab}. We also have set $p := h^{ab}p_{ab}$. From the definition (3.4) there follows

$$2Np_{ab} = 2h_{ab} + \rho h'_{ab}. \tag{3.8}$$

In the case where η_μ is timelike, the equations $H = 0$, $F_a = 0$; (3.8) and $F_{ab} = 0$, with appropriate sign changes, are referred to as the Arnowitt–Deser–Misner equations [16] in the general relativity literature.

In the next step we insert the expansions ($n \leq m$)

$$N = 1 + \frac{{}^1\sigma}{\rho}, \tag{3.9}$$

$$h_{ab} = {}^0h_{ab} + \frac{1}{\rho}{}^1h_{ab} + \cdots + \frac{1}{\rho^n}{}^nh_{ab} + O^3\left(\frac{1}{\rho^{n+1}}\right), \tag{3.10}$$

into (3.8) and (3.5, 6, 7). As a result we get similar expansions

$$p_{ab} = {}^0p_{ab} + \frac{1}{\rho}{}^1p_{ab} + \cdots, \tag{3.11}$$

$$H = {}^0H + \frac{1}{\rho}{}^1H + \cdots, \tag{3.12}$$

$$F_a = {}^0F_a + \frac{1}{\rho}{}^1F_a + \cdots, \tag{3.13}$$

$$F_{ab} = {}^0F_{ab} + \frac{1}{\rho}{}^1F_{ab} + \cdots, \tag{3.14}$$

where quantities with a zero depend only on ${}^0h_{ab}$. Our aim is to study the field equations successively in powers of $1/\rho$. We shall use the rule that indices of quantities with an order symbol are lowered and raised with the "unperturbed" metric ${}^0h_{ab}$, for example ${}^1F^a = {}^0h^{ab\,1}F_b$. Furthermore covariant derivatives acting on such quantities are also taken with respect to ${}^0h_{ab}$ and are denoted with the same symbol D_a. Let us first look at order zero. From (3.8) we see that

$${}^0P_{ab} = {}^0h_{ab}. \tag{3.15}$$

Thus ${}^0F_a = 0$ is identically satisfied. Similarly, from (3.9), ${}^0H = 0$ is satisfied iff

$${}^0\mathfrak{R} = {}^0h^{ab\,0}\mathfrak{R}_{ab} = 6. \tag{3.16}$$

Similarly, ${}^0F_{ab} = 0$ is valid iff

$${}^0\mathfrak{R}_{ab} = 2\,{}^0h_{ab}. \tag{3.17}$$

Note that (3.17) implies (3.16) and both are, in fact, valid on grounds of our initial assumption that ${}^0h_{ab}$ is the unit hyperboloid metric ${}^0h_{ab}$ on $\mathscr{H}$. Conversely, had we left ${}^0h_{ab}$ an unspecified Lorentz metric on the manifold $S^2 \times \mathbb{R}$ from the beginning, Einstein's equations, using (3.17) and the vanishing of the Weyl tensor in three dimensions, would have implied

$${}^0\mathfrak{R}_{abcd} = {}^0h_{ac}{}^0h_{bd} - {}^0h_{bc}{}^0h_{ad}, \tag{3.18}$$

and thus that ${}^0h_{ab}$ is the unit hyperboloid metric on $\mathscr{H}$. (This is in analogy to the following result about null infinity: The leading order of the field equations imply that $\mathscr{I}$ is a null hypersurface.)

Next we consider the first-order equations which—because of the occurrence of ${}^1\sigma$—are different from the higher ones. From (3.8) we find

$${}^1p_{ab} = \tfrac{1}{2}{}^1h_{ab} - {}^1\sigma{}^0h_{ab}. \tag{3.19}$$

From the definition of $\mathfrak{R}_{ab}$ it follows that

$$\mathfrak{R}_{ab} = {}^0\mathfrak{R}_{ab} + \frac{1}{\rho}[-\tfrac{1}{2}D^{2\,1}h_{ab} - \tfrac{1}{2}D_aD_b{}^1h + D^cD_{(a}{}^1h_{b)c}] + O^1\left(\frac{1}{\rho^2}\right), \tag{3.20}$$

where ${}^1h = {}^0h^{ab\,1}h_{ab}$ and $D^2 = D^aD_a$, D_a being, as remarked, taken from ${}^0h_{ab}$. For the inverse metric h^{ab} we have

$$h^{ab} = {}^0h^{ab} - \frac{1}{\rho}{}^1h^{ab} + O^3\left(\frac{1}{\rho^2}\right), \tag{3.21}$$

which implies

$$p^b_a = \delta^b_a + \frac{1}{\rho}[-\tfrac{1}{2}{}^1h^b_a - {}^1\sigma\delta^b_a] + O^2\left(\frac{1}{\rho^2}\right), \tag{3.22}$$

$$p^{ab} = {}^0h^{ab} + \frac{1}{\rho}[-\tfrac{3}{2}{}^1h^{ab} - {}^1\sigma{}^0h^{ab}] + O^2\left(\frac{1}{\rho^2}\right), \tag{3.23}$$

and, using (3.20)

$$\mathfrak{R} = {}^0\mathfrak{R} + \frac{1}{\rho}[-D^2{}^1h + D^aD^{b1}h_{ab} - 2{}^1h] + O^1\left(\frac{1}{\rho^2}\right). \tag{3.24}$$

Inserting into ${}^1H = 0$ we obtain

$$-D^2{}^1h + D^cD^{d1}h_{cd} + 12{}^1\sigma = 0. \tag{3.25}$$

Thus ${}^1F_a = 0$ gives

$$-\tfrac{1}{2}D_b{}^1h^b_a + \tfrac{1}{2}D_a{}^1h + 2D_a{}^1\sigma = 0, \tag{3.26}$$

and ${}^1F_{ab} = 0$ yields

$$\tfrac{1}{2}D^2{}^1h_{ab} + \tfrac{1}{2}D_aD_b{}^1h - D^cD_{(a}{}^1h_{b)c} + \tfrac{3}{2}({}^1h_{ab} - \tfrac{1}{3}{}^1h{}^0h_{ab}) + D_aD_b{}^1\sigma - 3{}^1\sigma{}^0h_{ab} = 0. \tag{3.27}$$

Equations (3.25, 26, 27) form a linear second-order system for $({}^1\sigma, {}^1h_{ab})$. As it stands, it is not hyperbolic. We will show, however, that it is equivalent to a pair of uncoupled hyperbolic equations for two scalars, first found by Geroch [6] by a completely different procedure.

Using the commutator formula (A.5) of the appendix in the third term of (3.27), we see that (3.25, 26, 27) are equivalent to

$$\tfrac{1}{2}D^2{}^1h_{ab} - \tfrac{1}{2}D_aD_b{}^1h - 3D_aD_b{}^1\sigma - 3{}^1\sigma{}^0h_{ab} - \tfrac{3}{2}({}^1h_{ab} - \tfrac{1}{3}{}^1h{}^0h_{ab}) = 0 \tag{3.28}$$

together with (3.26). Note that (3.28) implies

$$(D^2 + 3){}^1\sigma = 0. \tag{3.29}$$

Let us now introduce the following fields

$$k_{ab} := {}^1h_{ab} + 2{}^1\sigma{}^0h_{ab}, \tag{3.30}$$

$$t_{ab} := \varepsilon_a{}^{cd}D_ck_{db}. \tag{3.31}$$

(Besides a numerical factor t_{ab} is the magnetic part of the four dimensional Weyl tensor—taken with respect to the unit normal of the $\rho = \text{const}$ foliation—which is one of the basic objects considered by Ashtekar and Hansen [4].)

After contracting (3.31) with $\varepsilon_e{}^{ab}$, we easily see that (3.26) is equivalent to

$$t_{[ab]} = 0. \tag{3.32}$$

Similarly, (3.28), using (3.26), is equivalent to

$$D_{[a}t_{b]c} = 0. \tag{3.33}$$

A result due to Ashtekar [4] states that a symmetric t_{ab} satisfying (3.33) on $\mathscr{H}$ can be written as

$$t_{ab} = D_a D_b {}^1\beta + {}^0h_{ab}{}^1\beta, \tag{3.34}$$

for some function ${}^1\beta$ on $\mathscr{H}$. Because of (3.31), t_{ab} is tracefree, whence

$$(D^2 + 3){}^1\beta = 0. \tag{3.35}$$

From the tracelessness of t_{ab} and (3.33) we also infer that

$$D^a t_{ab} = 0. \tag{3.36}$$

We have thus obtained

Theorem (3.1). *Every solution of (3.25, 26, 27) defines a pair of functions on $\mathscr{H}$, ${}^1\sigma$ and ${}^1\beta$, which satisfy*

$$(D^2 + 3){}^1\sigma = 0 \tag{3.37}$$

$$(D^2 + 3){}^1\beta = 0 \tag{3.38}$$

If, conversely, any solution $({}^1\sigma, {}^1\beta)$ to (3.37, 38) would define a solution $({}^1\sigma, {}^1h_{ab})$ to (3.25, 26, 27), the existence question would be settled. However, this is only true under some additional assumption which arises as follows:

Let $\xi_a = D_a\xi$ be in the 4-parameter class of vector fields on $\mathscr{H}$ which satisfy

$$D_a\xi_b = -{}^0h_{ab}\xi. \tag{3.39}$$

and consider the quantity

$$Q(\xi^a) = \int_C t_{ab}\xi^a d^2S^b. \tag{3.40}$$

where C is any section of $\mathscr{H}$. Using (3.39) and (3.36), it follows (see Geroch [6] or Ashtekar and Hansen [4] or Sommers [7]) that Q is independent of C. Actually, from the definition (3.31) of t_{ab}, we find that Q constructed from our t_{ab} vanishes identically. Via (3.34) this gives rise to a further restriction on ${}^1\beta$, which can be described as follows: Take any spherical section S^2 of $\mathscr{H}$. Evaluate ${}^1\beta_{|S^2}$ and the normal derivative $(\partial/\partial n){}^1\beta_{|S^2}$, and expand in spherical harmonics. Then $Q(\xi^a)$ vanishes if and only if the first datum has vanishing $l = 0$-component and the second datum has vanishing $l = 1$-component.

Theorem (3.2). *Equations (3.37, 38) with ${}^1\beta$ such that $Q(\xi^a) = 0$ define, via (3.30, 31, 34), a solution $({}^1\sigma, {}^1h_{ab})$ to (3.25, 26, 27).*

Proof. All the steps in the proof of Theorem 3.1 were reversible except for Eq. (3.31), namely

$$\varepsilon_a{}^{cd} D_c k_{db} = t_{ab}. \tag{3.41}$$

We have to show that, given a field t_{ab} which is trace-free, divergence-free and such that $Q(\xi^a) = 0$, Eq. (3.41) can be solved for k_{ab}. The detailed proof of this [17] falls outside the scope of this paper. Roughly, one uses a $2+1$-splitting of (3.41) on $\mathscr{H}$ thus reducing the problem to certain equations on S^2 and an ordinary differential equation along the orthogonal congruence. Some lengthy computations reveal that all of these can consistently be solved where the vanishing of $Q(\xi^a)$ is used as an essential ingredient.

Remark. From Ashtekar's Lemma it follows that k_{ab}, whence ${}^1h_{ab}$, is determined by ${}^1\sigma, {}^1\beta$ only modulo addition of terms of the form

$${}^1h_{ab} \rightarrow {}^1h_{ab} + D_a D_b \omega + {}^0h_{al}\,\omega \tag{3.42}$$

for ω an arbitrary function on $\mathscr{H}$. This is precisely the gauge freedom discussed in Sect. 2.

That these transformations send solutions to solutions, is, of course, a manifestation of the tensorial nature of Einstein's equations.

We also point out that this gauge freedom could have been used as an alternative way of handling Eqs. (3.25, 26, 27). If one imposes the gauge condition ${}^1h = -6\,{}^1\sigma$, the system splits into an evolution part for ${}^1h_{ab}$ and ${}^1\sigma$, which is hyperbolic, and a constraint part which is compatible with the first.

4. The n^{th} Order Equations

In these equations we will meet two types of terms: firstly those which are nonlinear in ${}^{n'}h_{ab}$ with $n' < n$ and ${}^1\sigma$ and their derivatives up to second order. Secondly, the terms linear in ${}^nh_{ab}$ and its derivatives. It will be easy to write down the linear terms explicitly, whereas it is for general n rather cumbersome and, in fact, unnecessary for our purposes to do so for the nonlinear ones.

From (3.8) we infer ($n > 1$)

$${}^np_{ab} = \frac{2-n}{2}\,{}^nh_{ab} + \mathrm{NT}, \tag{4.1}$$

where NT stands for "nonlinear terms." In analogy to (3.20) we have the formula

$${}^n\mathfrak{R}_{ab} = -\tfrac{1}{2}D^2\,{}^nh_{ab} - \tfrac{1}{2}D_aD_b\,{}^nh + D^cD_{(a}\,{}^nh_{b)c} + \mathrm{NT}. \tag{4.2}$$

(Recall that on the right side of Eq. (4.2) all operations refer to ${}^0h_{ab}$ rather than h_{ab}.) Next we have to insert (4.1, 2) into the definitions of F_{ab}, F_a and H. Note that, in order to obtain ${}^nF_{ab}$, ${}^nF_a$ and ${}^nH$ we also have to use the zeroth-order terms of p_{ab} and $\mathfrak{R}_{ab}$, whereas the terms of order $0 < n' < n$ only contribute to the nonlinear pieces. After some algebra and using (A.5) there results

$$ {}^{n}F_{ab}=\tfrac{1}{2}D^{2n}h_{ab}+\tfrac{1}{2}D_aD_b{}^{n}h-D_{(a}D^{cn}h_{b)c}-\frac{n-2}{2}{}^{n}hh_{ab}+\frac{n^2-2n-2}{2}{}^{n}h_{ab}-{}^{n}J_{ab}, \tag{4.3} $$

$$ nF_a=-\frac{n}{2}D_b({}^{n}h^b_a-{}^{n}h\delta^b_a)-{}^{n}J_a, \tag{4.4} $$

$$ {}^{n}H=-D^{2n}h+D^cD^{dn}h_{cd}-2(1-n)^{n}h-{}^{n}J, \tag{4.5} $$

where ${}^{n}h:={}^{0}h^{abn}h_{ab}$ and the nonlinear terms have been given the more prominent name ${}^{n}J({}^{1}\sigma,{}^{1}h_{ab,\ldots}{}^{n-1}h_{ab})$ ("J" for "junk"). Using (4.4) and (4.5), all trace- and divergence-terms are now eliminated from (4.3). As a result we obtain ($1<n<m$)

$$ D^{2n}h_{ab}+(n^2-2n-2)^{n}h_{ab}=2^{n}J_{ab}+\frac{2}{n(n-1)}D_aD_b({}^{n}J+{}^{n}J^c_c)-\frac{4}{n}D_{(a}{}^{n}J_{b)} $$
$$ +\frac{2(n-2)}{n(n-1)}({}^{n}J+{}^{n}J^c_c){}^{0}h_{ab}, \tag{4.6} $$

$$ {}^{n}k:={}^{n}h-\frac{2}{n(n-1)}({}^{n}J+{}^{n}J^c_c)=0, \tag{4.7} $$

$$ {}^{n}k_a:=D_b{}^{n}h^b_a+\frac{2}{n}{}^{n}J_a-\frac{2}{n(n-1)}D_a({}^{n}J+{}^{n}J^c_c)=0. \tag{4.8} $$

Equation (4.6) alone is already a linear hyperbolic equation, hence allows an initial value formulation. Equations (4.7) and (4.8) on the other hand, constrain the value of ${}^{n}h_{ab}$ and its normal derivative on some initial slice in $\mathscr{H}$. But if we want to solve the whole set (4.6, 7, 8), we also have to maintain (4.7, 8) during evolution via (4.6). That this is, in fact, possible is shown by

Theorem (4.1). *Assumption*:

a) ${}^{1}\sigma,{}^{1}h_{ab,\ldots}{}^{n'}h_{ab}((1<n'<n)$ *satisfy the first (respectively n'^{th}) order field equations, and*

b) ${}^{n}h_{ab}$ *satisfies (4.6), and, on some initial slice S of $\mathscr{H}$, Eqs. (4.7, 8) and their normal derivatives.*

Assertion: ${}^{n}h_{ab}$ *solves (4.7) and (4.8) for all times.*

Remark. The time derivative of Eq. (4.8) contains second time derivatives of ${}^{n}h_{ab}$, which is disturbing at first sight. However, using the evolution equation (4.6) these can be eliminated. Thus the initial conditions used in assumption b) of Theorem (4.1) effectively only constrain ${}^{n}h_{ab}$ and its time derivative on S. All this is in complete analogy to the usual Cauchy problem in general relativity (to be more precise: Eqs. (4.7, 8), to which we refer as "constraint equations," play formally the same role as harmonicity condition in the normal Cauchy problem. Together with their time derivatives (4.7, 8) are, after removing second time derivatives of ${}^{n}h_{ab}$, analogous to the constraints there, written in the harmonic gauge. For all this, see Choquet–Bruhat [18]).

The proof of Theorem (4.1) follows from two lemmas.

Lemma. *F_{ab}, F_a and H satisfy the following identities, independently of any field equations*

$$\rho H' = 2ND_aF^a - 2NpH + 4(D_aN)F^a - 2pF^a_a + 2p_{ab}F^{ab} + 2H, \tag{4.9}$$

$$\rho F'^a = -(D^aN)H - \frac{N}{2}D^aH - 2NF^a - NpF^a - D^aF^b_b + D_bF^{ab} + 3F^a. \tag{4.10}$$

Proof. These identities are the contracted Bianchi identities $G_{\mu\nu;}{}^{\nu} = 0$ in 3+ 1-form. It is, however, easier to obtain them directly from the definition of F_{ab}, F_a and H. The essential ingredients are the identity

$$\mathfrak{R}' = (\mathfrak{R}_{ab} - D_aD_b)(h^{ab})' + D^2(h_{ab}(h^{ab})'), \tag{4.11}$$

which follows from the definition of the Ricci scalar and the following commutator formula

$$\begin{aligned}(D_bH^{ab})' - D_bH'^{ab} = {} & \tfrac{1}{2}[D_b(h^{ac}h'_{cd}) + D_d(h^{ac}h'_{cb}) - D^ah'_{bd}]H^{bd} \\ & + \tfrac{1}{2}D_d(h^{bc}h'_{bc})H^{ad},\end{aligned} \tag{4.12}$$

where H^{ab} is an arbitrary family (parametrized by ρ) of symmetric tensor fields on $\mathscr{H}$. When we now act on Eq. (3.5) for H with $\rho(\partial/\partial\rho)$, use (4.11) and (3.6, 7), Eq. (4.9) follows after some straightforward algebra. Similarly, differentiating (3.6) and using (4.12), we obtain (4.10). This proves the lemma.

Remark. The contracted Bianchi identities used in the Cauchy problem differ in one respect from (4.9, 10). In the former case, on the left side of the analogues to (4.9, 10) there appear the time derivatives $\partial/\partial t$ of the quantities playing the role of our H and F_a. Thus one can conclude that if $H = F_a = 0$ and the constraints are satisfied at some initial time, then they are always valid. In our case we have the singular operator $\rho(\partial/\partial\rho)$ on the left side or, if one likes, $-\rho'(\partial/\partial\rho')$, where $\rho' = 1/\rho$. From this we cannot conclude that if ${}^0H = H|_{\rho'=0}$ and ${}^0F = F_a|_{\rho'=0}$ vanish, they vanish for all ρ. Rather we have to expand (4.9) and (4.10) near $\rho' = 0$ or $\rho = \infty$, like we did before with the field equations.

We need another

Lemma. *Assume that the induction hypothesis of Theorem (4.1), namely condition* a) *is satisfied. Then the following equations are true*

$$(4-n)^nJ = 2D_c{}^nJ^c - 4{}^nJ^c_c, \tag{4.13}$$

$$(2-n)^nJ^a = -\tfrac{1}{2}D^{an}J - D^{an}J^c_c + D_c{}^nJ^{ac}. \tag{4.14}$$

Proof. From the induction hypothesis we infer

$$H = \frac{1}{\rho^n}{}^nH + O^1\left(\frac{1}{\rho^{n+1}}\right), \tag{4.15}$$

$$F_a = \frac{1}{\rho^n}{}^nF_a + O^1\left(\frac{1}{\rho^{n+1}}\right), \tag{4.16}$$

$$F_{ab} = \frac{1}{\rho^n}{}^nF_{ab} + O^1\left(\frac{1}{\rho^{n+1}}\right). \tag{4.17}$$

Because of our initial assumptions these relations may be differentiated once with respect to ρ. Inserting (4.15, 16, 17) into (4.9, 10) and using the zeroth order form of p_{ab} we easily get

$$(4-n)^n H = 2D_c{}^n F^c - 4^n F^c_{c,} \tag{4.18}$$

$$(2-n)^n F^a = -\tfrac{1}{2} D^{a\,n} H - D^{a\,n} F^c_c + D_c{}^n F^c. \tag{4.19}$$

Now insert (4.3, 4, 5) into (4.18, 19). Remarkably, all linear terms, i.e. all terms involving ${}^n h_{ab}$ add up to zero. This is easily seen for Eq. (4.18). For (4.19) this involves some commutator algebra based on the formulae of Appendix A which is tedious but straightforward. This proves the lemma.

We now have four relations satisfied by the inhomogeneous terms of Eqs. (4.6, 7, 8). This is just what we need for the

Proof of Theorem (4.1). Let us operate on ${}^n k$, defined in (4.7) with D^2 and use the trace of Eq. (4.6). Using, again, the definition of ${}^n k$, all junk-terms, using (4.18), drop out leaving us with

$$D^{2\,n} k + (n^2 - 2n - 2)^n k = 0. \tag{4.20}$$

The ${}^n k_a$ of Eq. (4.8) is treated similarly. We act on it with D^2 and compare with what we get when we take D^b of Eq. (4.6). Using our list of commutators and the definitions of ${}^n k$ and ${}^n k_a$ there results, using (4.19)

$$D^{2\,n} k_a + (n^2 - 2n + 2)^n k_a - 2D_a{}^n k = 0. \tag{4.21}$$

Equations (4.20, 21) are a set of linear, homogeneous, second order hyperbolic equations for ${}^n k$ and ${}^n k_a$. Consequently, if ${}^n k$, ${}^n k_a$ and their time derivatives vanish on some slice of $\mathscr{H}$, they vanish everywhere in $\mathscr{H}$ if only ${}^n h_{ab}$ satisfies the evolution equations ${}^n F_{ab} = 0$, because only these have been used to derive (4.20, 21).

5. Concluding Remarks

In this work we have introduced a class of space-times which possess a certain expansion near spatial infinity. We were able to show that our ansatz is compatible with Einstein's vacuum equations to all orders in the radial expansion parameter. Our viewpoint is different from that of earlier approaches to spacelike infinity (see Geroch [6], Ashtekar and Hansen [4], Sommers [7]). There more emphasis is laid on the geometrical framework and (or) a unification with null infinity, rather than making full use of the field equations. We hope that our approach will make it possible to investigate whether the geometrical assumptions in the AEFANSI-framework are dynamically justifiable. Of course, even our ansatz contains some *a priori* assumptions about dynamics. As far as the size of our space-time is concerned, this is justified by the solution to the "boost-problem" in general relativity due to Christodoulou and o'Murchadha [19].

More important is the possibility of expanding the metric in $1/\rho$ which leads to our hierarchy of equations. The fact that this hierarchy can consistently be solved strongly suggests a theorem along the following lines: Cauchy data which are radially smooth at spatial infinity of order m develop into a space-time satisfying our requirements for order m. Up to now our knowledge about existence of such space-times is limited to the examples mentioned in Sect. 1.

Appendix A

Here we collect a few facts about hyperboloids which are often used in the text. The Riemann tensor of the metric ${}^0h_{ab}$ on $\mathscr{H}$ is given by

$$ {}^0\mathfrak{R}_{abcd} = {}^0h_{ac}{}^0h_{bd} - {}^0h_{bc}{}^0h_{ad}. \tag{A.1} $$

Let t, t_a and $t_{ab} = t_{(ab)}$ be arbitrary fields on $\mathscr{H}$. Using the Ricci identities and its corollaries, we find

$$ [D_a, D^2]t = -2D_a t, \tag{A.2} $$

$$ [D_a, D_b]t_c = 2\,{}^0h_{c[a}t_{b]}, \tag{A.3} $$

$$ [D_a, D^2]t_b = 2\,{}^0h_{ab}D_c t^c - 4D_{(a}t_{b)}, \tag{A.4} $$

$$ [D_a, D_b]t_{cd} = 2\,{}^0h_{a(c}t_{d)b} - 2\,{}^0h_{b(c}t_{d)a}, \tag{A.5} $$

$$ [D_a, D^2]t_{bc} = 4\,{}^0h_{a(b}D^d t_{c)d} - 6D_{(a}t_{bc)}. \tag{A.6} $$

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