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The Lipatov Argument

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Abstract. Lipatov's argument gives a formula for evaluating asymptotically the large order perturbation coefficients for the anharmonic oscillator or (ϕ^4) quantum field models. We give a partial justification of the argument which enables us to prove that the radius of convergence of the Borel transform of the pressure for lattice ϕ^4 models given by

$$\exp\left[\inf_{\phi}\left\{\tfrac{1}{2}\sum_{j}[(\nabla\phi)^2(j)+\phi(j)^2]-\log\sum\phi(j)^4\right\}-2\right].$$

Let $E(\lambda)$ be the ground state energy for the anharmonic oscillator

$$H(\lambda)=-\tfrac{1}{2}\frac{d^2}{dx^2}+\frac{x^2}{2}+\lambda x^4-\tfrac{1}{2}.$$

It is well known that $E(\lambda)$ has an asymptotic but divergent series in λ

$$E(\lambda)\approx\sum_{n=0}^{\infty}a_n\lambda^n. \tag{1}$$

We shall discuss the behavior of a_n for large n.

In 1973 Bender and Wu [2] developed W.K.B. techniques to obtain asymptotics of the form

$$a_n\approx C_0C_1^n n^\alpha n!\left(1+\frac{O(1)}{n}\right) \tag{2}$$

with explicit expressions for C_0, C_1, and α. Recently Benassi et al. [1] have rigorously established (2) along the lines of Bender and Wu. Several years later Lipatov [5] developed steepest descent methods for functional integrals which he and Brezin et al. [3] applied to quantum field models to obtain results analogous

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to (2). In [4] it is shown that these asymptotics together with the Borel transform yield impressive numerical calculations of critical exponents.

This note provides a partial justification of Lipatov's method. We shall show how Laplace asymptotics combined with some simple inequalities on the graphs contributing to a_n are sufficient (modulo technicalities discussed later) to compute the radius of convergence of the Borel transform

$$B(t) = \sum \frac{a_n}{n!} t^n .$$

The radius of convergence is

$$\lim_{n\to\infty} \left|\frac{a_n}{n!}\right|^{-1/n} = C_1^{-1} = \exp \inf F(\phi), \tag{3}$$

where

$$F(\phi) = \tfrac{1}{2} \int [(\nabla\phi)^2(x) + \phi^2(x)]\,dx - \log \int \phi^4(x)\,dx - 2 . \tag{4}$$

Actually here one can explicitly solve the corresponding Euler equation to show that the minimizing ϕ is proportional to $\operatorname{sech} x$. Our methods might be sharpened to obtain α and C_0 in (2) and we expect them to apply to the ϕ^4 model in two space time dimensions. In fact we shall prove the analogue of (3) for lattice ϕ^4 models in any dimension where $E(\lambda)$ in (1) is replaced by the pressure

$$p(\lambda) = \lim_{\Lambda \uparrow Z^\nu} \frac{1}{|\Lambda|} \log \int \exp\left[-\lambda \sum_{j\in\Lambda} \phi^4(\mathrm{j})\right] \mathrm{d}\mu(\phi),$$

where $d\mu$ is the lattice free field with covariance $(-\Delta+1)^{-1}$ and Δ is the lattice laplacian.

We now briefly discuss Lipatov's argument. We shall concentrate on the anharmonic oscillator and discuss other models in the remarks. Let $\Omega_0 = \exp - x^2/2$. By the spectral theorem

$$E(\lambda) = \lim_{T\to\infty} -\frac{\log}{T} \langle \Omega_0, e^{TH(\lambda)} \Omega_0 \rangle_{L^2(R)} . \tag{5}$$

Let $d\mu$ be the Gaussian measure with covariance $(-d^2/dx^2+1)^{-1}$ i.e. an Ornstein Uhlenbeck process. Lipatov now fixes T in (5) and studies the large order perturbation coefficients b_n^T of

$$\langle \Omega_0, e^{-TH(\lambda)} \Omega_0 \rangle = \int \exp\left[-\lambda \int_0^T \phi^4(s)\,ds\right] d\mu$$

by applying steepest descent jointly in ϕ and λ to the formal relation

$$b_n^T = \frac{1}{n!} \frac{1}{2\pi i} \oint \int e^{-\lambda \int_0^T \phi^4(s)ds} \lambda^{-n-1}\, d\mu\, d\lambda .$$

This contour integral representation is very useful for interactions in which the coupling constant does not appear linearly (e.g. in a double well) but is difficult to justify. Instead we use the simple identity

$$b_n^T = \frac{1}{n!} \int \left[\int_0^T \phi^4(s) ds \right]^n d\mu . \tag{6}$$

The following lemma shows that for fixed T the log in (5) does not affect the large n asymptotics. Hence it suffices to study b_n^T.

Lemma 1. *Let $f(\lambda)$ be infinitely differentiable for $\lambda > 0$ with a Taylor expansion $\sum_{n \geqq 1} b_n \lambda^n$. If*

$$b_n = c_0 n^{1/\alpha} c_1^n n! \left(1 + O\left(\frac{1}{n}\right)\right). \tag{7}$$

Then the corresponding coefficients a_n of $\log(1 + f(\lambda))$ *also satisfy* (7).

We omit the elementary proof, but we shall present the proof of a closely related lemma at the end of this article.

Now once the asymptotics of b_n^T have been obtained one then takes the $T \to \infty$ limit and (2) follows.

There are two major mathematical difficulties in the above outline. The first one is to obtain large n asymptotics for fixed T by Laplace's method in function space. For Weiner integrals such methods have been investigated by Schilder and Pincus [6, 7]. See also [8] for a discussion of the particular case considered here. However, to obtain asymptotics as sharp as (7) present mathematical techniques require isolated non-degenerate minima. Notice that for periodic boundary conditions there is a one parameter family of minima.

The second major problem concerns the interchange of the large n and T limits which was crucial to the above argument. The main purpose of this note is to show how to resolve this difficulty by studying the graphs which contribute to a_n in perturbation theory.

By standard perturbation theory a_n is the sum of all connected graphs γ having n vertices and with precisely 4 lines attached to each vertex. To each line of γ joining the i^{th} vertex to the j^{th} vertex one associates a factor of

$$G(x_i - x_j) = (-\Delta + 1)^{-1}(x_i, y_j).$$

Thus

$$a_n = \lim_{V \to \infty} \frac{1}{V} \sum_{\gamma} \int_{-V/2}^{+V/2} \Pi_\gamma G(x_i - x_j) dx_1 \dots dx_n , \tag{8}$$

where Π_γ ranges over the lines of γ. Now fix an interval $[-T/2, T/2]$ and let G_D, G_P denote the Green's function with Dirichlet and periodic boundary conditions on the boundary of $I_m = [(m-1)T/2, (m+1)T/2]$, $m \in \mathbb{Z}$. Note that

$$\begin{aligned} &0 \leqq G_D(x, y) \leqq G(x, y) \\ &0 \leqq G_P(x, y) = \sum_{n=-\infty}^{+\infty} G(x - y + nT) \qquad x, y \in I_0 . \end{aligned} \tag{9}$$

Let $d\mu_X$ be the Gaussian measure with covariance G_X and define

$$a_n^X=(n!T)^{-1}\frac{d^n}{d\lambda^n}\bigg|_{\lambda=0}\log\left\{\int\exp\left[-\lambda\int_{-T/2}^{T/2}\phi^4(s)ds\right]d\mu_X\right\}$$
$$=\frac{1}{T}\sum_\gamma\int_{-T/2}^{T/2}\Pi_\gamma G_X(x_i-x_j)dx_1\ldots dx_n$$

and also let

$$b_n^X=\frac{1}{n!}\int\left[\int_{-T/2}^{+T/2}\phi^4(s)ds\right]^n d\mu_X.$$

Lemma 2. *For all n and T*

$$a_n^D\leqq a_n\leqq a_n^P. \tag{10}$$

This lemma enables us to fix T and analyze a_n^X or equivalently $b_n^{T,X}$ for large n. Suppose that one can prove

$$b_n^{T,X}=(C_1^X)^n C_0^X n^\alpha n!\left(1+O\left(\frac{1}{n}\right)\right),$$

where

$$C_1^X=\exp-\left\{\inf_\phi\tfrac{1}{2}\int_{-T/2}^{T/2}(\nabla\phi)^2(x)+\phi^2(x)dx-\log\int_{-T/2}^{T/2}\phi^4(x)dx+2\right\}.$$

When $X=P$ we identify $T/2$ and $-T/2$ and when $X=D$ ϕ is required to vanish at $\pm T/2$. The above result is elementary to establish for lattice ϕ^4 models since the corresponding integrals are finite dimensional. For the anharmonic oscillator the methods of [7] yield this result except in the case of periodic data where there are a continuous family of minima which violates a technical condition of [7]. After dividing both sides of (10) by $n!$ and taking the n^{th} root we have

$$C_1^{D,T}\leqq\lim_{n\to\infty}\left|\frac{a_n}{n!}\right|^{1/n}\leqq C_1^{P,T}.$$

We shall show later that for lattice models

$$\lim_{T\to\infty}C_1^{D,T}=\lim_{T\to\infty}C_1^{P,T}=C_1 \tag{11}$$

which gives the desired result (3).

Proof of Lemma. If we replace G by G_D in (8), the resulting expression is clearly smaller by (9). Since $G_D(x_i,x_j)$ vanishes whenever x_i and x_j belong to distinct intervals I_m we see that by translation invariance with respect to nT we can replace V by T. Thus the lower bound holds.

To prove the upper bound we use the obvious identity (for $V=MT$ and $x_1=0$)

$$\int_{-V}^{V}\Pi_\gamma G(x_i-x_j)dx_2\ldots dx_n$$
$$=\sum_{n_j\in\mathbb{Z}}\int_{-T/2}^{T/2}\Pi_\gamma G(x_i-x_j+(n_i-n_j)T)dx_2\ldots dx_n, \tag{12}$$

where $n_1 = 0$ and $|n_j| \leqq M$. By (9) the corresponding periodic graph is (with $x_1 = 0$)

$$\sum_{m_{ij} \in \mathbb{Z}} \int_{-T/2}^{T/2} \Pi_\gamma G(x_i - x_j + m_{ij} T) dx_2 \ldots dx_n . \tag{13}$$

It is easy to see that each term of (12) appears in (13) (but not conversely). Since all terms are positive the proof is complete.

Remarks. The proof of the above lemma also applies with only minor modifications to ϕ^4 models on a lattice $\mathbb{Z}^\nu$ and to the continuum $(\phi^4)_2$ model. In the case of $(\phi^4)_2$ we normal order the interaction with respect to the underlying Gaussian measure $d\mu$ or $d\mu_X$. The graphs are described as before except that lines linking a vertex to itself are not allowed. The methods of this note are not restricted to the study of the ground state energy or pressure. They apply equally well to the perturbation theory of Schwinger or correlation functions. By arguments analogous to those of Lemma 1 it suffices so show that

$$\int \prod_i \phi(x_i) \Big[\sum_{j \in \Lambda} \phi^4(j) \Big]^n d\mu_X = F(x) C_0 C^n n^\alpha n! \left(1 + O\left(\frac{1}{n}\right)\right).$$

The proof of Lemma 2 depends heavily on the fact that all graphs contributing to a fixed order have the same sign. For this reason we cannot provide a similar proof in the case of the double well anharmonic oscillator. For the $(\phi^4)_3$ model there is also a difficulty arising from the mass counter term which makes the signs of the corresponding graphs difficult to determine.

The rest of this article is devoted to the ϕ^4 model on a lattice $\mathbb{Z}^\nu$. Let Δ_X now denote the finite difference Laplacian with boundary conditions $X = P$ or D and define

$$F_X^T(\phi) = \langle \phi, (-\Delta_X + 1)\phi \rangle - \log\Big(\sum_{j \in \Lambda} \phi(j)^4 \Big),$$

where $\Lambda = \left[-\frac{T}{2}, \frac{T}{2} \right]^\nu \cap \mathbb{Z}^\nu$. By scaling $\phi(j) \to \sqrt{n}\,\phi(j)$ we see that

$$\begin{aligned} n!\, b_n^X &= \int [\sum \phi(j)^4]^n d\mu_X \\ &= n^{|\Lambda|/2} n^{2n} Z_\Lambda^{-1} \int \exp[-n F_X^T(\phi)] \prod_{j \in \Lambda} d\phi(j), \end{aligned}$$

where

$$Z_\Lambda = \int \exp[-\langle \phi, (-\Delta_X + 1)\phi \rangle] \prod_{j \in \Lambda} d\phi(j).$$

There is a positive measure $d\alpha(t)$ such that

$$\int \exp[-n F_X^T(\phi)] \prod_{j \in \Lambda} d\phi(j) = \int e^{-nt} d\alpha(t).$$

Clearly

$$\begin{aligned} \Big(\int e^{-nt} d\alpha(t)\Big)^{1/n} &\to e^{-(\inf \operatorname{supp} d\alpha(t))} \\ &= \exp - \inf_\phi (F_X^T(\phi)). \end{aligned}$$

However this estimate alone is insufficient to yield the corresponding estimate on a_n. Higher order asymptotics in finite volume are difficult to establish because we need to know uniqueness and non-degeneracy of the minima of $F_X^T(\phi)$. Nevertheless, the following lemma will enable us to obtain the desired asymptotics on $a_n^{X,T}$.

Lemma 3. *Let $U(\lambda)$ be a C^∞ function of $\lambda \geqq 0$ with a Taylor series*

$$U(\lambda)=\sum_{n=1}^{\infty} b_n \lambda^n .$$

If there is a positive measure $d\alpha(t)$ whose support is bounded from below such that $\int e^{-t} d\alpha(t) < \infty$ and

$$b_n = n^p \frac{n^{2n}}{n!} \int e^{-tn} d\alpha(t) .$$

Then the Taylor series of $\log(1+U(t)) = \sum a_n \lambda^n$ has coefficients which satisfy

$$a_n = b_n \left(1 + O\left(\frac{1}{n}\right)\right).$$

Proof. Using the series for $\log(1+x)$ it is easy to show that

$$a_n = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m} \sum_{\substack{n_1+\ldots+n_m=n \\ n_i \geqq 1}} \prod^{m} b_{n_i} .$$

To prove the lemma we must bound

$$b_n^{-1} \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{\substack{\sum n_i = n \\ n_i \geqq 1}} \prod_{i=1}^{m} b_{n_i}$$

$$= \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{\substack{\sum n_i = n \\ n_i \geqq 1}} \frac{\prod^{m} n_i^{2n_i} (n_i!)^{-1} n_i^p \int e^{-tn_i} d\alpha(t)}{n^{2n} (n!)^{-1} n^p \int e^{-tn} d\alpha(t)} . \tag{14}$$

By a change of variables we may assume

$$0 = \inf \operatorname{supp} d\alpha(t)$$

hence

$$\mathrm{Const} \geqq \int e^{-tn} d\alpha(t) \geqq e^{-n\varepsilon_n} ,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Now using the above bound and the log convexity of $\int e^{-nt} d\alpha(t)$ we have

$$\frac{\int e^{-n_1 t} d\alpha(t)}{\int e^{-nt} d\alpha(t)} \leqq \mathrm{Const}(1+\varepsilon_n)^{n-n_1} .$$

These estimates combined with Stirling's formula yield

$$\frac{\prod^m b_{n_i}}{b_n} \leqq \frac{\mathrm{Const}^m}{m} \frac{\prod n_i! n_i^{p+1/2}}{n! n^{p+1/2}} (1+\varepsilon)^{(n-\max(n_i))}.$$

In order to bound the sum over n_i let us relabel the index i so that n_1 is maximum of n_i i.e.,

$$n_1 = n - \sum_2^m n_i \geqq n_j \qquad j = 2, 3, \ldots, m.$$

Now observe that for $|\varepsilon| \leqq \frac{1}{4}$

$$\sum_{r/2 \geqq q \geqq 1} q!(r-q)!(1+\varepsilon)^{2q} \leqq (r-1)! \,\mathrm{Const}.$$

Iterated application of this inequality yields a bound on (14)

$$\frac{\mathrm{Const}^m}{n!} \sum_{\substack{n - \sum_2^m n_i \geqq n_j \\ n_i \geqq 1 \quad i = 2, 3, \ldots, m}} \prod_2^m n_i!(1+\varepsilon)^{2n_i}(n - n_2 - \ldots - n_m)!$$

$$\leqq \mathrm{Const}^m \frac{(n-m+1)!}{n!}.$$

The sum over $m \geqq 2$ shows

$$a_n = b_n \left(1 + \sum_{m \geqq 2}^{n} \frac{(n-m+1)!}{n!} \mathrm{Const}^m\right)$$

$$= b_n \left(1 + O\left(\frac{1}{n}\right)\right).$$

To conclude this note we verify (11) for ϕ^4 theories on a lattice $\mathbb{Z}^\nu$. Since

$$\sum \phi(j)^4 \leqq \left(\sum \phi(j)^2\right)^2$$

it is clear that there is a constant M independent of T such that

$$\tfrac{1}{2} \sum \phi^2(j) - M \leqq \sum_{|j_i| \leqq T/2} \phi(j)^2 + (\nabla\phi)^2 - 2 \log \sum \phi(j)^4 \equiv F_X^T(\phi). \tag{15}$$

When $X = P$ we interpret the sum as a sum over the torus. Now let $\phi_P(j)$ minimize F_P^T among periodic functions. By (15) we see that $\sum \phi_P(j)^2 \leqq \mathrm{Const}$ hence there is a $j_1^*, j_2^* \ldots j_\nu^*$ such that for each $i \leqq \nu$

$$\sum_{j:\, j_i = j_i^*} \phi_P(j)^2 + \sum_{j:\, j_i = j_i^*} (\nabla\phi_P)(j)^2 \leqq \frac{\mathrm{Const}}{T}.$$

Since translates of ϕ_P also minimize F_P we can choose $j_i^* = \pm T/2$. Now define

$$\begin{aligned} \phi'(j) &= \phi_P(j) \qquad |j_i| < T/2 \quad \text{for all} \quad i \\ &= 0 \qquad\quad\; |j_i| = T/2 \quad \text{some} \quad i. \end{aligned}$$

From (10) we have

$$F_P(\phi_P) \leqq F_D(\phi_D) \leqq F_D(\phi')$$

since ϕ' satisfies Dirichlet boundary conditions. Since

$$|F_P(\phi_P) - F_D(\phi')| \leqq \frac{\text{Const}}{T}.$$

The first equality of (11) follows. The second can be established similarly.

Remarks. From recent work of Gidas, Ni, and Nirenberg, the ϕ which minimizes $F(\phi)$ of (4) decays exponentially at infinity (provided that the dimension is less than or equal to 4). Using their methods one expects that

$$C_1^{T,P} - C_1^{T,D} \tag{16}$$

goes to zero exponentially fast with T.

In order to establish the coefficients C_0 and α of (2) we propose that one take T to depend weakly on n, e.g. $T = (n^\varepsilon)$. Assuming (16) goes to zero exponentially fast in T we have

$$[C_1^{T,X}]^n = C_1^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

It then remains to justify a modified Laplace expansion in which $T \approx (n^\varepsilon)$. Formally one obtains (2) but with $O\left(\frac{1}{n}\right)$ replaced by $O(n^{-\varepsilon})$, for small $\varepsilon > 0$.

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