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A Commutativity Theorem of Partial Differential Operators

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Abstract. Let $u=u(x,t)$ be a function of x and t, and $u_i=\mathscr{D}^i u$, $\mathscr{D}=d/dx$, $i=0,1,2,\ldots$, be its derivatives with respect to x. Denote by $\mathbf{W}_n$ the set $\{f\,|\,f=f(u,u_1,\ldots,u_n),\ (\partial/\partial u_n)f\neq 0\}$, where $f(u,\ldots,u_n)$ are polynomials of u_i with constant coefficients. To any $f\in\mathbf{W}=\bigcup\limits_{n=2}^{\infty}\mathbf{W}_n$, we relate it with an operator $\mathscr{U}(f)=\sum\limits_{i\geqq 0}(\mathscr{D}^i f)\partial/\partial u_i$. In this paper we prove that: $\mathscr{U}(f)$ commutes with $\mathscr{U}(g)$ if they commute respectively with $\mathscr{U}(h)$, provided f, g, $h\in\mathbf{W}$. Relating to this commutativity theorem, we prove that, if an evolution equation $u_t=f(u,\ldots,u_n)$ possesses nontrivial symmetries (or conservation laws for a class of polynomials f), then $f=Cu_n+f_1(u,\ldots,u_r)$, where $C=\text{const}$, and $r<n$. In the end of this paper, we state a related open problem whose solution would be of much value to the theory of soliton.

1. Introduction

The soliton [1], being a particle-like solution of the nonlinear wave equation, has been now applied widely in various fields of physics. In the recent years, a number of interesting mathematical problems have arisen in the study of soliton, one of them is, among other things, the commutativity of differential operators [2, 3]. Let $\mathscr{A}=a_0\mathscr{D}^n+\ldots+a_{n-1}\mathscr{D}+a_n$, $a_0\neq 0$, $\mathscr{D}=d/dx$ be a differential operator, and $\mathbf{C}(\mathscr{A})$ be the set of all linear operators which commute with $\mathscr{A}$. A pronounced results [4] is the fact that $\mathbf{C}(\mathscr{A})$ is a commutative ring. In this paper we established a similar result concerning the partial differential operators $\mathscr{U}(f)=\sum\limits_{i\geqq 0}(\mathscr{D}^i f)\partial/\partial u_i$, where $f=f(u,\ldots,u_n)$ are polynomials of $u_i=\mathscr{D}^i u$ with constant coefficients, and $u=u(x,t)$ is a sufficiently smooth function of x and t. We discuss further the application of this result to the study of symmetries and conservation laws of nonlinear evolution equations.

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2. Notation

Let $u=u(x,t)$ be a function of x and t, and

$$u_i=\mathscr{D}^i u\,,\qquad \mathscr{D}=d/dx$$

be its derivatives with respect to x. Throughout this paper, f, g, h will stand for polynomials of u_i with constant coefficients and without constant term [i.e., say, $f(0)=0$]. The small letters $i, j, k, l, p, q, a, b, c, d$ will stand for nonnegative integers, and the capital letter C solely for constants. For convenience, the binomial coefficient will be understood as

$$\binom{k}{i}=k!/(i!(k-i)!),\qquad (k\geqq i\geqq 0)\,;\qquad \binom{k}{i}=0,\ \text{(otherwise)}.$$

Let

$$\mathbf{W}_n=\{f|f=f(u,u_1,\dots,u_n),\ (\partial/\partial u_n)f\neq 0\} \tag{2.1}$$

and $\mathbf{W}=\bigcup\limits_{n=2}^{\infty}\mathbf{W}_n$. For convenience, we agree that the constant $C\in\mathbf{W}_0$. To any $f\in\mathbf{W}_k$, we relate it with the following operators [5, 6]:

$$\mathscr{V}(f)=\sum_i(\partial_i f)\mathscr{D}^i,\qquad \mathscr{U}(f)=\sum_i(\mathscr{D}^i f)\partial_i \tag{2.2}$$

and

$$\mathscr{V}_j(f)=\sum_i\binom{i}{j}(\partial_i f)\mathscr{D}^{i-j}, \tag{2.3}$$

here and always below, the summation in $\sum\limits_i$ is over all nonnegative integers i, and

$$\partial_i=\partial/\partial u_i,\quad (i\geqq 0)\,;\qquad \partial_i=0,\quad (i<0)\,.$$

It may be noted that $\mathscr{V}_0(f)=\mathscr{V}(f)$ and $\mathscr{U}(f)g=\mathscr{V}(g)f$. We introduce furthermore the operation

$$[f,g]=\mathscr{V}(f)g-\mathscr{V}(g)f\,. \tag{2.4}$$

It is easy to see that

$$\mathscr{U}(f)\mathscr{U}(g)-\mathscr{U}(g)\mathscr{U}(f)=\mathscr{U}([g,f])=\sum_{j\geqq 0}\sum_i[f_i(\partial_i g_j)-g_i(\partial_i f_j)]\partial_j\,. \tag{2.5}$$

Hence an operator $\mathscr{U}(f)$ commutes with $\mathscr{U}(g)$ iff $[f,g]=0$.

Consider an evolution equation

$$u_t=f(u,u_1,\dots,u_k)\,,\qquad f\in\mathbf{W}_k\,, \tag{2.6}$$

if there exists an infinitesimal transformation $u\to v=u+eg$, where $g\in\mathbf{W}_l$ and e is an infinitesimal parameter, such that

$$d/de(v_t-f(v))|_{e=0}=0$$

holds for solutions $u(x,t)$ of Eq. (2.6), then g is called a symmetry of order l of (2.6).

It is known [5] that

Proposition 1. *In order that g is a symmetry of* (2.6), *it is necessary and sufficient that* $[f,g]=0$, *or equivalently* $\mathcal{U}(f)$ *commutes with* $\mathcal{U}(g)$.

Since it always holds that $[f,u_1]=[f,f]=0$, for arbitrary f, thus we call the symmetries $g=C_1u_1+C_2f$ as trivial.

Let

$$u_K^A=u_{k_1}^{a_1}u_{k_2}^{a_2}\dots u_{k_p}^{a_p}, \qquad u_L^B=u_{l_1}^{b_1}u_{l_2}^{b_2}\dots u_{l_q}^{b_q} \tag{2.7}$$
$$(k_1>k_2>\dots>k_p\geqq 0, \quad l_1>l_2>\dots l_q>0$$
$$p,a_1\dots,a_p\geqq 1, \quad q,b_1,\dots,b_q\geqq 1)$$

be two monomials of u_i, we introduce lexicography order $\ll$ among monomials, that is, $u_K^A\ll u_L^B$ if $k_1<l_1$, or $k_1=l_1$ but $a_1<b_1$, or $k_1=l_1$, $a_1=b_1$ but $k_2<l_2$, and so on. For convenience, we agree that $0\ll u_K^A$. To a nonzero polynomial $f=\sum C_{AK}u_K^A$, we denote by $M(f)$ the monomial which is of the highest order among $C_{AK}u_K^A$, and call it dominant, e.g., $M(3uu_1+2u_2^2u_4+u_3^3)=2u_2^2u_4$. It is obvious from the definition that

$$f=g \quad \text{implies} \quad M(f)=M(g). \tag{2.8}$$

The following commutative formula [7] will be used frequently in this paper:

$$\partial_p\mathcal{D}^q=\sum_i\binom{q}{i}\mathcal{D}^{q-i}\partial_{p-i}. \tag{2.9}$$

3. A Necessary Condition for the Existence of Nontrivial Symmetries

Lemma 1

$$\partial_k[f,g]=[\partial_kf,g]+[f,\partial_kg]+\sum_{j\geqq 1}[\mathcal{V}_j(f)\partial_{k-j}g-\mathcal{V}_j(g)\partial_{k-j}f]. \tag{3.1}$$

Proof. It is easy to see by (2.2) and (2.9) that

$$\partial_k(\mathcal{V}(f)g)=\mathcal{V}(\partial_kf)g+\mathcal{V}(f)(\partial_kg)+\sum_{j\geqq 1}\mathcal{V}_j(f)(\partial_{k-j}g). \tag{3.2}$$

a similar equation for $\partial_k(\mathcal{V}(g)f)$ can be also derived, from which together with (2.4) and (2.3), (3.1) follows.

Corollary. $f\in\mathbf{W}_k$, $g\in\mathbf{W}_l$, $k,l\geqq 1$ *imply* $[f,g]\in\mathbf{W}_r$, $r<k+l$,

Proof. From (3.1), *it is easily seen that*

$$\partial_{k+l}[f,g]=\sum_{j\geqq 1}[\mathcal{V}_j(f)\partial_{k+l-j}g-\mathcal{V}_j(g)\partial_{k+l-j}f]=0.$$

Lemma 2. *If* $[f,g]=0$ *for* $f\in\mathbf{W}_k$, $g\in\mathbf{W}_l$, $k,l\geqq 2$, *then*

$$(\partial_lg)^k=C(\partial_kf)^l, \qquad C\neq 0. \tag{3.3}$$

Proof. From (3.1) we have

$$0=\partial_{k+l-1}[f,g]=k(\partial_kf)\mathcal{D}(\partial_lg)-l(\partial_lg)\mathcal{D}(\partial_kf),$$

hence

$$(k\mathscr{D}(\partial_l g))/(\partial_l g) = (l\mathscr{D}(\partial_k f))/(\partial_k f),$$

(3.3) then follows upon integrating.

Corrollary. *Suppose that* $f \in \mathbf{W}_k$, $g \in \mathbf{W}_l$, $k, l \geqq 2$ *and* $[f, g] = 0$, *then* $f = C_1 u_k + f_1$, $f_1 \in \mathbf{W}_{k'}$, $k' < k$ *iff* $g = C_2 u_l + g_1$, $g_1 \in \mathbf{W}_{l'}$, $l' < l$.

Proof. It is a immediate consequence of (3.3).

Lemma 3. *If* $f, g \in \mathbf{W}_k$, $k \geqq 2$, *and* $[f, g] = 0$, *then there exists a nonzero constant* C *such that* $f - Cg \in \mathbf{W}_{k'}$, $k' < k$.

Proof. By Lemma 2, a constant C can be chosen such that $\partial_k(f - Cg) = 0$, thus $f - Cg \in \mathbf{W}_{k'}$, $k' < k$.

Corollary. *If an evolution equation* (2.6) *possesses a nontrivial symmetry* $g \in \mathbf{W}_k$, *then it possesses also a nontrivial symmetry* $h \in \mathbf{W}_{k'}$, $k' < k$.

Proof. Let C be the constant as stated in Lemma 3, and take $h = f - Cg$, then it holds obviously that $[h, f] = 0$, and the fact that g is nontrivial implies that h is also nontrivial.

Lemma 4. *If* $f \in \mathbf{W}_k$, $g \in \mathbf{W}_l$, $k, l \geqq 3$ *and* $[f, g] = 0$ *then*

$$\binom{k-1}{1}(\partial_{k-1} f)\mathscr{D}(\partial_l g) + \binom{k}{1}(\partial_k f)\mathscr{D}(\partial_{l-1} g) + \binom{k}{2}(\partial_k f)\mathscr{D}^2(\partial_l g)$$
$$= \binom{l-1}{1}(\partial_{l-1} g)\mathscr{D}(\partial_k f) + \binom{l}{1}(\partial_l g)\mathscr{D}(\partial_{k-1} f) + \binom{l}{2}(\partial_l g)\mathscr{D}^2(\partial_k f). \tag{3.4}$$

Proof. From (3.1)

$$0 = \partial_{k+l-2}[f, g] = \sum_{j=k-2}^{k} \mathscr{V}_j(f)(\partial_{k+l-2-j} g) - \sum_{j=l-2}^{l} \mathscr{V}_j(g)(\partial_{k+l-2-j} f),$$

from which (3.4) follows.

Lemma 5. *Let*

$$f = u_K^A + f_1, \quad f_1 \ll u_K^A; \qquad g = u_L^B + g_1, \quad g_1 \ll u_L^B, \tag{3.5}$$

where u_K^A *and* u_L^B *are defined as* (2.7), *and* k_1, $l_1 \geqq 2$, *then* $[f, g] = 0$ *imply that*

$$u_{k_1}^{(a_1-1)l_1} u_{k_2}^{a_2 l_1} \dots u_{k_p}^{a_p l_1} = C u_{l_1}^{(b_1-1)k_1} u_{l_2}^{b_2 k_1} \dots u_{l_q}^{b_q k_1}, \qquad C \neq 0 \tag{3.6}$$

Proof. From (3.3) and (2.8) we have

$$M((\partial_{l_1} g)^{k_1}) = CM((\partial_{k_1} f)^{l_1}),$$

from which (3.6) follows.

Now we proceed to prove the following

Theorem A. *A necessary condition for an evolution equation* (2.6) *to possess a nontrivial symmetry is* $f = Cu_k + f_1$, *where* $f_1 \ll u_k$, *i.e.,* $f_1 = f_1(u, u_1, \dots, u_{k'})$, $k' < k$.

By virtue of Proposition 1, the Theorem A can be stated equivalently as

Theorem A*. *Suppose that* $f \in \mathbf{W}_k$, $g \in \mathbf{W}_l$, $k \geqq 2$, $l \leqq 1$, $g \neq C_1 f + C_2 u_1$, *then the necessary conditions for* $\mathscr{U}(f)$ *being commuted with* $\mathscr{U}(g)$ *are* $f = C_1 u_k + f_1$, *and* $g = C_2 u_1 + g_1$, *where* $f_1 \ll u_k$ *and* $g_1 \ll u_l$.

The idea of the proof of this theorem is straightforward, but the whole discussion is tedious, since we must verify carefully all the possible cases. To reduce the length we shall, in some minor cases, pass over a series of simple argument and purely quote the conclusions.

Proof. Let $f \in \mathbf{W}_{k_1}$, $g \in \mathbf{W}_{l_1}$, since $C_1 C_2 [f, g] = [C_1 f, C_2 g]$, we may assume that f and g take the form of (3.5). Now by hypothesis that $[f, g] = 0$ and the Corollary of Lemma 3 we need only to discuss the cases (I). k_1, $l_1 \geqq 2$, $k_1 \neq l_1$; and (II). $k_1 \geqq 2$, $l_1 \leqq 1$. (By symmetry, the discussion of the case $k_1 \leqq 1$, $l_1 \geqq 2$ is similar.)

Case I. k_1, $l_1 \geqq 2$, $k_1 \neq l_1$.
(Ia) a_1, $b_1 \geqq 2$

In this case, the (3.6) implies $k_1 = l_1$, which contradicts with the hypothesis $k_1 \neq l_1$, and hence is impossible.

(Ib) $a_1 = 1$, $b_1 \geqq 2$. (By symmetry, the discussion of the case $b_1 = 1$, $a_1 \geqq 2$ is similar.)

In this case, (3.6) reads

$$u_{k_2}^{a_2 l_1} \dots u_{k_p}^{a_p l_1} = C u_{l_1}^{(b_1 - 1)k_1} u_{l_2}^{b_2 k_1} \dots u_{l_q}^{b_q k_1},$$

hence it must hold that

$$q = p - 1, \qquad (b_1 - 1) = a_2 l_1 \tag{3.7}$$

$$l_i = k_{i+1}, \qquad b_i k_1 = a_{i+1} l_1, \qquad (i = 2, \dots, q).$$

(Iba) $k_1, l_1 \geqq 3$

(i) $k_2 < k_1 - 1$, $l_2 < l_1 - 1$. In this case we have, by Lemma 4 and (2.8), that

$$M\left(\binom{k_1}{2}(\partial_{k_1} f)\mathscr{D}^2(\partial_{l_1} g)\right) = M\left(\binom{l_1}{2}(\partial_{l_1} g)\mathscr{D}^2(\partial_{k_1} f)\right), \tag{3.8}$$

from which it is easy to deduce that $\binom{k_1}{2} b_1 (b_1 - 1) = \binom{l_1}{2} b_1 a_2$, by means of (3.7) that $(b_1 - 1)k_1 = a_2 l_1$, we get $k_1 = l_1$, which contradicts also with the hypothesis.

(ii) $k_2 < k_1 - 1$, $l_2 = l_1 - 1$. In this case, (3.4) reads

$$\binom{k_1}{1}(\partial_{k_1} f)\mathscr{D}(\partial_{l_2} g) + \binom{k_1}{2}(\partial_{k_1} f)\mathscr{D}^2(\partial_{l_1} g)$$

$$= \binom{l_2}{1}(\partial_{l_2} g)\mathscr{D}(\partial_{k_1} f) + \binom{l_1}{2}(\partial_{l_1} g)\mathscr{D}^2(\partial_{k_1} f).$$

But it is easy to see $M(\mathscr{D}^2(\partial_{l_1} g)) \gg M(\mathscr{D}(\partial_{l_2} g))$ and by virtue of $u_{k_2 + 2} = u_{l_1 + 2}$,

$$M\left(\binom{l_1}{2}(\partial_{l_1} g)\mathscr{D}^2(\partial_{k_1} f)\right) \gg M\left(\binom{l_2}{1}(\partial_{l_2} g)\mathscr{D}(\partial_{k_1} f)\right),$$

hence the Eq. (3.8) holds again, the case is therefore impossible by the same reason.

(iii) $k_2=k_1-1$. By the similar argument we get, in this case, that

$$\binom{k_1}{2} b_1(b_1-1)u_{l_1+2}u_{l_1}^{a_2+b_1-2}u_{l_2}^{a_3+b_2}\dots u_{l_q}^{a_{q+1}+b_q}$$

$$=\left[\binom{l_1}{1}+\binom{l_1}{2}\right] b_1 a_2 u_{k_1+1} u_{k_2}^{b_1+a_2-2} u_{k_3}^{b_2+a_3}\dots u_{k_p}^{b_{p-1}+a_p}, \tag{3.9}$$

from which we get $\binom{k_1}{2}(b_1-1)=\binom{l_1+1}{2}a_2$, this together with (3.7) imply $k_1-1=l_1+1$ or $k_1=l_1+2=k_2+2$, which contradicts again with hypothesis that $k_2=k_1-1$.

Since k_1, $l_1 \geqq 2$ and $k_1>k_2=l_1$ in the case (Ib), it remains to discuss the following subcases:

(Ibb) $k_1 \geqq 3$, $l_1=2$

By Lemma 1 we have then

$$\partial_{k_1}[f,g]=[\partial_{k_1}f,g]+\sum_{j=k_1-2}^{k_1}\mathscr{V}_j(f)(\partial_{k_1-j}g)-\sum_{j=1}^{2}\mathscr{V}_j(g)(\partial_{k_1-j}f). \tag{3.10}$$

By setting

$$f^*=\partial_{k_1}f=u_{k_2}^{a_2}\dots u_{k_p}^{a_p}+f_2, \qquad f_2 \ll u_{k_2}^{a_2}\dots u_{k_p}^{a_p},$$

(3.10) can be reduced to

$$[f^*,g]+\left[\binom{k_1-1}{1}(\partial_{k_1-1}f)\mathscr{D}+\binom{k_1}{2}f^*\mathscr{D}^2\right](\partial_2 g)$$

$$+\binom{k_1}{1}f^*\mathscr{D}(\partial_1 g)+f^*(\partial_0 g)-2(\partial_2 g)\mathscr{D}(\partial_{k_1-1}f)=0. \tag{3.11}$$

(i) $k_2=k_1-1$. Since $k_2=l_2=2$, so $k_1=k_2+1=3$. Applying the operator ∂_3^2 to both sides of (3.11), and making use of (2.9), we can deduce, as easily verified, that

$$(\partial_2 f^*)(\partial_2^2 g)+f^*(\partial_2^3 g)=(\partial_2 g)(\partial_2^2 f^*).$$

The comparison of the coefficients of the dominant terms in both sides yields that

$$a_2 b_1(b_1-1)+b_1(b_1-1)(b_1-2)=b_1 a_2(a_2-1), \tag{3.12}$$

which together with (3.7) imply that $a_2^2+3a_2=0$, hence $a_2=0$ and $b_1=1$, which contradicts with the hypothesis that $b_1 \geqslant 2$.

(ii) $k_2<k_1-1$. In this case, by the similar argument it can be concluded from (3.11) that $a_2^2=(k_1-2)a_2$, since $a_2 \neq 0$ as showed above, so $a_2=k_1-2$, and from (3.7), $(b_1-1)k_1=2(k_1-2)$ or $4=(3-b_1)k_1$, but $b_1 \geqq 2$, so $b_1=2$, hence $k_1=4$, $a_2=2$. Applying the operator ∂_3 to both sides of (3.11), and then comparing the coefficients of u_2^3, we get $36b_2=0$, hence $b_2=a_2=0$. Comparing further the coefficients of u_3^2 we get $b_3=a_3=0$. In other words it must hold that $f=u_4+f_{1_0}$ and $g=u_2^2+g_1$, which contradict with the corollary of Lemma 2, and thus are impossible.

(Ic) $a_1=b_1=1$

This hypothesis together with $p=q=1$ are just the desired conclusion, so we assume that $p+q\geqq 3$ and proceed to show it is impossible. When $a_1=b_1=1$, (3.6) reads

$$u_{k_2}^{a_2l_1}\dots u_{k_p}^{a_pl_1}=Cu_{l_2}^{b_2k_1}\dots u_{l_q}^{b_qk_1}, \tag{3.13}$$

from which we get $p=q$, accordingly p, $q\geqq 2$, and

$$k_i=l_i, \quad a_il_1=b_ik_1, \quad (i=2,3,\dots,p)$$

(Ica) k_1, $l_1\geqq 3$

This case can be further divided into two subcases: (i) $k_2<k_1-1$, $l_2<l_1-1$; and (ii) $k_2<k_1-1$, $l_2=l_1-1$. By symmetry, the subcase that $k_2=k_1-1$, $l_2<l_1-1$ is similar to (ii), and since $k_2=l_2$, the another subcase that $k_2=k_1-1$, $l_2=l_1-1$ is excluded by the hypothesis $l_1\neq k_1$. The discussion of the subcase (i) and (ii) are similar to that of case (Iba), so we omit the detail.

Case II. $k_1\geqq 2$, $l_1\leqq 1$.

In this case, it is easy to deduce, from Lemma 1, that

$$\partial_{k_1}[f,g]=[\partial_{k_1}f,g]+(\partial_{k_1}f)[(\partial_0 g)+k_1\mathscr{D}\partial_1 g]. \tag{3.14}$$

Noting that $\partial_{k_i}(\partial_0 g)=\partial_{k_i}\mathscr{D}\partial_1 g=0$ when $k_i>2$, we deduce in general that

$$\begin{aligned}\partial_{k_1}^{a_1}\dots\partial_{k_r}^{a_r}[f,g]&=[\partial_{k_1}^{a_1}\dots\partial_{k_r}^{a_r}f,g]+(\partial_{k_1}^{a_1}\dots\partial_{k_r}^{a_r}f)\\&\cdot\left[\left(\sum_{j=1}^{r}a_j\right)(\partial_0 g)+\left(\sum_{j=1}^{r}a_jk_j\right)\mathscr{D}\partial_1 g\right],\end{aligned} \tag{3.15}$$

when $k_r>1$. Hence, if we set

$$u_K^A=u_{k_1}^{a_1}\dots u_{k_r}^{a_r}u_2^{d_1}u_1^{d_2}u^{d_3},$$

$$k_1,\dots,k_r>2, \quad a_1,\dots,a_r\geqq 1, \quad d_1,d_2,d_3\geqq 0$$

and denote that $C_1=a_1+\dots+a_r$, $C_2=a_1k_1+\dots+a_rk_r$ and

$$h=\partial_{k_1}^{a_1}\dots\partial_{k_r}^{a_r}f/(a_1!\dots a_r!)=u_2^{d_1}u_1^{d_2}u^{d_3}+f_2, \quad f_2\ll u_2^{d_1}u_1^{d_2}u^{d_3},$$

then we have

$$[h,g]+h(C_1(\partial_0 g)+C_2\mathscr{D}\partial_1 g)=0. \tag{3.16}$$

(IIa) $d_1+d_2+d_3=0$

In this case, $h=\text{const}$, hence (3.16) implies

$$C_1(\partial_0 g)+C_2\mathscr{D}\partial_1 g=0. \tag{3.17}$$

Applying the operator ∂_2 to both sides of (3.17), we get $\partial_1^2 g=0$, consequently $g=u_1g_1+g_2$, $g_1,g_2\in\mathbf{W}_0$. Applying the operator ∂_1 again to (3.17), we get $(C_1+C_2)(\partial_0 g_1)=0$. Since it is obvious that $C_1+C_2>0$, hence $\partial_0 g_1=0$ and $g_1=\text{const}$, $g=Cu_1+g_2$, which contradicts with the hypothesis that $q\geqq 2$.

(IIb) $d_1+d_2+d_3>0$

The comparison of the coefficients of $u_2^{d_1+1}$ in both sides of (3.16) then implies $\partial_1^2 g=0$, after a little more careful but similar discussion, we can arrive at the same conclution that $g=Cu_1+g_2$, which is thus also contradictory. The proof of the Theorem A is now completed.

4. The Commutativity Theorem

We have in [5] established the relationship between symmetries and conservation laws of the following nonlinear evolution equation

$$u_t=f(u,u_1,\ldots,u_{2l+1}), \tag{4.1}$$

where $f=\mathscr{D}\mathscr{Y}h$, $h\in W_1$, $\mathscr{Y}=\sum_i(-\mathscr{D})^i\partial_i$ [8, 9]. Basing on this relation we can easily show that in order (4.1) possesses four or more conservation laws it is necessary and sufficient that this equation possesses nontrivial symmetries. Therefore from Theorem A we deduce the following

Theorem B. *In order that* (4.1) *possesses four or more conservation laws, it is necessary that*

$$f(u,u_1,\ldots,u_{2l+1})=Cu_{2l+1}+f_1(u,\ldots,u_r), \qquad r<2l+1.$$

We now prove the main theorem of this paper.

Theorem C. *Suppose that f, g, $h\in\mathbf{W}$, then $\mathscr{U}(f)$ commutes with $\mathscr{U}(g)$ if they commute respectively with $\mathscr{U}(h)$. In other words if we denote by $[\mathscr{A},\mathscr{B}]=\mathscr{A}\mathscr{B}-\mathscr{B}\mathscr{A}$, the commutator of operators $\mathscr{A}$ and $\mathscr{B}$, then*

$$[\mathscr{U}(f),\mathscr{U}(h)]=[\mathscr{U}(g),\mathscr{U}(h)]=0 \quad \text{imply} \quad [\mathscr{U}(f),\mathscr{U}(g)]=0. \tag{4.2}$$

Proof. By (2.5), it is equivalent to show

$$[f,h]=[g,h]=0 \quad \text{imply} \quad [f,g]=0. \tag{4.3}$$

Now by Theorem A and the supposition that $[f,h]=0$, it can be concluded that either (i) $h=C_1f+C_2u_1$, $C_1\neq 0$; or (ii) $f=C_1u_k+f_1$, $h=C_3u_r+h_1$, $f_1\in\mathbf{W}_{k'}$, $h_1\in\mathbf{W}_{r'}$, $k'<k$, $r'<r$. In the case (i), $0=[g,h]=C_1[g,f]+C_2[g,u_1]=C_1[g,f]$, so $[g,f]=0$. In the case (ii), by collorary of Lemma 2 and the supposition that $[g,h]=0$, we see $g=C_2u_l+g_1$, $g_1\in\mathbf{W}_{l'}$, $l'<l$. Since $[u_k,u_l]=0$, for arbitrary k and l, hence if $[f,g]$ is nonzero, any monomial of $[f,g]$ is of degree bigger than one, in particular

$$[f,g]\neq Cu_p+h_2, \qquad h_2\ll u_p. \tag{4.4}$$

But, however, by the Jacobi identity that $[h,[f,g]]+[f,[g,h]]+[g,[h,f]]=0$, we have $[h,[f,g]]=0$ whenever $[g,h]=[f,h]=0$, hence by Theorem A and the fact that $h=C_3u_r+h_1$, we must have $[f,g]=Cu_p+h_2$, $h_2\ll u_p$, which contradicts with (3.4), therefore $[f,g]=0$ as desired.

5. An Open Problem

Let $\mathscr{A}=\sum_i a_i\mathscr{D}^i$, $\mathscr{B}=\sum_i b_i\mathscr{D}^i$ and $\mathscr{C}=\sum_i c_i\mathscr{D}^i$, where a_i, b_i, c_i are in general functions of x. One of the most pronounced results in the theory of linear ordinary differential operators is [4] that

$$[\mathscr{A},\mathscr{C}]=[\mathscr{B},\mathscr{C}]=0 \quad \text{imply} \quad [\mathscr{A},\mathscr{B}]=0 \tag{5.1}$$

The similarity between (5.1) and (4.2) leads us naturely to the problem: can the Theorem C be extended further? For example, can we unite the two statements (5.1) and (4.2) into a general theorem?

In the theory of linear ordinary differential operators one obtained [5], besides (5.1), a sufficient and necessary condition for the existence of nontrivial differential operators which commute with a given operator $\mathscr{A}$. But, as regards operator $\mathscr{U}(f)$, the Theorem A givess only a necessary condition, hence it remains an open problem that

What is the necessary and sufficient condition for the existence of nontrivial operators $\mathscr{U}(g)$ which commutes with a given $\mathscr{U}(f)$?

The solution of this problem would be of much value to the theory of soliton, because the commutativity of $\mathscr{U}(f)$'s is closely related, as mentioned above, to the existence of nontrivial symmetries, especially to the existence of an infinite number of conservation laws, which is in turn closely related to the soliton solution of an evolution equation.

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References

1. Scott, A.C., et al.: IEEE Proceedings **61**, 1443–1483 (1973)
2. Drifeld, V.G.: Funct. Anal. **11**, 11–14 (1977) (in Russian)
3. Krieshevel, E.M.: Funct. Anal. **12**, 20–31 (1978) (in Russian)
4. Amitsur, S.A.: Pacific J. Math. **8**, 1–10 (1958)
5. Tu, G.Z., Cheng, M.Z.: Kexue Tongbao **24**, 913–917 (1979) (in Chinese)
6. Tu, G.Z.: Lett. Math. Phys. **3**, 387–393 (1979)
7. Kruskal, M.D., et al.: J. Math. Phys. **11**, 952–960 (1970)
8. Olver, P.J.: J. Math. Phys. **18**, 1212–1215 (1977)
9. Tu, G.Z.: Nature J. **2**, 531 (1979) (in Chinese)

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