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Some Nonabelian Toy Models in the Large N Limit

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Abstract. Schwinger–Dyson equations are used to study the large N limit of $U(N)$ gauge theory on several small lattices. Explicit solutions are found which are beyond the reach of existing steepest descent technique. They show a phase transition in a three placquette model at coupling $g^2N = 3$, resembling the known transition in the one placquette model, and lending support to expectations of a similar transition in the four dimensional lattice theory.

1. Introduction

The $U(\infty)$ lattice gauge theory differs qualitatively from the $U(N)$ or $SU(N)$ theories in having an infinite number of internal degrees of freedom per unit volume of space-time. Gross and Witten [1], using the steepest descent technique of [2], solved exactly the large N limit of $U(N)$ gauge theory on a single placquette and found peculiarities attributable to just this difference. The free energy and correlation functions depend analytically on the coupling constant except at a single critical value, which marks a continuous transition between weak and strong coupling phases. The average eigenvalue distribution of the placquette variable (an $N \times N$ unitary matrix) covers the entire unit circle in the strong coupling regime, but for small coupling constant is excluded from a neighborhood of -1.

This paper presents some new exact results for $U(\infty)$ gauge theories on small lattices. The main result is for a three placquette model which consists essentially of two unitary matrices governed by the action $S(U_1, U_2) = -\beta N\,\mathrm{tr}(U_1 + U_2 + U_1U_2^* + \text{adjoints})$. The steepest descent method (in the form used in [1] and [2]) fails here because the number of true degrees of freedom goes as N^2 rather than N. We look instead to the Schwinger–Dyson equations recently derived for lattice gauge theories [3, 4]. Because of symmetries special to the three placquette lattice, the $N = \infty$ Schwinger–Dyson equations close on a manageable subset of the correlation functions. Extending a technique suggested by Foerster [5],

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and employing an ansatz suggested by numerical calculation of the strong coupling expansion, we find in closed form the unique solution of these equations which is consistent with the strong coupling expansion and analytic in the coupling constant in a neighborhood of infinity. This solution cannot be continued below a critical value of the coupling constant, and the behavior of the one placquette spectral density is qualitatively similar to that found by Gross and Witten.

The same techniques are used to calculate the spectral density at $N = \infty$ of the product of two independent placquette variables with action $S = -\beta N \operatorname{tr}(U_1 + U_2 + \text{adjoints})$. Again qualitative aspects of the one placquette model are confirmed.

In only one small respect does the one placquette model seem to be atypical. The phase transition there is signaled by a change in the asymptotic behavior at large k of the k-th Fourier coefficient of the average spectral distribution of the placquette variable. In the strong coupling phase the coefficients vanish identically for large k; in the weak coupling phase they diminish as a power of k. But the strong coupling behavior is anomalous; in the more complicated models the Fourier coefficients decay exponentially in k, the rate of decay vanishing at the transition point.

These results are presented to serve three purposes: to add confidence that the phase transition found in the one placquette model will also be present in more realistic models; to refine slightly expectations of the location and characteristics of the transition; and to give complete, non-terminating strong coupling expansions for correlation functions in a $U(\infty)$ lattice gauge model, to be used as sources and test cases for conjectures on the general structure of the strong coupling, $1/N$ double expansion.

The organization of the paper is as follows. In Sect. 2 the use of the Schwinger–Dyson equations to calculate correlation functions is demonstrated on the one placquette model. The results of Gross and Witten are reproduced. Section 3 contains a derivation of the Schwinger–Dyson equations for the three placquette model. In Sect. 4 the strong coupling solution is presented and some of its properties discussed. Section 5 gives the spectral density calculation for two independent placquette variables. Section 6 is a discussion of the results.

2. The one Placquette Model

We are interested in the probability measure on $U(N)$ given by

$$d\mu(U) = Z^{-1} \exp[\beta N(\operatorname{tr} U + \operatorname{tr} U^*)] dU \tag{1}$$

where dU is Haar measure, $\beta = 1/(g^2 N)$, g the standard coupling constant, and Z provides normalization. Conjugation by unitary matrices leaves $d\mu$ unchanged so it is sufficient to consider expectation values $\langle f(U) \rangle$ of functions invariant under conjugation. These are generated by the functions $N^{-1} \operatorname{tr}(U^k)$. On evidence from strong and weak coupling expansions we expect factorization in the large N limit. That is, if f and g are invariant functions then $\langle fg \rangle = \langle f \rangle \langle g \rangle$ at infinite N. So it is enough to find $\langle N^{-1} \operatorname{tr}(U^k) \rangle$. These numbers are real because $d\mu$

is invariant under replacement of U by its complex conjugate. And, since $d\mu$ is invariant under $U \to U^{-1}$, only $k > 0$ need be considered.

Foerster pointed out [5] that the $N = \infty$ Schwinger–Dyson equations for the one placquette model become algebraic when expressed in terms of the analytic function

$$R(z) = \langle N^{-1} \operatorname{tr}[(1 - zU)^{-1}] \rangle, \tag{2}$$

which completely describes the model in the large N limit. $R(z)$ is holomorphic in the interior of the unit disk because as a power series in z all its coefficients $\langle N^{-1} \operatorname{tr}(U^k) \rangle$ lie between -1 and 1.

The Schwinger–Dyson equations are efficiently derived using a device due to Guth [6]. Let X be a skew adjoint $N \times N$ matrix. Start from the quantity

$$\int d\mu(U) N^{-1} \operatorname{tr}[X(1 - zU)^{-1}], \tag{3}$$

then change variables from U to $e^{tX}U$. Haar measure is invariant under left multiplication, so

$$\frac{d}{dt/t=0} \int dU \exp[\beta N \operatorname{tr}(e^{tX}U + U^* e^{-tX})] N^{-1} \operatorname{tr}[X(1 - ze^{tX}U)^{-1}] = 0 \tag{4}$$

or

$$\begin{aligned} &\int d\mu \{\beta N \operatorname{tr}[X(U - U^*)] N^{-1} \operatorname{tr}[X(1 - zU)^{-1}] \\ &\quad + N^{-1} \operatorname{tr}[X(1 - zU)^{-1} zXU(1 - zU)^{-1}]\} = 0. \end{aligned} \tag{5}$$

Contract with the invariant quadratic form on the Lie algebra of $U(N)$ (i.e., use $\sum_a (X_a)^i_j (X_a)^k_1 = N^{-1} \delta^i_1 \delta^k_j$ for an appropriately normalized basis $\{X_a\}$ of the skew adjoint matrices) to get

$$\begin{aligned} &\int d\mu \{\beta N^{-1} \operatorname{tr}[(U - U^*)(1 - zU)^{-1}] \\ &\quad + N^{-1} \operatorname{tr}[(1 - zU)^{-1}] z N^{-1} \operatorname{tr}[U(1 - zU)^{-1}]\} = 0 \end{aligned} \tag{6}$$

or

$$\begin{aligned} &\beta(z^{-1} - z)R(z) - \beta z^{-1} - \beta < N^{-1} \operatorname{tr} U^* > - R(z) \\ &\quad + \langle [N^{-1} \operatorname{tr}(1 - zU)^{-1}]^2 \rangle = 0 \end{aligned} \tag{7}$$

In the large N limit this becomes

$$R(z)^2 + [\beta(z^{-1} - z) - 1]R(z) - \beta R'(0) - \beta z^{-1} = 0 \tag{8}$$

or

$$R(z) = \tfrac{1}{2}[1 + \beta(z - z^{-1}) + \sqrt{F(z)}\,] \tag{9}$$

where

$$F(z) = [\beta(z + z^{-1}) + 1]^2 + 4\beta(R'(0) - \beta). \tag{10}$$

$R(z)$ must be analytic inside the unit circle, so the unknown $R'(0)$ must be chosen to ensure that $F(z)$ has no zeros or poles of odd order there. When $0 < \beta \leqq 1/2$ the only possibility is $R'(0) = \beta$, giving $R(z) = 1 + \beta z$. When $1/2 < \beta$

the analyticity condition is less restrictive, forcing only

$$1-(4\beta)^{-1} \leqq R'(0) \leqq \beta. \tag{11}$$

A unique solution is obtained by noting that the boundary value of $R(z)$ on the unit circle must satisfy the positivity condition

$$2\,\mathrm{Re}(R(z))-1 \geqq 0 \tag{12}$$

or equivalently

$$\mathrm{Re}\sqrt{F(z)} \geqq 0. \tag{13}$$

This is just the positivity of spectral distributions, and can be thought of as following from

$$N^{-1}\,\mathrm{tr}\,[(\sum_i c_i U^i)(\sum_j c_j U^j)^*] \geqq 0 \text{ for all } \{c_i\}, \tag{14}$$

which implies

$$\oint dz(2\,\mathrm{Re}(R(z))-1)|c(z)|^2 \geqq 0 \text{ for all } c(z), \tag{15}$$

where the integral is taken around the unit circle.

We determine when (11) and (13) can be satisfied simultaneously. Under conditions (11), the line $\{z : F(z) \text{ real negative}\}$ divides the unit disk in two. The square root in (9) must lie on different sheets on either side of this line. $F(-1)$ and $F(1)$ are both non-negative real but on opposite sheets on the Riemann surface of the square-root, so $\sqrt{F(-1)}$ and $\sqrt{F(1)}$ cannot both be positive. $F(1)$ is always positive so $F(-1)$ must be made zero. This requires

$$R'(0) = 1-(4\beta)^{-1}. \tag{16}$$

giving

$$R(z) = \tfrac{1}{2}[1+\beta(z-z^{-1})+\beta(1+z^{-1})\sqrt{z^2+2(\beta^{-1}-1)z+1}]. \tag{17}$$

$\sqrt{F(z)}$ is imaginary along the arc $\mathrm{Re}(z) \leqq 1-\beta^{-1}, |z|=1$, so the spectral density $2\,\mathrm{Re}(R(z))-1$ vanishes there. As $\beta \to \infty$ this arc grows to fill the whole circle, and $R(z)$ approaches $(1-z)^{-1}$, corresponding to a spectral distribution concentrated at $z=1$.

The correlation functions of the one placquette model are analytic in β except at the critical value $\beta=\frac{1}{2}$. At $\beta=\frac{1}{2}$ they have continuous first derivatives but not second derivatives. A signal of the transition is the asymptotic behavior for large k of the Fourier coefficients $\langle N^{-1}\,\mathrm{tr}\,U^k\rangle$ of the one placquette spectral density. In the strong coupling regime, $\beta \leqq \frac{1}{2}$, the large k Fourier coefficients vanish identically. In the weak coupling regime they decay as $k^{-3/2}$. To see this, use the residue formula to write

$$\langle N^{-1}\,\mathrm{tr}(U^k)\rangle = \oint dz\,\frac{R(z)}{z^{k+1}} \tag{18}$$

where the contour of integration is a small circle around the origin. Deform the integration path until it surrounds the branch cut of $\sqrt{F(z)}$, then use steepest descent to find the large k behavior.

3. Schwinger–Dyson Equations for the Three Placquette Model

Consider a lattice consisting of two vertices, three links connecting them, and three placquettes, each with a different pair of links as boundary. The link variables are U_1, U_2, and U_3 and the action is

$$S(U_1, U_2, U_3) = -\beta N \operatorname{tr}(U_1 U_2^* + U_2 U_3^* + U_3 U_1^* + \text{adjoints}). \tag{19}$$

Gauge invariance can be used to eliminate U_3, giving a two matrix version:

$$S(U_1, U_2) = -\beta N \operatorname{tr}(U_1 + U_2 + U_1 U_2^* + \text{adjoints}), \tag{20}$$

but the three matrix version better reveals the symmetries of the model. The measure of interest is

$$d\mu(U_1, U_2, U_3) = Z^{-1} \exp[-S(U_1, U_2, U_3)] dU_1 dU_2 dU_3. \tag{21}$$

The gauge transformations take $(U_1, U_2, U_3) \to (V^* U_1 W, V^* U_2 W, V^* U_3 W)$, V and W in $U(N)$, and leave $d\mu$ invariant. The factorization assumption says that $\langle fg \rangle = \langle f \rangle \langle g \rangle + O(N^{-2})$ for gauge invariant functions f and g with nontrivial large N limits. Thus the only expectation values of interest at $N = \infty$ are of the form $\langle N^{-1} \operatorname{tr} U(L) \rangle$, where $U(L)$ is the product of link variables along a closed loop L.

The first step in applying the analytic-algebraic technique is to notice that the $N = \infty$ Schwinger–Dyson equations close on the correlation functions generated by

$$R(w, z) = \langle N^{-1} \operatorname{tr}[(1 - wU_1 U_2^*)^{-1} (1 - zU_1 U_3^*)^{-1}] \rangle. \tag{22}$$

To see this, start from the quantity

$$\langle N^{-1} \operatorname{tr}[X(1 - wU_1 U_2^*)^{-1} (1 - zU_1 U_3^*)^{-1}] \rangle, \tag{23}$$

replace U_1 with $e^{tX} U_1$ and proceed as in (4)-(6) to arrive at

$$\beta A + B + C = 0 \tag{24}$$

where

$$A = \langle N^{-1} \operatorname{tr}[(U_1 U_2^* + U_1 U_3^* - U_2 U_1^* - U_3 U_1^*)(1 - wU_1 U_2^*)^{-1} (1 - zU_1 U_3^*)^{-1}] \rangle \tag{25}$$

$$B = \langle N^{-1} \operatorname{tr}(1 - wU_1 U_2^*)^{-1} N^{-1} \operatorname{tr}[wU_1 U_2^* (1 - wU_1 U_2^*)^{-1} (1 - zU_1 U_3^*)^{-1}] \rangle \tag{26}$$

$$C = \langle N^{-1} \operatorname{tr}[(1 - wU_1 U_2^*)^{-1} (1 - zU_1 U_3^*)^{-1}] \operatorname{tr}[zU_1 U_3^* (1 - zU_1 U_3^*)^{-1}] \rangle \tag{27}$$

Some algebraic manipulation gives

$$A = (w^{-1} + z^{-1} - w - z) R(w, z) - w^{-1} R(0, z) - z^{-1} R(w, 0) - A_1 A_2 \tag{28}$$

where

$$A_1 = \langle N^{-1} \operatorname{tr}[U_2 U_1^* (1 - zU_1 U_3^*)^{-1}] \rangle \tag{29}$$

$$A_2 = \langle N^{-1} \operatorname{tr}[U_3 U_1^* (1 - wU_1 U_2^*)^{-1}] \rangle. \tag{30}$$

Cyclicity of the trace gives

$$A_1 = \langle N^{-1}\,\mathrm{tr}(U_2U_1^*)\rangle + \langle N^{-1}\,\mathrm{tr}[zU_3^*U_2(1 - zU_3^*U_1)^{-1}]\rangle. \tag{31}$$

The symmetries $U_1 \to U_2$, $U_2 \to U_1$ and $U_1 \to U_3^*$, $U_2 \to U_2^*$, $U_3 \to U_1^*$ then yield

$$A_1 = R_1(0, 0) + zR_1(0, z) \tag{32}$$

where

$$R_1(w, z) = \frac{\partial}{\partial w} R(w, z). \tag{33}$$

Similarly

$$A_2 = R_2(0, 0) + wR_2(w, 0). \tag{34}$$

More algebraic manipulation combined with the factorization assumption for $N = \infty$ gives

$$B = R(w, 0)R(w, z) - R(w, 0)R(0, z) \tag{35}$$

and

$$C = R(0, z)R(w, z) - R(w, z). \tag{36}$$

Write $R(w) = R(w, 0)$, note that the symmetry $U_2 \to U_3$, $U_3 \to U_2$ implies $R(w, z) = R(z, w)$, and collect all of the above into the $N = \infty$ algebraic Schwinger–Dyson equation

$$\begin{aligned}[][R(w) + R(z) - 1 + \beta(w^{-1} + z^{-1} - w - z)]R(w, z) &= R(w)R(z) \\ + \beta[w^{-1}R(z) + z^{-1}R(w) + zR_1(0, z) &+ wR_2(w, 0) + 2R'(0)]. \end{aligned} \tag{37}$$

Equation (37) refers only to information contained in $R(w, z)$; this is the closure property claimed above.

A more useful form of (37) is obtained by the following manipulations. Expand both sides of (37) in power series in w and use $R_1(w, z) = R_2(z, w)$ to get

$$R_1(0, z) = (1 - z)^{-1}[\beta^{-1}(R(z) - R(z)^2) + z^{-1}(1 - R(z)) + zR(z) + 2R'(0)]. \tag{38}$$

Define

$$D(z) = R(z) + \beta(z^{-1} - z) - 1/2. \tag{39}$$

Use (38) and (39) and a considerable amount of algebra to rewrite (31) as

$$\begin{aligned} R(w, z) = 1/2 + \beta(1 + w + z) &- (1 - z)^{-1}zD(z) - (1 - w)^{-1}wD(w) \\ &+ (1 - w)^{-1}(1 - z)^{-1}(1 - wz)S(w, z) \end{aligned} \tag{40}$$

where

$$S(w, z) = [D(w) + D(z)]^{-1}T(w, z) \tag{41}$$

and

$$T(w, z) = D(w)D(z) - \beta^2(1 - w)(1 - z)(1 + w^{-1}z^{-1}) + 2\beta R'(0) + \beta + 1/4. \tag{42}$$

Now the problem is to find $D(z)$ such that $zD(z)$ is holomorphic in the interior of unit disk value β at the origin, and such that $R(w, z)$, given by (40)–(42), is holomorphic in the bidisk $|w| < 1, |z| < 1$. A necessary condition is that $T(w, z)$ vanish on the curve C defined by $D(w) + D(z) = 0$.

4. A Solution of the Schwinger–Dyson Equation

There is a clue to a solution in the strong coupling expansion. If $R(w, z)$ is expanded in a triple Taylor series in w, z and β we know that $\beta^k w^1 z^m$ will appear only if $k \geqq 1 + m$. Using this fact and the Schwinger–Dyson Eq. (37) we can calculate the coefficients of the triple Taylor series recursively. We notice among the terms contributing to $R(z)$, i.e. those with $1 = 0$, a curious pattern holding to quite high order: after βz, only terms with $k - 2m$ divisible by three appear. Take as an ansatz that this is the exact truth, or, what is equivalent, that $D(\beta u)$ depends only on u and β^3. We find that there is a unique solution to our problem for which this ansatz holds.

Define the curve C' in (u, v) by $D(\beta u) + D(\beta v) = 0$. By hypothesis, C' depends only on β^3. Rearranging (42),

$$\begin{aligned} T(\beta u, \beta v) = {} & [D(\beta u)D(\beta v) + \beta^3(u + v) - u^{-1}v^{-1}] + \beta[u + v - \beta^3 uv] \\ & + 2\beta R'(0) + \tfrac{1}{4} + \beta - 2\beta^2. \end{aligned} \tag{43}$$

$T(\beta u, \beta v)$ can vanish along C' only if both expressions in square brackets are functions only of β there. Equivalently, along $C: D(w) + D(z) = 0$, we must have

$$D(w)D(z) + \beta^2(w + z - w^{-1}z^{-1}) + a(\beta) = 0 \tag{44}$$

and

$$w^{-1} + z^{-1} - wz + \beta^{-1}b(\beta) = 0 \tag{45}$$

where $a(\beta)$ and $b(\beta)$ actually depend only on β^3. Substituting for $D(w)$ and w^{-1} in (44) we find that, on C,

$$D(z)^2 = \beta^2 z^{-2} + b\beta z^{-1} + a + \beta^2 z. \tag{46}$$

Near the origin the curve C looks like $w = -z + 0(z^2)$. Thus (46) must hold for all z near 0. But $zD(z)$ is analytic inside the unit circle, so (46) must be identically true. This is compatible with (39) only if $b = 1$. We are left with

$$D(z) = \beta z^{-1}\sqrt{P(z)} \tag{47}$$

where

$$P(z) = z^3 + a\beta^{-2}z^2 + \beta^{-1}z + 1. \tag{48}$$

To see the constraints on $a(\beta)$ imposed by the analyticity of $R(w, z)$, use (47) and (48) in (40)–(42) and simplify to arrive at

$$R(w, z) = \tfrac{1}{2} + \beta(1 + w + z) + \beta(w - z)^{-1}[(1 + z)\sqrt{P(w)} - (1 + w)\sqrt{P(z)}]. \tag{49}$$

$R(w, z)$ given by (49) is analytic whenever $\sqrt{P(w)}$ and $\sqrt{P(z)}$ are. So the only constraint on $a(\beta)$ is that $P(z)$ must have no zeros of odd order inside the unit circle. Let the three zeros of $P(z)$ be z_1, z_2 and z_3. Since $z_1 z_2 z_3 = -1$ the only admissable possibilities are:

$$z_2 = z_3, |z_2| \leqq 1 \tag{50}$$

$$\text{or} \quad |z_1| = |z_2| = |z_3| = 1. \tag{51}$$

Alternative (50) is equivalent to

$$a(\beta) = \beta^2(z_2^{-2} - 2z_2) \tag{52}$$

where $z_2(z)$ must satisfy

$$Q(z_2) = z_2^3 - \beta^{-1} z_2 - 2 = 0, \ |z_2| \leqq 1. \tag{53}$$

For $1/3 < \beta$ or $\beta < -1$ all three roots of $Q(z)$ lie outside the unit circle, making (53) impossible to satisfy. But for $-1 \leqq \beta \leqq 1/3$ there is exactly one root inside:

$$z_2(\beta) = -2(3\beta)^{-1/2} \sin\left[\tfrac{1}{3}\sin^{-1}((3\beta)^{3/2})\right], \quad 0 \leqq \sin^{-1}(\cdot) \leqq \frac{\pi}{2}. \tag{54}$$

It is easily seen that $a(\beta)$ defined by (52) and (54) depends only on β^3 and is analytic in β near 0. It follows that $R(w, z)$ defined by (47)–(49) is also analytic in β near 0. The condition $k \geqq 1 + m$ in the triple Taylor series $R(w, z) = \sum \beta^k w^1 z^m$ is verified by noting that $a(0) = \frac{1}{4}$ so $R(\beta^{-1}u, \beta^{-1}v) = 1 + u + v + uv + 0(\beta)$. Since the Taylor series coefficients of $R(w, z)$ can be calculated recursively from (37), given this fact about the strong coupling expansion, there can be no other solution of the Schwinger–Dyson equation (37) compatible with the strong coupling expansion and analytic in β near 0.

The second alternative, (51), requires $a(\beta) = (\beta^*)^{-1}\beta^2$ and, for real β, either $\beta \leqq -1$ or $1/3 \leqq \beta$. This contradicts the original ansatz, but the derivation of (49) depended only on (38), not on the particular form of $a(\beta)$. So $a(\beta) = (\beta^*)^{-1}\beta^2$ does give a solution of (37) outside the strong coupling regime. Unfortunately it must be discarded for violation of the spectral positivity condition

$$\mathrm{Re}(D(z)) \geqq 0, \quad |z| = 1, \tag{55}$$

on the boundary value of $D(z)$.

The solution determined by (50) gives a one placquette spectral density $2\,\mathrm{Re}(R(z)) - 1, |z| = 1$ which is strictly positive as long as $-1 < \beta < 1/3$. Its Fourier coefficients $\langle N^{-1} \mathrm{tr}(U_1 U_2^*)^k \rangle$ go as $(1 + z_2^{-3})k^{-3/2} \exp[-k(\ln(z_2^{-2}))]$ for large k. At $\beta = 1/3$, where $z_2 = -1$, the spectral density acquires a zero at $z = -1$ and stops being smooth there, and the Fourier coefficients go as $k^{-5/2}$. At $\beta = -1$ the spectral density has two zeros, one at $z = -1$ and the other at $z = 1$. The second zero reflects the frustration caused by negative β.

5. The Two Placquette Model

This is a model of two independent unitary matrices. It can be thought of as a piece of the full two dimensional lattice model or as the three placquette model with one placquette, but no links, left out. The action (in terms of the independent matrices) is

$$S(U_1, U_2) = -\beta N \,\mathrm{tr}(U_1 + U_2 + \text{adjoints}) \tag{56}$$

We sketch here a computation of the spectral distribution for the product matrix $U_1 U_2$, i.e., of

$$M(z) = \langle N^{-1} \mathrm{tr}(1 - zU_1 U_2)^{-1} \rangle. \tag{57}$$

The $N = \infty$ spectral distributions for U_1 and U_2 are the same as for the unitary matrix of the one placquette model, so

$$M(z) = \int dV N^{-1} \mathrm{tr}[(1 - zUVUV^*)^{-1}] \tag{58}$$

where U has the spectral distribution found in sect. 2. Thus $M(z)$ could in principle be calculated directly, but there seems no convenient way to do so.

Instead we again look for a convenient set of correlation functions which include those generated by $M(z)$ and on which the Schwinger–Dyson equations close. We use

$$M(w,z) = \langle N^{-1}\,\mathrm{tr}(1 - wU_1)^{-1}(1 - zU_1U_2)^{-1}\rangle . \tag{59}$$

The Schwinger–Dyson equations are obtained by now familiar manipulations of the expressions

$$\langle N^{-1}\,\mathrm{tr}\,X(1 - wU_1)^{-1}(1 - zU_1U_2)^{-1}\rangle \tag{60}$$

and

$$\langle N^{-1}\,\mathrm{tr}\,X(1 - wU_1)^{-1}(1 - zU_2U_1)^{-1}\rangle , \tag{61}$$

after replacing U_1 with $e^{tX}U_1$. They lead directly to the results

$$\beta \leqq \tfrac{1}{2} : (2M(z) - 1)^2 = (1 + 4\beta^2 z) \tag{62}$$

$$\beta > \tfrac{1}{2} : \qquad = 4\beta^2 z^{-1}\{\tfrac{1}{2}(z+1)^2 - 2yz + (z-1)[\tfrac{1}{4}(z+1)^2 - y^2 z]^{1/2}\}, \tag{63}$$

where $y = 1 - (2\beta)^{-1}$.

In the strong coupling regime, $\beta < \frac{1}{2}$, the spectral density $\mathrm{Re}(2M(z) - 1), |z| = 1$ is smooth and positive. The Fourier coefficients $\langle N^{-1}\,\mathrm{tr}(U_1U_2)^k\rangle$ go for large k as $k^{-3/2}\exp[-k(\ln(4\beta^2))]$. When $\beta \geqq \frac{1}{2}$ the spectral density vanishes along the closed arc $\mathrm{Re}(z) \leqq -1 + 2y^2$, and the Fourier coefficients go as $k^{-3/2}$.

6. Discussion

Note first that the correlation functions of the one placquette model at $N = \infty$ are completely determined by the Schwinger–Dyson equation (8) and the positivity condition (12). The steepest descent technique is more powerful, however, because the Schwinger–Dyson equations have nothing to say about the free energy.

The situation for the three placquette model is not so clear. Equations (40)–(42) state that the joint generating function $R(w,z)$ is determined by the one placquette generating function $R(z)$. From the rest of the Schwinger–Dyson equations it can be argued all of the correlation functions can be calculated once $R(z)$ is known. But it is not clear what analyticity and positivity conditions beyond (40)–(42) and (55) are needed to fix $R(z)$ uniquely.

The strong coupling solution presented here, in (47)–(54), requires information beyond the Schwinger–Dyson equations: the lowest order in the strong coupling expansion at which each correlation function can begin, and analyticity of the $N = \infty$ correlation functions in β near 0. Moreover, to actually find the solution required an ansatz which does not seem at all obvious.

It might have been wondered whether the inter-placquette interaction in the three placquette model (20), which renders the steepest descent technique unusable, would have a qualitative effect on the phase transition. It seems not to. The strong coupling solution for $R(z)$ in the three placquette model looks much like that in the one placquette model. The corresponding spectral density $2\,\mathrm{Re}(R(z)) - 1, |z| = 1$ is analytic and positive for $\beta < \beta_c$, starting constant at $\beta = 0$ and becoming more and more biased towards eigenvalue $z = 1$ as β increases, until, at $\beta = \beta_c$, a zero occurs in the density at $z = -1$, and differentiability fails there. Actually this was to be expected. The interaction between placquettes strengthens the ordering effect of the action, so outgh to encourage the phase transition to occur at an even smaller value of β_c than in a model with independent placquettes. This is exactly what happens: $\beta_c = \frac{1}{3}$ in the three placquette model, $\frac{1}{2}$ in the one placquette model.

The two placquette calculation gives the spectral distribution of the product of two matrices whose individual spectral distributions are fixed but which are subject to extreme relative disordering, as described in (58). It seems possible that the additional disorder would produce a smooth spectral density even in the weak coupling phase. But this does not happen. Whenever the individual placquette matrices are forbidden a range of eigenvalues around -1 the product matrix is also; the extra disorder is expressed only in a broadening of the range of eigenvalues covered by the product. Because this model maximizes relative dis-disorder among the placquette variables making up a loop variable, the result suggests that the change in spectral properties seen in the one placquette model will occur simultaneously for all loop variables in the general lattice model.

In all three models the strong coupling ($\beta < \beta_c$) spectral densities are smooth and positive, the k-th Fourier coefficient going asymptotically to $c(\beta)k^{-3/2} \times \exp[-r(\beta)k]$. (In the one placquette model $c(\beta) = 0$.) At $\beta = \beta_c, r(\beta)$ vanishes. In the one and two placquette models, $c(\beta_c)$ is nonzero and the Fourier coefficients go as $k^{-3/2}$ for $\beta \geqq \beta_c$. But in the three placquette model $c(\beta_c) = 0$ and the Fourier coefficients go as $k^{-5/2}$ at the transition point.

Considering the ordering influence of the lattice action in more than two dimensions it is plausible that an order-disorder transition should always occur at $(g^2 N)_c > 2$. The three placquette example at least encourages this expectation. And it is plausible that the strong coupling phase should be characterized by spectral densities with exponentially decaying Fourier coefficients (corresponding to generating functions $R(z)$ analytic in disks of radii greater than 1.) But the simple global analytic structures of the toy model generating functions at and beyond the transition to the weak coupling phase seem tied to the fact that they are solutions of a finite set of algebraic equations in a finite number of variables. This does not seem likely to be the case for even slightly larger lattices.

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