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Born Series for (2 Cluster) → (2 Cluster) Scattering of Two, Three, and Four Particle Schrödinger Operators

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Abstract. We investigate elastic and inelastic (2 cluster) → (2 cluster) scattering for classes of two, three, and four body Schrödinger operators $H = H_0 + \sum_{i<j} V_{ij}$. Formulas are derived for those generalized eigenfunctions of H which correspond asymptotically in the past to two freely moving clusters. With these eigenfunctions, we establish a formula for the (2 cluster) → (2 cluster) T-matrix and prove the convergence of a Born series for the T-matrix at high energy.

1. Introduction

In this paper we investigate (2 cluster) → (2 cluster) scattering of certain classes of two, three, and four particle Schrödinger operators. We begin by finding those generalized eigenfunctions of the Hamiltonian, $\phi_\alpha(X, k)$, which correspond to two cluster initial channels α. Using these eigenfunctions we prove the validity of a formula for the physicists' T-matrix for (2 cluster)→(2 cluster) elastic and inelastic processes. We then prove the convergence of a Born series expansion for the T-matrix at high energy.

In the two body case, we have little to say which is new. Eigenfunction expansions for two body Hamiltonians are developed in [2, 11, 14, 17, 20]. The two body T-matrix formula is proved in [17, 20]. Also, for each potential V in certain classes, there exists $E_0 < \infty$, such that the Born series converges for energies in (E_0, ∞) [1, 6, 15, 17, 20, 24]. However, our methods and the closely related methods of [15] are presently the only methods which can be used to estimate E_0.

Our principal new results deal with 3 and 4 body systems. Previous authors [5, 13, 19, and references therein] have obtained the asymptotic behavior and distributional Fourier transforms of the generalized eigenfunctions. They have not proved Born series convergence nor the validity of the T-matrix formula [Eq. (1.2)].

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Theorems 1.1–1.3 are precise statements of our main results:

Theorem 1.1. *Let* $m \geqq 3$ *and* $N \leqq 4$. *Let* $H = H_0 + \sum_{i<j} \lambda_{ij} V_{ij}$ *be an* N *particle Schrödinger operator (with the center of mass motion removed) on* $\mathscr{H} = L^2(\mathbb{R}^{(N-1)m})$. *Assume each* V_{ij} *may be factored as* $V_{ij} = U_{ij} W_{ij}$, *so that:*

i) *each* U_{ij} *and* W_{ij} *is dilation analytic in some strip,*

ii) $(1+x_{ij}^2)^\nu U_{ij}(x_{ij})$ *and* $(1+x_{ij}^2)^\nu W_{ij}(x_{ij})$ *belong to* $L^p(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$ *for some* $p > m$ *and* $\nu > m/4$,

iii) *bound state energies of three body subsystems are non-positive when* $N = 4$.

Then, for generic couplings $\{\lambda_{ij}\}$, there is a closed set $\mathscr{E} \subseteqq \mathbb{R}$ of measure zero, such that the following hold for each channel α, whose cluster decomposition $D(\alpha)$ contains exactly two clusters.

(a) Let E_α denote the threshold energy corresponding to α, and let M_α be the reduced mass associated with the coordinate between the centers of mass of the clusters of α. If $k \in \mathbb{R}^m$ satisfies $k^2/2M_\alpha + E_\alpha \notin \mathscr{E}$, then, for sufficiently large δ, there exists $\phi_\alpha(X,k) \in L^2_{-\delta}(\mathbb{R}^{(N-1)m}, dX) = \{f : (1+X^2)^{-\delta/2} f(X) \in L^2(\mathbb{R}^{(N-1)m})\}$, such that $f \in L^2_\delta(\mathbb{R}^{(N-1)m})$ implies $f^\#_\alpha(k) = (2\pi)^{-m/2} \int \overline{\phi_\alpha(X,k)} f(X) dX$ satisfies $f^\#_\alpha = ((\Omega^+_\alpha)^* f)^\wedge$.

(b) $f \mapsto f^\#_\alpha$ extends uniquely to a partial isometry (also denoted $f \mapsto f^\#_\alpha$) with initial subspace $\operatorname{Ran} \Omega^+_\alpha \subseteqq \mathscr{H}$ and final subspace $\mathscr{H}_\alpha = L^2(\mathbb{R}^m)$. If $f \in \mathscr{H}_\alpha$, then $\hat{f} = (\Omega^+_\alpha f)^\#_\alpha$.

(c) If $f \in D(H)$, then $(Hf)^\#_\alpha(k) = (k^2/2M_\alpha + E_\alpha) f^\#_\alpha(k)$.

Remarks. 1. Balslev [3] and Simon [21] have given sufficient conditions for hypothesis iii) to hold (see also [8, Theorem II.10]). Yukawa potentials, generalized Yukawa potentials, and potentials of the form $r^{-1}(1+\varepsilon r)^{-m+1-\varepsilon}$ satisfy all the hypothesis of Theorem 1.1.

2. The generic couplings are precisely those for which no cluster Hamiltonian has a threshold resonance or threshold bound state. This set of couplings is large in the sense that its complement is a closed set of measure zero [8, Sect. VI].

3. Note that $f^\#_\alpha \equiv 0$ if f is orthogonal to $\operatorname{Ran} \Omega^+_\alpha$.

Definition. Assume the hypotheses of Theorem 1.1, and assume $\{\lambda_{ij} = 1\}$ is a set of generic couplings. Let α and β both be two cluster channels, and let $\phi^0_\alpha(X,k) = e^{ik\zeta} \psi_\alpha(x_\alpha)$. Here ψ_α is the tensor product of the bound states of the clusters of α; ζ is the coordinate between the centers of mass of the clusters of α; and $X = (x_\alpha, \zeta)$.

Let $V = \sum_{i<j} V_{ij}$, and let $V_{D(\alpha)}$ be the sum of all V_{ij} with i and j in the same cluster of $D(\alpha)$. If $k'^2/2M_\beta + E_\beta \notin \mathscr{E}$, then we define

$$T_{\alpha,\beta}(k,k') = (2\pi)^{-m} \int_{\mathbb{R}^{(N-1)m}} \overline{\phi^0_\alpha(X,k)} [V(X) - V_{D(\alpha)}(X)] \phi_\beta(X,k') dX \,. \tag{1.1}$$

Theorem 1.2. *Assume the hypotheses of Theorem 1.1, and assume* $\{\lambda_{ij} = 1\}$ *is a set of generic couplings. Then* $T_{\alpha,\beta}(k,k')$ *is continuous in* k *and* k' *for* $k'^2/2M_\beta + E_\beta \notin \mathscr{E}$, *whenever both* α *and* β *are two cluster channels. Suppose* $f \in \mathscr{H}_\alpha = L^2(\mathbb{R}^m)$ *and* $g \in \mathscr{H}_\beta = L^2(\mathbb{R}^m)$ *are chosen so that* $\hat{f}$ *and* $\hat{g}$ *are* C^∞ *with compact support, and assume*

$k' \in \operatorname{supp} \hat{g}$ implies $k'^2/2M_\beta + E_\beta \notin \mathscr{E}$. If $S_{\alpha,\beta}: \mathscr{H}_\beta \to \mathscr{H}_\alpha$ denotes the block of the S-matrix of $H = H_0 + \sum_{i<j} V_{ij}$ for scattering from channel β to channel α, then

$$\langle f, S_{\alpha,\beta} g \rangle - \delta_{\alpha,\beta} \langle f, g \rangle$$

$$= -2\pi i \int_{\mathbb{R}^{2m}} \overline{\hat{f}(k)} \hat{g}(k') \delta(k^2/2M_\alpha + E_\alpha - k'^2/2M_\beta - E_\beta) T_{\alpha,\beta}(k, k') dk dk' . \tag{1.2}$$

The Born series for the two body T-matrix arises from the substitution in Eq. (1.1) of a series for $\phi_\beta(X, k')$. This series for $\phi_\beta(X, k')$ is obtained by expanding $(1 - M(z))^{-1}$ in Eq. (2.2) as a geometric series, and substituting the result in Eq. (3.1) for $\phi_\beta(X, k')$. If $M(z) = W(z - H_0)^{-1} U$ has norm less than 1, the Born series converges. Various authors [1, 6, 17, 20, 24] prove $\|M(z)\| \to 0$ as $\operatorname{Re} z \to \infty$. Our slightly stronger result (also obtained in [15]) is $\|M(z)\| \leqq C(\operatorname{Re} z)^{-\varepsilon}$, where C and ε depend on U and W. Thus, given a potential, this allows the computation of an energy above which the two body Born series converges. Estimates of the error in the n^{th} Born approximation can also be calculated.

For $N = 3$ and $N = 4$ we mimic the above procedure to obtain expansions which we call the Born series. We expand the 3 and 4 body Eq. (2.2) by expanding $(1 - M(z))^{-1}$ as a geometric series, and substitute the result in Eq. (3.1) for $\phi_\beta(X, k')$. This series for ϕ_β is then substituted into Eq. (1.1), thus giving us a series for $T_{\alpha,\beta}(k, k')$.

For example, if $N = 3$, $D(\beta) = \{\{1, 2\}, \{3\}\}$, and $E = k'^2/2M_\beta + E_\beta$, then $\phi_\beta(X, k') = \sum_{n=0}^{\infty} \phi_\beta^{(n)}(X, k')$, where the first few terms are:

$$\phi_\beta^{(0)}(X, k') = \phi_\beta^0(X, k') = e^{ik'\zeta} \psi_\beta(x_\beta) ,$$

$$\begin{aligned} \phi_\beta^{(1)}(X, k') = [&(E + i0 - H_{13})^{-1} V_{13} \phi_\beta^0(\cdot, k') \\ &+ (E + i0 - H_{23})^{-1} V_{23} \phi_\beta^0(\cdot, k')](X) , \end{aligned}$$

$$\begin{aligned} \phi_\beta^{(2)}(X, k') = [&(E + i0 - H_{12})^{-1} V_{12} (E + i0 - H_{13})^{-1} V_{13} \phi_\beta^0(\cdot, k') \\ &+ (E + i0 - H_{23})^{-1} V_{23} (E + i0 - H_{13})^{-1} V_{13} \phi_\beta^0(\cdot, k') \\ &+ (E + i0 - H_{12})^{-1} V_{12} (E + i0 - H_{23})^{-1} V_{23} \phi_\beta^0(\cdot, k') \\ &+ (E + i0 - H_{13})^{-1} V_{13} (E + i0 - H_{23})^{-1} V_{23} \phi_\beta^0(\cdot, k')](X) . \end{aligned}$$

The n^{th} term for $T_{\alpha,\beta}(k, k')$ is given by

$$(2\pi)^{-m} \int_{\mathbb{R}^{(N-1)m}} \overline{\phi_\alpha^0(X, k)} [V(X) - V_{D(\alpha)}(X)] \phi_\beta^{(n)}(X, k') dX .$$

For $N = 3$, this Born series is sometimes called the Faddeev-Watson series. It is physically motivated by decomposing the three body scattering into sequences of two body collisions. For $N = 4$, the situation is much more complicated because the barely connected perturbation diagrams contributing to $M(z)$ are more complicated.

Theorem 1.3. *Assume the hypotheses of Theorem 1.1, and assume* $\{\lambda_{ij}=1\}$ *is a set of generic couplings. If* $M(z)$ *is defined by Eq.* (2.2), *then*

a) *for* $N=2$, *the corresponding* $M(z)$ *satisfies* $\|M(z)\| \leqq C(\operatorname{Re} z)^{-\varepsilon}$,

b) *the* $N=3$ *body* $M(z)$ *satisfies* $\|(M(z))^n\| \leqq (c \operatorname{Re} z)^{-[n/2]\varepsilon}$, *where* $[n/2]$ *is the greatest integer less than or equal to* $n/2$,

c) *the* $N=4$ *body* $M(z)$ *satisfies* $\lim_{\operatorname{Re} z \to \infty} \|(M(z))^n\| = 0$ *for* $n \geqq 2$.

So, for $N \leqq 4$, *the Born series for the* (2 cluster)$\to$(2 cluster) *T-matrix converges at high energies.*

Remarks. 1. The fact that $M(z)$ contains only connected perturbation diagrams is crucial for this theorem. Disconnected diagrams *cannot* tend to zero in norm as $z \to \infty$ along the positive real axis.

2. The proof of b) also holds for the potentials of Ginibre and Moulin [7]. As a consequence, there is no high energy singular continuous spectrum for these potentials. Previously, only the absence of negative energy singular continuous spectrum has been proved [7].

Throughout the paper we use results of [8]. Section 2 establishes notation and recalls a few essential results of [8]. Section 3 contains the proofs of Theorems 1.1 and 1.2. These proofs are similar to their 2-body analogs, except that certain formulas from [8] are required. The proofs of Theorem 1.3 for $N=2$, 3, and 4 are given in Sects. 4, 5, and 6, respectively. The principal methods involved in these sections are those of [8] and [4].

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2. Preliminaries

Throughout this paper, we use results and notation from [8]. Those technical devices not discussed below (such as clustered Jacobi coordinates, dilation analyticity, etc.), are discussed in [8], as well as the references to [8]. We will, however, recall a few definitions and facts.

The Schrödinger operator for N particles in m dimensions is

$$\tilde{H} = -\sum_{i=1}^{N} (2m_i)^{-1} \Delta_i + \sum_{i<j} V_{ij}(r_i - r_j)$$

on $L^2(\mathbb{R}^{Nm})$. We remove the center of mass motion from $\tilde{H}$ to obtain $H = H_0 + \sum_{i<j} V_{ij}$ on $\mathscr{H} = L^2(\mathbb{R}^{(N-1)m})$.

A *cluster decomposition* $D = \{C_i\}_{i=1}^{k}$ is a partition of $\{1, 2, \ldots, N\}$ into k disjoint clusters C_i. $H_D = H_0 + V_D$, where V_D is the sum of all V_{ij} with i and j in the same cluster of D. $\mathscr{H}$ may be decomposed as $\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \ldots \otimes \mathscr{H}_k \otimes \mathscr{H}(D)$ so that $H_D = h_1 \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes h_2 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes h_k \otimes 1 + 1 \otimes \ldots \otimes 1 \otimes K_D$. The Hamiltonian h_i corresponds to the energy of the particles in cluster C_i alone. K_D is the kinetic energy of the centers of mass of the clusters of D.

For each i, we choose eigenfunctions $\eta_j^{(i)}$ of h_i so that $\{\eta_j^{(i)}\}$ is an orthonormal basis for the subspace of $\mathscr{H}_i$ generated by the eigenfunctions of h_i.

A *channel* α is a cluster decomposition $D(\alpha)$ together with an eigenfunction $\eta^{(i)}\in\{\eta_j^{(i)}\}$ for each h_i. We define $E_\alpha=\sum_{i=1}^{k} E_i$, where $h_i\eta^{(i)}=E_i\eta^{(i)}$, and let $\psi_\alpha=\eta^{(1)}\otimes\eta^{(2)}\otimes\ldots\otimes\eta^{(k)}$. $P_\alpha:\mathscr{H}\to\mathscr{H}$ is the orthogonal projection onto all vectors of the form $\psi_\alpha\otimes\phi$, where $\phi\in\mathscr{H}(D(\alpha))$ is arbitrary. $\mathscr{H}_\alpha=\mathscr{H}(D(\alpha))$ is identified with the range of P_α by identifying ϕ with $\psi_\alpha\otimes\phi$. We let P_D be the sum of all P_α with $D(\alpha)=D$.

Using this identification and letting $T_\alpha=1\otimes1\otimes\ldots\otimes1\otimes K_{D(\alpha)}+E_\alpha$, we define the *channel wave operators* $\Omega_\alpha^\pm:\mathscr{H}_\alpha\to\mathscr{H}$ by

$$\Omega_\alpha^\pm=\underset{t\to\mp\infty}{\text{strong-limit}}\, e^{itH}e^{-itT_\alpha}P_\alpha\,.$$

When α is a two cluster channel, we let M_α denote the reduced mass associated with the coordinate between the centers of mass of the clusters of $D(\alpha)$.

In Sect. III of [8] the multiparticle limiting absorption principle is used to obtain expressions for $((\Omega_\alpha^\pm)^*f)\hat{}$ for a dense set of f's. To obtain a partial eigenfunction expansion we require a stronger version of this absorption principle. The only difference between the version below and the version in [8] is that we require $\mu>m/2$ rather than $\mu>1$.

Definition. $L_\delta^p(\mathbb{R}^n)=\{f:(1+x^2)^{\delta/2}f(x)\in L^p(\mathbb{R}^n)\}$.

Definition. Let H be an N particle Hamiltonian on $L^2(\mathbb{R}^{(N-1)m})$, with $m\geqq3$. The *strong multiparticle limiting absorption principle* holds for H if

$$(z-H)^{-1}=\sum_D(z-H_D)^{-1}P_D\sum_{\ell=1}^{L(D)}F_{\ell,D}Z_{\ell,D}(z)\,,$$

where:

(a) there exists δ_0 such that $\phi\in L_{\delta_0}^2(\mathbb{R}^{(N-1)m})$ implies $Z_{\ell,D}(z)\phi$ is an $L^2(\mathbb{R}^{(N-1)m})$ valued meromorphic function in $\mathbb{C}\backslash\sigma_{\text{ess}}(H)$, with continuous extensions to $\sigma_{\text{ess}}(H)$ from above and below in the complement of a closed set $\mathscr{E}$ of measure zero;

(b) for each ℓ and D, $F_{\ell,D}$ maps $L^2(\mathbb{R}^{(N-1)m})$ into $L_\mu^p(\mathbb{R}^m)\otimes L^2(\mathbb{R}^{(N-2)m})$ for some $\mu>m/2$ and $p\in(1,2]$, where the first factor denotes functions of a Jacobi coordinate for the motion of the centers of mass of clusters of D.

Lemma 2.1. *Suppose the strong multiparticle limiting absorption principle holds for H on $L^2(\mathbb{R}^{(N-1)m})$, where $m\geqq3$. Let α be a 2 cluster channel, and assume $\Omega_\alpha^\pm$ exist. If $\phi\in L_{\delta_0}^2(\mathbb{R}^{(N-1)m})$, then, for all $k\in\mathbb{R}^m$ with $k^2/2M_\alpha+E_\alpha\notin\mathscr{E}$,*

$$((\Omega_\alpha^\pm)^*\phi)\hat{}(k)=\left(P_\alpha\sum_{\ell=1}^{L(D(\alpha))}F_{\ell,D(\alpha)}Z_{\ell,D(\alpha)}(k^2/2M_\alpha\mp i0)\phi\right)\hat{}(k)\,. \tag{2.1}$$

(Note that both sides are Fourier Transforms of functions on $\mathbb{R}^m$. We have identified $\mathscr{H}_\alpha$ and $P_\alpha\mathscr{H}$.)

Proof. Proposition III.6 of [8] shows that the result is valid if both sides are viewed as L^2 functions on the sphere of radius $|k|$ (see also [5, 7, 10, 23]). Note that the right-hand side of (2.1) is actually continuous whenever $k^2/2M_\alpha+E_\alpha\notin\mathscr{E}$. This is because $\mu>m/2$ in the strong multiparticle limiting absorption principle implies

$$P_\alpha\sum_{\ell=1}^{L(D(\alpha))}F_{\ell,D(\alpha)}Z_{\ell,D(\alpha)}(k^2/2M_\alpha+E_\alpha\mp i0)\phi$$

has the form $\phi_\alpha \otimes f$ with $f \in L^p_\mu(\mathbb{R}^m) \subseteqq L^1(\mathbb{R}^m)$, and because $Z_{\ell, D(\alpha)}(z)$ is continuous for $z \notin \mathscr{E}$. $\square$

Lemma 2.2. *The hypotheses of Theorem 1.1 imply the existence of the channel wave operators for H. Furthermore, for generic couplings, the strong multiparticle limiting absorption principle holds for H.*

Proof. The existence of $\Omega_\alpha^\pm$ is proved in [18, 22]. If we choose $\gamma > m/2$ in the definition of the ϱ functions (defined at the beginning of Sect. V of [8]), then Sect. V of [8] proves the strong multiparticle limiting absorption principle for H with generic couplings. This γ may be chosen greater than $m/2$ because Hypothesis (ii) of Theorem 1.1 requires $v > m/2$ (see the beginning of Sect. V of [8] for a discussion of the allowed values of γ). $\square$

Under the conditions of Lemma 2.2, the formulas of Sect. IV of [8] show that each non-trivial operator $Z_{\ell,D}(z)$ may be represented as

$$Z_{\ell,D}(z) = B_{\ell,D}(z)(1 - M(z))^{-1} C(z) . \tag{2.2}$$

For $N=2$, $M(z) = W(z - H_0)^{-1} U$. For $N=3$ and $N=4$ the operators $M(z)$ are given by Eqs, (IV.6) and (IV.10) of [8], respectively.

The 3 and 4 body operators $M(z)$ contain factors ϱ_{ij}, ϱ_{ijk}, $\varrho_{ij,kl}$, etc. The choice of these factors is rather arbitrary, and we use a different choice from that of [8]. At the beginning of Sect. V of [8], each ϱ function is chosen to have the form $\varrho(y) = (1 + y^2)^{-\gamma/2}$, where y is some particular coordinate, and $1 < \gamma < \delta$. The δ depends on the potentials, and for the potentials of Theorem 1.1 of this paper, we have $\delta > m/2$.

For this paper, we fix γ satisfying $m/2 < \gamma < \delta$, and then make the following definitions for the ϱ functions.

Definition. Let ξ be the coordinate from the center of mass of particles i and j to particle k. Let ζ be the coordinate from the center of mass of particles i, j, and k to particle ℓ. Let η be the coordinate from the center of mass of particles i and j to the center of mass of particles k and ℓ. We define

$$\left.\begin{aligned} \varrho_{ij,k}(\xi) &= (1+\xi^2)^{-\gamma/2} \\ \varrho_{ijk}(\zeta) &= (1+\zeta^2)^{-\gamma/2} \\ \varrho_{ij,k\ell}(\eta) &= (1+\eta^2)^{-\gamma/2} \end{aligned}\right\} . \tag{2.3}$$

3. Two Cluster Eigenfunctions and the (2→2) T-Matrix

In this section we prove Theorems 1.1 and 1.2. We begin by constructing the 2 cluster generalized eigenfunctions.

Definition. Assume the strong multiparticle limiting absorption principle for the N body Hamiltonian H. Let α be a channel for H such that $D = D(\alpha)$ has exactly two clusters. Let $\phi_\alpha^0(X, k) = e^{ik\cdot\zeta}\psi_\alpha(x_\alpha)$, where $X = (x_\alpha, \zeta)$ denotes a set of clustered Jacobi coordinates for D. Whenever $k^2/2M_\alpha + E_\alpha \notin \mathscr{E}$, we define

$$\phi_\alpha(X, k) = \sum_{\ell=1}^{L(D)} (Z_{\ell,D}(k^2/2M_\alpha + E_\alpha - i0))^* F^*_{\ell,D} \phi_\alpha^0(X, k) . \tag{3.1}$$

Remark. In the above definition, $\psi_\alpha \in L^2(\mathbb{R}^{(N-2)m})$ and $e^{ik\cdot} \in L^\infty(\mathbb{R}^m) \subseteq L^2_{-\delta}(\mathbb{R}^m)$ for $\delta > m/2$. Consequently, $\phi^0_\alpha(\cdot, k) \in L^2_{-\delta}(\mathbb{R}^{(N-1)m})$, and $F^*_{\ell,D}\phi^0_\alpha(\cdot, k) \in L^2(\mathbb{R}^{(N-2)m})$. Thus $(Z_{\ell,D}(k^2/2M_\alpha + E_\alpha - i0))^* F^*_{\ell,D}\phi^0_\alpha(\cdot, k) \in L^2_{-\delta_0}(\mathbb{R}^{(N-1)m})$, and

$$\phi_\alpha(\cdot, k) \in L^2_{-\delta_0}(\mathbb{R}^{(N-1)m}) .$$

So, the following definition makes sense.

Definition. Let $\phi_\alpha(X, k)$ be defined as above. For $f \in L^2_{\delta_0}(\mathbb{R}^{(N-1)m})$, we define $f^\#_\alpha(k) = (2\pi)^{-m/2} \int_{\mathbb{R}^{(N-1)m}} \overline{\phi_\alpha(X, k)} f(X) dX$ for all $k \in \mathbb{R}^m$ with $k^2/2M_\alpha + E_\alpha \notin \mathscr{E}$.

Lemma 3.1. *Choose H and α as above, and assume Ω^+_α exists. If $f \in L^2_{\delta_0}(\mathbb{R}^{(N-1)m})$, then $((\Omega^+_\alpha)^* f)\hat{} = f^\#_\alpha$.*

Proof. If $k^2/2M_\alpha + E_\alpha \notin \mathscr{E}$, then Lemma 2.1 shows

$$\begin{aligned}
&((\Omega^+_\alpha)^* f)\hat{}(k) \\
&= (2\pi)^{-m/2} \int \overline{\phi^0_\alpha(X, k)} \left[\sum_{\ell=1}^{L(D)} F_{\ell,D} Z_{\ell,D}(k^2/2M_\alpha + E_\alpha - i0) f \right](X) dX \\
&= (2\pi)^{-m/2} \int \left[\sum_{\ell=1}^{L(D)} \overline{((Z_{\ell,D}(k^2/2M_\alpha + E_\alpha - i0))^* F^*_{\ell,D}\phi^0_\alpha(\cdot, k))(X)} \right] f(X) dX \\
&= f^\#_\alpha(k) .
\end{aligned}$$

Note that the integrals converge absolutely because $\mu > m/2$ in the strong multiparticle limiting absorption principle. □

Proof of Theorem 1.1. By Lemma 2.2, the hypotheses of Theorem 1.1 imply that the strong multiparticle limiting absorption principle holds for generic couplings. Also, the channel wave operators exist. Thus, Lemma 3.1 implies (a).

By (a), $f^\#_\alpha = ((\Omega^+_\alpha)^* f)\hat{}$ for f in a dense subset of $\mathscr{H}$. $(\Omega^+_\alpha)^*$ is a partial isometry with initial subspace $\operatorname{Ran} \Omega^+_\alpha$ and final subspace $\mathscr{H}_\alpha$. Since the Fourier transform is unitary on $\mathscr{H}_\alpha$, (b) follows

Since $T_\alpha(\Omega^+_\alpha)^* \supset (\Omega^+_\alpha)^* H$ (see [12, p. 532]), the following computation imples (c). Let $f \in D(H)$

$$\begin{aligned}
(Hf)^\#_\alpha(k) &= ((\Omega^+_\alpha)^* Hf)\hat{}(k) \\
&= (T_\alpha(\Omega^+_\alpha)^* f)\hat{}(k) \\
&= (k^2/2M_\alpha + E_\alpha)((\Omega^+_\alpha)^* f)\hat{}(k) \\
&= (k^2/2M_\alpha + E_\alpha) f^\#_\alpha(k) . \quad \square
\end{aligned}$$

Lemma 3.2. *Assume the hypotheses of Theorem 1.2. For $k'^2/2M_\beta + E_\beta \notin \mathscr{E}$, define $\phi_\beta(X, k')$ by Eq.* (3.1). *Then the integral* (1.1) *for $T_{\alpha\beta}(k, k')$ is absolutely convergent. Moreover, $T_{\alpha,\beta}(k. k')$ is continuous in k and k' whenever $k'^2/2M_\beta + E_\beta \notin \mathscr{E}$.*

Proof. Choose γ as in the definitions of the ϱ functions [Eq. (2.3)]. Let i and j belong to different clusters of $D(\alpha)$. Then,

$$U_{ij}\overline{\phi^0_\alpha(X, k)} = (U_{ij} P_{D(\alpha)}(1 + \zeta^2_\alpha)^{\gamma/2})((1 + \zeta^2_\alpha)^{-\gamma/2}\overline{\phi^0_\alpha(X, k)}) ,$$

where $X = (x_\alpha, \zeta_\alpha)$. Since $(1 + \zeta^2_\alpha)^{-\gamma/2}\overline{\phi^0_\alpha(X, k)} \in \mathscr{H}$, Lemma V.4 of [8] shows $U_{ij}\overline{\phi^0_\alpha(\cdot, k)} \in L^2_\delta(\mathbb{R}^{(N-2)m}, dx_\alpha) \otimes L^2(\mathbb{R}^m, d\zeta_\alpha)$ for all positive δ. To prove $T_{\alpha,\beta}$ is well defined, it is therefore sufficient to prove

$$W_{ij}\phi_\beta(\cdot, k') \in L^2_{-\delta}(\mathbb{R}^{(N-2)m}, dx_\alpha) \otimes L^2(\mathbb{R}^m, d\zeta_\alpha) \tag{3.2}$$

for some δ.

In view of Eq. (3.1) and the remark thereafter, (3.2) follows if

$$(1+x_\alpha^2)^{-\delta/2} W_{ij}(Z_{\ell,D(\beta)}(k'^2/2M_\beta + E_\beta - i0))^*$$

is a bounded operator on H. Equation (2.2) and the boundedness [8, Sect. V] of $B(z)(1-M(z))^{-1}$ for $z \notin \mathscr{E}$ show that it suffices to prove

$$(1+x_\alpha^2)^{-\delta/2} W_{ij}(C(z-i0))^*$$

is bounded for $z \in \mathbb{R}$. For $N=2$ this follows from Lemma II.3 of [8]. For $N=3$ it follows from Corollary V.5 and Lemmas V.4 and V.8 of [8]. For $N=4$ the result is proved by mimicking the proofs of Propositions V.15, 16, 25, 26, and using Corollary V.5 and Lemmas V.4 and V.8 of [8].

To prove the continuity we note that $T_{\alpha\beta}(k,k')$ is a sum of terms of the form

$$\langle U_{ij}\phi_\alpha^0(\cdot, k), (W_{ij}(Z_{\ell,D(\beta)}(k'^2/2M_\beta + E_\beta - i0))^*) F^*_{\ell,D(\beta)}\phi_\beta^0(\cdot, k')\rangle \,.$$

Every such term is continuous because $U_{ij}\phi_\alpha^0(\cdot, k) \in L^2_\delta \otimes L^2$ depends continuously on k; $F^*_{\ell,D(\beta)}\phi_\beta^0(\cdot, k') \in L^2$ depends continuously on k'; and

$$(W_{ij}(Z_{\ell,D(\beta)}(k'^2/2M_\beta + E_\beta - i0))^*) : L^2 \to L^2_{-\delta} \otimes L^2$$

is strongly continuous in k when $k'^2/2M_\beta + E_\beta \notin \mathscr{E}$.

Proof of Theorem 1.2. By Lemma 3.2 we need only establish Eq. (1.2). This is done by mimicking the two body methods of [17, 20].

By the orthogonality of channels [16]

$$\begin{aligned}
&\langle f, S_{\alpha\beta} g\rangle - \delta_{\alpha\beta}\langle f, g\rangle \\
&= \langle (\Omega_\alpha^- - \Omega_\alpha^+) f, \Omega_\beta^+ g\rangle \\
&= \lim_{S\to\infty} (-i) \int_{-S}^{S} \langle e^{itH}(V - V_{D(\alpha)}) e^{-itT_\alpha} f \otimes \psi_\alpha, \Omega_\beta^+ g\rangle dt \\
&= \lim_{\varepsilon\to 0} (-i) \int_{-\infty}^{\infty} \langle e^{itH}(V - V_{D(\alpha)}) e^{-itT_\alpha} f \otimes \psi_\alpha, \Omega_\beta^+ g\rangle e^{-\varepsilon|t|} dt \,.
\end{aligned} \tag{3.3}$$

Here we have used the fundamental theorem of calculus and an Abelian limit formula [16, 20].

We now use Theorem 1.1 (b) to express the last integrand as

$$\begin{aligned}
&\langle (e^{itH}(V - V_{D(\alpha)}) e^{-itT_\alpha} f \otimes \psi_\alpha)^\#_\beta, (\Omega_\beta^+ g)^\#_\beta\rangle_{\mathscr{H}_\beta} e^{-\varepsilon|t|} \\
&= \int \overline{((V - V_{D(\alpha)}) e^{-itT_\alpha} f \otimes \psi_\alpha)^\#_\beta(k')} e^{-it(k'^2/2M_\beta + E_\beta)} \hat{g}(k') dk' e^{-\varepsilon|t|} \,.
\end{aligned} \tag{3.4}$$

To obtain this second expression, we have again used Theorem 1.1 (b) and the fact that $(e^{itH}\psi)^\#_\beta(k) = e^{it(k^2/2M_\beta + E_\beta)} \psi^\#_\beta(k)$. This is proved by using $e^{itH}\Omega_\beta^+ = \Omega_\beta^+ e^{itT_\beta}$ and mimicking the proof of Theorem 1.1 (c).

To evaluate the integral (3.4), we compute

$$\begin{aligned}
&((V-V_{D(\alpha)})e^{-itT_\alpha}f\otimes\psi_\alpha)^\#_\beta(k') \\
&=(2\pi)^{-m/2}((V-V_{D(\alpha)})e^{-itT_\alpha}\int \hat{f}(k)e^{ik\cdot}\otimes\psi_\alpha dk)^\#_\beta(k') \\
&=(2\pi)^{-m/2}(\int e^{-it(k^2/2M_\alpha+E_\alpha)}\hat{f}(k)(V-V_{D(\alpha)})\phi^0_\alpha(\cdot,k)dk)^\#_\beta(k') \\
&=(2\pi)^{-m}\int\int e^{-it(k^2/2M_\alpha+E_\alpha)}\hat{f}(k)\overline{\phi_\beta(X,k')} \\
&\quad\cdot[V(X)-V_{D(\alpha)}(X)]\phi^0_\alpha(X,k)dkdX \\
&=\int e^{-it(k^2/2M_\alpha+E_\alpha)}\hat{f}(k)\overline{T_{\alpha\beta}(k,k')}dk\,.
\end{aligned}\tag{3.5}$$

In the last step, we have used Fubini's theorem, Lemma 3.2 and the fact that $f\in\mathscr{S}$.

Substituting (3.5) in (3.4) and using the result in (3.3), we obtain

$$\begin{aligned}
&\langle f,S_{\alpha\beta}g\rangle-\delta_{\alpha\beta}\langle f,g\rangle \\
&=\lim_{\varepsilon\to0}(-i)\int_{-\infty}^{\infty}\int\int e^{-it(k'^2/2M_\beta+E_\beta-k^2/2M_\alpha-E_\alpha)}e^{-\varepsilon|t|} \\
&\quad\cdot\overline{\hat{f}(k)}\hat{g}(k')T_{\alpha\beta}(k,k')dkdk'dt\,.
\end{aligned}$$

The theorem now follows by applying Fubini's theorem, computing the t integral explicitly, and evaluating the limit. □

4. Two Body Born Series Convergence

In this section we prove Theorem 1.3 for $N=2$ and obtain estimates to be used in Sects. 6 and 7. Our analysis is based on the following lemma, which has been proved by Ginibre and Moulin [7] and Herbst [9 (Appendix)]. Rauch [15] has proved a similar result.

Lemma 4.1. *Suppose $\delta>1/2$. For each $M>0$ there exists a constant c such that the norm of $(z+\Delta)^{-1}:L^2_\delta(\mathbb{R}^n)\to L^2_{-\delta}(\mathbb{R}^n)$ is dominated by $c|z|^{-1/2}$, whenever $|z|>M$.*

Proof of Theorem 1.3 for $N=2$. We need ony show

$$\|W(z-H_0)^{-1}U\|\leqq C|z|^{-\varepsilon}$$

for large $|z|$ because Lemma II.3 of [8] shows

$$\|W(z-H_0)^{-1}U\|\leqq C_0\,.$$

Suppose first that F and G belong to $L^\infty_\delta(\mathbb{R}^m)$ for some $\delta>1$. Then $F:L^2_{-\delta}\to L^2$ and $G:L^2\to L^2_\delta$ have norms $\|F\|_{L^\infty_\delta}$ and $\|G\|_{L^\infty_\delta}$, respectively. Thus, Lemma 4.1 implies

$$\|F(z-H_0)^{-1}G\|\leqq C_1\|F\|_{L^\infty_\delta}\|G\|_{L^\infty_\delta}|z|^{-1/2}\tag{4.1}$$

whenever $|z|>M$.

Next, let $F\in L^\infty_\delta(\mathbb{R}^m)$ and $G\in L^r_\delta(\mathbb{R}^m)$, with $r>m\geqq3$ and $\delta>1$. Lemma II.3 of [8] shows

$$\|F(z-H_0)^{-1}G\|\leqq C_2\|F\|_{L^\infty_\delta}\|G\|_{L^r_\delta}\,.\tag{4.2}$$

By interpolating [16] between (4.1) and (4.2), we obtain

$$\|F(z-H_0)^{-1}G\| \leqq C_3(z)^{-\alpha}\|F\|_{L^\infty_\delta}\|G\|_{L^p_\delta} \tag{4.3}$$

for $p>m$, $\delta>1$, and $0<\alpha<(1-m/p)/2$.

Finally, if $F\in L^S_\delta(\mathbb{R}^m)$ and $G\in L^p_\delta(\mathbb{R}^m)$, with $p>m\geqq 3, S>m$, and $\delta>1$, then Lemma II.3 of [8] shows

$$\|F(z-H_0)^{-1}G\| \leqq C_4\|F\|_{L^S_\delta}\|G\|_{L^p_\delta}\,. \tag{4.4}$$

By interpolating between (4.3) and (4.4), we obtain

$$\|F(z-H_0)^{-1}G\| \leqq C_5|z|^{-\beta}\|F\|_{L^q_\delta}\|G\|_{L^p_\delta} \tag{4.5}$$

for $p>m$, $q>m$, $\delta>1$, and $0\leqq\beta<(1-m/p)(1-m/q)/2$.

Since U and W belong to $L^p_\delta(\mathbb{R}^m)+L^\infty_\delta(\mathbb{R}^m)$ with $p>m$ and $\delta>1$, (4.5) implies Theorem 1.3 for $N=2$. □

Remark. For $N=2$, dilation analyticity of U and W has not been used. Thus, we can take $U=|V|^{1/2}$ and $W=V/U$, where we take $W(x)=0$ when $V(x)=U(x)=0$. In this case, the above proof shows $\|W(z-H_0)^{-1}U\|\leqq C|z|^{-\beta}$, with $0\leqq\beta<(1-m/2r)^2/2$, if $V\in L^r_\delta(\mathbb{R}^m)+L^\infty_\delta(\mathbb{R}^m)$ with $r>m/2$ and $\delta>1$. In particular, if V is a Yukawa potential, β may be taken arbitrarily close to 1/8.

Corollary 4.2. *Assume the hypotheses of Theorem 1.1, and let $N=2$. Assume $1\notin\sigma(W(0-H_0)^{-1}U)$, and let P denote the orthogonal projection onto the eigenvectors of H. Then $W(1-P)(z-H)^{-1}U$ is uniformly bounded and uniformly norm continuous in the closed cut plane, cut along $[0,\infty)$. Moreover, there exists C such that for all $z\in\mathbb{C}$, $\|W(z-H)^{-1}U\|\leqq C|\mathrm{Im}\,z|^{-1}$.*

Proof. The first assertion implies the second because

$$\|WP(z-H)^{-1}U\|\leqq\|WP\|\,\|P(z-H)^{-1}\|\,\|PU\|\leqq C|\mathrm{Im}\,z|^{-1}\,.$$

If z is restricted to a compact set, then the proof of Lemma V.7 of [8] implies the first result. So, it suffices to prove

$$\lim_{|z|\to\infty}\|W(1-P)(z-H)^{-1}U\|=0\,.$$

To prove this, we write

$$\begin{aligned} &W(1-P)(z-H)^{-1}U\\ &\quad=W(1-P)(z-H_0)^{-1}U(1+[1-W(z-H_0)^{-1}U]^{-1}W(z-H_0)^{-1}U)\,. \end{aligned} \tag{4.6}$$

Theorem 1.3 for $N=2$ shows that $\lim\limits_{|z|\to\infty}\|W(z-H_0)^{-1}U\|$, and the proof of this result shows $\lim\limits_{|z|\to\infty}\|(1+x^2)^{-1}(z-H_0)^{-1}U\|=0$. Thus, $\|W(1-P)(z-H_0)^{-1}U\|$ $\leqq\|W(z-H_0)^{-1}U\|+\|WP(1+x^2)\|\,\|(1+x^2)^{-1}(z-H_0)^{-1}U\|$ tends to zero as $|z|\to\infty$ ($\|WP(1+x^2)\|<\infty$ by [8, Corollary V.5]). The corollary now follows from (4.6) by using geometric series to compute $[1-W(z-H_0)^{-1}U]^{-1}$. □

5. Three Body Born Series Convergence

In this section we prove Theorem 1.3 for $N=3$ and establish some technical results to be used in Sect. 6.

Lemma 5.1. *Let $\delta > 1/2$. For each $M > 0$ there exists C such that*

$$\|(z+\Delta)^{-1}(1+x^2)^{-\delta/2}\| \leqq C|\mathrm{Im}\, z|^{-1/2}|\mathrm{Re}\, z|^{-1/4},$$

whenever $|z| > M$.

Proof. Let $A(z) = (z+\Delta)^{-1}(1+x^2)^{-\delta/2}$. Then

$$\begin{aligned}\|A(z)\|^2 &= \|A(z)^* A(z)\| \\ &= \tfrac{1}{2}|\mathrm{Im}\, z|^{-1}\|(1+x^2)^{-\delta/2}[(z+\Delta)^{-1} - (\bar{z}+\Delta)^{-1}](1+x^2)^{-\delta/2}\|.\end{aligned}$$

Apply Lemma 4.1. □

Lemma 5.2 (Balslev [4]). *Suppose $\Omega \subseteq \mathbb{C}$ and $A(z)$ is a function from Ω to the bounded operators on $L^2(\mathbb{R}^n)$. If f denotes multiplication by a real bounded function $f(x)$, then $\|fA(z)\| \leqq \|A(z)\|^{(1-2^{-p})}\|f^{2^p}A(z)\|^{2^{-p}}$, for all $z \in \Omega$ and all integers $p \geqq 0$.*

Proof. $\|fA(z)\| = \|A(z)^* f^2 A(z)\|^{1/2} \leqq \|A(z)\|^{1/2}\|f^2 A(z)\|^{1/2}$. The lemma follows by iteration. □

Definition. Let D be a cluster decomposition for an N body system. Let $\xi_1^1, \ldots, \xi_{n(1)-1}^1, \xi_1^2, \ldots, \xi_{n(k)-1}^k, \zeta_1, \ldots, \zeta_{k-1}$ be clustered Jacobi coordinates for D, where the ξ's are coordinates within the clusters and the ζ's are coordinates between the clusters. We define

$$r_D = ((\xi_1^1)^2 + \ldots + (\xi_{n(1)-1}^1)^2 + (\xi_1^2)^2 + \ldots + (\xi_{n(k)-1}^k)^2)^{1/2}.$$

Lemma 5.3 (Balslev [4]). *Suppose $\alpha > 1/2 > \beta$, and D is a cluster decomposition of an N body system. There exists C, such that*

$$\|(1+r_D^2)^{-\alpha/2}(z-H_0)^{-1}\psi\| < C|\mathrm{Im}\, z|^{-1}\|(z-H_0)^{-1}(1+r_D^2)^{-\beta/2}\psi\|$$

for all $\psi \in \mathscr{H}$ and all $z \in \mathbb{C}\backslash\mathbb{R}$ satisfying $|\mathrm{Im}\, z| \leqq 1$.

Proof. One obtains this result by keeping track of the $\mathrm{Im}\, z$ dependence in Lemma 2.2 of [4]. □

Lemma 5.4. *Let $H = H_0 + \sum_{i<j} V_{ij}$ be a 3 body Hamiltonian on $L^2(\mathbb{R}^m)$ with $m \geqq 3$. Assume each $V_{ij} = U_{ij}W_{ij}$, where U_{ij} and W_{ij} belong to $L_\delta^q(\mathbb{R}^m) + L_\delta^\infty(\mathbb{R}^m)$ for some $p > m$ and $\delta > 1$. Let i, j, and k be distinct. For each $M_1 > 0$ and $M_2 > 0$, there exist C and $\varepsilon > 0$ such that $|\mathrm{Re}\, z| > M_1$ and $|\mathrm{Im}\, z| < M_2$ imply $\|W_{ij}(z-H_0)^{-1}U_{ik}\| \leqq C|\mathrm{Im}\, z|^{-2}|\mathrm{Re}\, z|^{-\varepsilon}$.*

Proof. Assume first that W_{ij} and U_{ik} belong to L_δ^∞. Then $W_{ij}(1+x_{ij}^2)^{\delta/2}$ and $(1+x_{ik}^2)^{\delta/2}U_{ik}$ are bounded, and it is sufficient to consider

$$(1+x_{ij}^2)^{-\delta/2}(z-H_0)^{-1}(1+x_{ik}^2)^{-\delta/2}.$$

However, Lemma 5.3 shows that for $\beta = 3/8$,

$$\begin{aligned}&\|(1+x_{ij}^2)^{-\delta/2}(z-H_0)^{-1}(1+x_{ik}^2)^{-\delta/2}\| \\ &\leqq C_1|\mathrm{Im}\, z|^{-1}\|(z-H_0)^{-1}(1+x_{ij}^2)^{-\beta/2}(1+x_{ik}^2)^{-\delta/2}\| \\ &\leqq C_1|\mathrm{Im}\, z|^{-1}\|(1+x_{ij}^2)^{-\beta/2}(1+x_{ik}^2)^{-\beta/2}(\bar{z}-H_0)^{-1}\| \\ &\leqq C_2|\mathrm{Im}\, z|^{-1}\|(1+X^2)^{-\beta/4}(\bar{z}-H_0)^{-1}\|.\end{aligned}$$

In the last two steps, we have taken adjoints and used

$$(1+X^2)^2 \geqq C_3(1+x_{ij}^2)(1+x_{ik}^2).$$

Lemmas 5.1 and 5.2 now show

$$\begin{aligned}
&\|(1+x_{ij}^2)^{-\delta/2}(z-H_0)^{-1}(1+x_{ik}^2)^{-\delta/2}\| \\
&\quad \leqq C_2|\mathrm{Im}\, z|^{-1}\|(\bar{z}-H_0)^{-1}\|^{7/8}\|(1+X^2)^{-2\beta}(\bar{z}-H_0)^{-1}\|^{1/8} \\
&\quad = C_4|\mathrm{Im}\, z|^{-31/16}|\mathrm{Re}\, z|^{-1/32} \qquad (5.1)
\end{aligned}$$

for $|\mathrm{Re}\, z| > M_1$ and $|\mathrm{Im}\, z| < M_2$. Thus the lemma holds when W_{ij} and U_{ik} belong to L_δ^∞.

If $r > m$, $s > m$, and $\delta > 1$, Lemma II.3 of [8] shows $\|F(x_{ij})(z-H_0)^{-1}G(x_{ik})\| \leqq C_5\|F\|_{L_\delta^r}\|G\|_{L_\delta^s}$. Using this result and (5.1), the proof of the lemma is completed by imitating the interpolation argument of the proof of Theorem 1.3 for $N=2$ (see Sect. 4). □

Corollary 5.5. *Under the conditions of Lemma 5.4, there exists $\varepsilon > 0$ so that $\|W_{ij}(x \pm i0 - H_0)^{-1}U_{ik}\| \leqq Cx^{-\varepsilon}$.*

Proof. The proof of Lemma II.3 of [8] shows that $A(z) = W_{ij}(z-H_0)^{-1}U_{ik}$ is uniformly bounded and uniformly Hölder continuous of order $\alpha > 0$ for z in the closed cut plane, cut along $[0, \infty)$ (see [7, Proposition (5.1); 22, Lemma 3.6]). Using this and Lemma 5.4, we obtain

$$\begin{aligned}
\|A(x)\| &\leqq \|A(x \pm iy)\| + C_1 y^\alpha \\
&\leqq C_2 x^{-\beta} y^{-2} + C_1 y^\alpha
\end{aligned}$$

for some $\beta > 0$ and all $y \in (0, 1]$. Choosing $y = x^{-\nu}$ with $\nu = \beta/(\alpha+2)$, and taking x large, we obtain

$$\|A(x)\| \leqq C_2 x^{-\varepsilon} + C_1 x^{-\varepsilon} = Cx^{-\varepsilon},$$

where $\varepsilon = \alpha\nu$. Since $A(x)$ is uniformly bounded for small x, the corollary follows. □

Proof of Theorem 1.3 for $N=3$. $M(z)$ consists of nine blocks of 2×2 matrices. The three blocks on the diagonal are zero, and the remaining 6 blocks all have the same form. Within these blocks, we will show that each entry, except for the z independent entry, $\varrho_{ij}^{-1}P_{ij}U_{ik}$, is dominated by $C(\mathrm{Re}\, z)^{-\varepsilon}$ for some $\varepsilon > 0$. Matrix multiplication then shows $\|(M(z))^n\| \leqq (C\,\mathrm{Re}\, z)^{-[n/2]\varepsilon}$ ($[n/2]$ is the greatest integer less than or equal to $n/2$).

Thus, we must study three types of entries with the assumption that the couplings are generic.

Type 1. $W_{ij}(1-P_{ij})(z-H_{ij})^{-1}U_{ik}$.
This entry equals

$$\begin{aligned}
&W_{ij}(1-P_{ij})(z-H_0)^{-1}U_{ik} \\
&\quad + (W_{ij}(1-P_{ij})(z-H_{ij})^{-1}U_{ij})(W_{ij}(z-H_0)^{-1}U_{ik}).
\end{aligned}$$

Corollary 4.2 implies the uniform boundedness of the first factor of the second term. Corollary 5.5 shows the second factor of the second term is dominated by $C_1(\operatorname{Re} z)^{-\varepsilon}$. Similarly, Corollary V.5 of [8] and Corollary 5.5 show that the first term is dominated by

$$\begin{aligned}&\| W_{ij}(z-H_0)^{-1}U_{ik}\| \\ &\quad + \| W_{ij}P_{ij}(1+x_{ij}^2)\| \; \|(1+x_{ij}^2)^{-1}(z-H_0)^{-1}U_{ik}\| \leqq C_2(\operatorname{Re} z)^{-\varepsilon} .\end{aligned}$$

Type 2. $W_{ij}(1-P_{ij})(z-H_{ij})^{-1}V_{ik}P_{ik}(z-H_{ik})^{-1}\varrho_{ik}$.
This entry equals [8, Lemma V.12]

$$\begin{aligned}&W_{ij}(1-P_{ij})\,[(z-H_{ik})^{-1}-(z-H_0)^{-1}]P_{ik}\varrho_{ik} \\ &\quad + (W_{ij}(1-P_{ij})(z-H_{ij})^{-1}U_{ij})(W_{ij}[(z-H_{ik})^{-1}-(z-H_0)^{-1}]P_{ik}\varrho_{ik}) . \qquad (5.2)\end{aligned}$$

As above, $W_{ij}(1-P_{ij})(z-H_{ij})^{-1}U_{ij}$ is uniformly bounded, and

$$\begin{aligned}&\| W_{ij}(z-H_0)^{-1}P_{ik}\varrho_{ik}\| \\ &\quad \leqq \| W_{ij}(z-H_0)^{-1}(1+x_{ik}^2)^{-1}\| \; \|(1+x_{ik}^2)P_{ik}\varrho_{ik}\| \\ &\quad \leqq C_3(\operatorname{Re} z)^{-\varepsilon} .\end{aligned}$$

Thus,

$$\begin{aligned}&\| W_{ij}(1-P_{ij})(z-H_0)^{-1}P_{ik}\varrho_{ik}\| \\ &\leqq \| W_{ij}(z-H_0)^{-1}P_{ik}\varrho_{ik}\| \\ &\quad + \| W_{ij}P_{ij}(1+x_{ij}^2)\| \; \|(1+x_{ij}^2)^{-1}(z-H_0)^{-1}P_{ik}\varrho_{ik}\| \\ &\quad \leqq C_4(\operatorname{Re} z)^{-\varepsilon} .\end{aligned}$$

Next, $W_{ij}(z-H_{ik})^{-1}P_{ik}\varrho_{ik}$ may be written as

$$(W_{ij}P_{ik}\varrho_{ik}^{-1})(\varrho_{ik}P_{ik}(z-H_{ik})^{-1}\varrho_{ik}) .$$

Assuming for convenience that there is only one bound state of the ik subsystem, we see that the second factor equals $\varrho_{ik}(z-H_0^{\xi}-E_{ik})^{-1}\varrho_{ik}P_{ik}$. Lemma 4.1 shows the norm of this factor is dominated by $C_5(\operatorname{Re} z)^{-1/2}$. The first factor above is uniformly bounded [8, Lemma V.4]. Similarly, we see that

$$\| W_{ij}(1-P_{ij})(z-H_{ik})^{-1}P_{ik}\varrho_{ik}\| \leqq C_6(\operatorname{Re} z)^{-1/2} .$$

So, using (5.2), the entries of type 2 fall off as $(\operatorname{Re} z)^{-\varepsilon}$.

Type 3. $\varrho_{ij}^{-1}P_{ij}V_{ik}(z-H_{ik})^{-1}P_{ik}\varrho_{ik}$.
This equals $(\varrho_{ij}^{-1}P_{ij}V_{ik}P_{ik}\varrho_{ik}^{-1})(\varrho_{ik}(z-H_{ik})^{-1}P_{ik}\varrho_{ik})$.

The proof of Lemma V.11 of [8] shows that the first factor is bounded. The second factor was studied in our analysis of type 2 terms, above. Its norm is less than $C_5(\operatorname{Re} z)^{-1/2}$.

Thus, the three types of entries fall off as $(\operatorname{Re} z)^{-\varepsilon}$, and the proof is complete. □

Corollary 5.6. *Assume the hypotheses of Theorem 1.1, and let $N=3$. For generic couplings, $M(z)$ is uniformly bounded and uniformly norm continuous in the closed cut plane cut along $[0,\infty)$. There exists n such that $(1-M(z))^{-1}$ also has these* *properties when* $|z|>n$. *Furthermore, for any* $\varepsilon>0$, *there exists* C_ε *such that* $\|W_{ij}(z-H)^{-1}U_{k\ell}\|\leqq C_\varepsilon$ *whenever* $|\mathrm{Im}\, z|\geqq\varepsilon$. *Here ij and* $k\ell$ *are any pairs in the three body problem.*

Proof. Lemma V.13 of [8] and Theorem 1.3 imply the first two results. To prove $|\mathrm{Im}\, z|\geqq\varepsilon$ implies $\|W_{ij}(z-H)^{-1}U_{k\ell}\|\leqq C_\varepsilon$, we use Eq. (IV.6) of [8] to write $W_{ij}(z-H)^{-1}U_{k\ell}=I(z)+II(z)$, where $I(z)=W_{ij}[(z-H_{12})^{-1}+(z-H_{13})^{-1}+(z-H_{23})^{-1}-2(z-H_0)^{-1}]U_{k\ell}$ and $II(z)=W_{ij}A(z)(1-M(z))^{-1}M(z)C(z)U_{k\ell}$.

To bound $I(z)$, we replace $(z-H_{12})^{-1}$ by $(z-H_0)^{-1}+(z-H_0)^{-1}V_{12}(z-H_0)^{-1}+(z-H_0)^{-1}V_{12}(z-H_{12})^{-1}V_{12}(z-H_0)^{-1}$, and replace $(z-H_{13})^{-1}$ and $(z-H_{23})^{-1}$ by the corresponding expressions. Lemma II.3 of [8] and Corollary 4.2 then show $I(z)$ is uniformly bounded for $|\mathrm{Im}\, z|\geqq\varepsilon$.

The same methods uniformly bound $W_{ij}A(z)$ and $M(z)C(z)U_{k\ell}$ for $|\mathrm{Im}\, z|\geqq\varepsilon$. For large z, $(1-M(z))^{-1}$ is uniformly bounded, and $(1-M(z))^{-1}$ has no singularities in $\mathbb{C}\backslash\mathbb{R}$ [8, Proposition V.2]. Thus, $(1-M(z))^{-1}$ is uniformly bounded for $|\mathrm{Im}\, z|\geqq\varepsilon$, and the corollary is proved. □

6. Four Body Born Series Convergence

In this section we study the large $\mathrm{Re}\, z$ behavior of the four particle operator $M(z)$. To prove $\lim_{\mathrm{Re}\, z\to\infty}\|(M(z))^n\|=0$, and enormous number of terms must be considered. So, for the sake of brevity, we will only sketch proofs.

As preparation for the proof of Theorem 1.3 for $N=4$, some technical results must be established. First, we study the convergence of the Born series for the cluster Hamiltonian $H_{ij,k\ell}$. Second, we prove some approximation lemmas; and third, we state a special case of a lemma of Balslev. We are then prepared to study the full four body operator $M(z)$.

Lemma 6.1. *Assume the hypotheses of Theorem 1.1. Let* $M(z)$ *denote the operator of Eq.* (V.6) *of* [8], *corresponding to* $H_{ij,k\ell}$. *Assume the motion of the center of mass of particles i and j relative to the center of mass of particles k and* ℓ *has been removed* $\left[\text{so that } M(z) \text{ acts on } \bigoplus_{i=1}^{4} L^2(\mathbb{R}^{2m})\right]$. *Then, for generic couplings,*

a) $M(z)$ *is uniformly bounded in the closed cut plane cut along* $\sigma_{\mathrm{ess}}(H_{ij,k\ell})$;

b) $(M(z))^2$ *is uniformly continuous in that region; and*

c) $\lim_{\mathrm{Re}\, z\to\infty}\|(M(z))^2\|=0$.

Sketch of Proof. Part a) is proved by combining Corollary 4.2 with the proofs of Lemmas V.12, 20, and 21 of [8].

To prove b), we first remark that the proof of Lemma V.9 of [8] shows uniform norm continuity of the operators considered in that lemma. This implies uniform continuity in Lemma V.12 of [8]. Using this and Corollary 4.2 in the proof of Lemma V.23 of [8], we obtain b).

c) In view of b), we need only show $\lim_{x\to\infty} \|(M(x\pm iy))^2\| = 0$ for each $y>0$. Using Corollary 4.2 and $G_{ij} = G_0 + G_{ij} V_{ij} G_0$, we see that it suffices to prove $\lim_{x\to\infty} \| W_{ij}(x\pm iy - H_0)^{-1} U_{k\ell} \| = 0$.

Furthermore, Lemma II.3 of [8] shows that it is enough to prove the limit is zero when W_{ij} and $U_{k\ell}$ belong to $L_\delta^\infty(\mathbb{R}^m)$ for $\delta>1$. However, by mimicking the proof of Lemma 5.4, we see that this follows from Lemmas 5.1–5.3. □

Corollary 6.2. *Assume the hypotheses of Lemma 6.1. Then* $W_{ij}(z-H_{ij,k\ell})^{-1} U_{ij}$ *and* $W_{ij}(z-H_{ij,k\ell})^{-1} U_{k\ell}$ *are uniformly bounded for* $|\mathrm{Im}\, z| \geqq \varepsilon$.

Proof. Use Eq. (V.6) of [8] for $W(z-H_{ij,k\ell})^{-1} U$. The poles of $(1-M(z))^{-1}$ are real [8], and $\lim_{|z|\to\infty} \|(M(z))^2\| = 0$ by Lemma 6.1.

Thus, $\|(1-M(z))^{-1}\| = \|(1+M(z))(1-(M(z))^2)^{-1}\|$ is bounded for $|\mathrm{Im}\, z| \geqq \varepsilon$. Lemma II.3 of [8] and Corollary 4.2 bound the other terms. □

Lemma 6.3. *Assume the hypotheses of Theorem 1.1. Let* $\{r_1, \dots, r_{n-1}\}$ *and* $\{r'_1, \dots, r'_{n-1}\}$ *be two (possibly identical) choices of clustered Jacobi coordinates, and let D be a cluster decomposition. Suppose* F_1 *and* F_2 *belong to* $L_\delta^p(\mathbb{R}^m) + L_\delta^\infty(\mathbb{R}^m)$ *with* $p>m$, $\delta>1$. *If* $|\mathrm{Im}\, z| \geqq \varepsilon$ *implies* $\| W_{ij}(z-H_D)^{-1} U_{k\ell} \| \leqq C_\varepsilon$ *whenever i and j belong to the same cluster of D, and k and* ℓ *belong to the same cluster of D, then there exists* C'_ε *such that*

$$\|F_1(r_1)(z-H_D)^{-1} F_2(r'_1)\| \leqq C'_\varepsilon \|F_1\|_{L_\delta^p + L_\delta^\infty} \|F_2\|_{L_\delta^p + L_\delta^\infty}$$

whenever $|\mathrm{Im}\, z| \geqq \varepsilon$.

$$(\|f\|_{L_\delta^p + L_\delta^\infty} = \inf\{\|g\|_{L_\delta^p} + \|h\|_{L_\delta^\infty} : f = g+h\}) .$$

Proof.

$$\begin{aligned} F_1(z-H_D)^{-1} F_2 &= F_1(z-H_0)^{-1} F_2 + F_1(z-H_0)^{-1} V_D (z-H_0)^{-1} F_2 \\ &\quad + F_1(z-H_0)^{-1} V_D (z-H_D)^{-1} V_D (z-H_0)^{-1} F_2 . \end{aligned}$$

Apply Lemma II.3 of [8]. □

Lemma 6.4. *Assume the hypotheses of Lemma 6.3. Suppose* $U_{ij}^{(n)}$ *and* $W_{ij}^{(n)}$ *converge to* U_{ij} *and* W_{ij}, *respectively, in the norm of* $L_\delta^p + L_\delta^\infty$ *described in Lemma 6.3. Let D be a cluster decomposition, and let* $H_D^{(n)} = H_0 + V_D^{(n)}$. *Then* $F_1(r_1)(z-H_D^{(n)})^{-1} F_2(r'_1)$ *converges in norm to* $F_1(r_1)(z-H_D)^{-1} F_2(r'_1)$, *uniformly in the region* $|\mathrm{Im}\, z| \geqq \varepsilon$.

Proof. Let ℓ be the number of pairs $\alpha = (ij)$, such that i and j belong to the same cluster of D. For all such pairs α and β, define

$$X_\alpha^{(n)} = F_1(r_1)(z-H_D)^{-1} \tilde{U}_\alpha^{(n)}, \quad Y_{\alpha\beta}^{(n)} = \tilde{W}_\alpha^{(n)} (z-H_D)^{-1} \tilde{U}_\beta^{(n)} ,$$

and

$$Z_\beta^{(n)} = \tilde{W}_\beta^{(n)} (z-H_D)^{-1} F_2(r'_1), \quad \text{where} \quad \tilde{U}_\alpha^{(n)} = |V_\alpha^{(n)} - V_\alpha|^{1/2} ,$$

and

$$\tilde{W}_\alpha^{(n)} = (V_\alpha^{(n)} - V_\alpha)/\tilde{U}_\alpha^{(n)} \quad (\tilde{W}_\alpha^{(n)}(x) = 0 \quad \text{if} \quad \tilde{U}_\alpha^{(n)}(x) = 0) .$$

We let $X^{(n)}(z)$, $Y^{(n)}(z)$, and $Z^{(n)}(z)$ denote the $1 \times \ell, \ell \times \ell$, and $\ell \times 1$ matrices with these entries. Then,

$$\begin{aligned} &\|F_1(z-H_D)^{-1}F_2 - F_1(z-H_D^{(n)})^{-1}F_2\| \\ &\quad = \|X^{(n)}(z)(1-Y^{(n)}(z))^{-1}Z^{(n)}(z)\| \\ &\quad \leqq \|X^{(n)}(z)\|_{\mathscr{H}^\ell \to \mathscr{H}} \|(1-Y^{(n)}(z))^{-1}\|_{\mathscr{H}^\ell \to \mathscr{H}^\ell} \|Z^{(n)}(z)\|_{\mathscr{H} \to \mathscr{H}^\ell} . \end{aligned}$$

Lemma 6.3 shows that $\|X^{(n)}(z)\|$, $\|Y^{(n)}(z)\|$, and $\|Z^{(n)}(z)\|$ tend to 0 uniformly for $|\mathrm{Im}\, z| \geqq \varepsilon$ as $n \to \infty$. So, the lemma follows by using geometric series. □

Lemma 6.5 (Balslev [4]). *Assume the hypotheses of Theorem 1.1, and assume all U_{ij} and W_{ij} belong to $L_\delta^\infty(\mathbb{R}^m)$ for some $\delta > 1$. Let D_1 and D_2 be cluster decompositions, and let $A(z)$ be a uniformly bounded analytic operator valued function for $|\mathrm{Im}\, z| \geqq \varepsilon$. If $\alpha > 1/2 > \beta$ then there exist $\mu > 0$, $v > 0$, such that $|\mathrm{Im}\, z| \geqq \varepsilon$ implies*

$$\|(1+r_{D_1}^2)^{-\alpha/2}(z-H_{D_2})^{-1}V_{ij}A(z)\| \leqq C\|(1+r_{D_1}^2)^{-\beta/2}(1+r_{ij}^2)^{-\mu}A(z)\|^v .$$

Proof. This is a special case of Lemma 5.2 of [4]. The proof in [4] can be simplified in our case by using self-adjointness and the boundedness of V_{ij}. □

Sketch of the Proof of Theorem 1.3 for $N=4$

Step 1. Uniform Continuity. The proofs of Lemmas II.3 and V.9 of [8] show that the norm continuity concluded in those lemmas is uniform. Using this uniformity, Lemma 6.1, and Corollaries 4.2 and 5.6, we can establish uniform norm continuity (rather than just norm continuity) and uniform bondedness in Lemmas V.6–V.14, V.22 and Propositions V.15–V.20, V.26, V.29, V.31, and V.32 of [8]. This is done by going through the proofs of these lemmas and propositions and systematically inserting the uniform estimates.

In Propositions V.25, 27, 28, and 30 of [8], this method yields less information because these propositions depend on Lemma V.24. We can only conclude that each operator studied is a sum of terms, each of which is either uniformly bounded and uniformly norm continuous, or is the product of a uniformly bounded, strongly continuous factor and a uniformly bounded, uniformly continuous compact factor. For example, in Proposition V.25 of [8],

$$W_{ij}(1-P_{ij})(z-H_{ij})^{-1}V_{k\ell}(1-P_{ij,k\ell})(z-H_{ij,k\ell})^{-1}U_{ik}$$

equals $I(z)+II(z)+III(z)+IV(z)$, where [Eq. (V.6) of [8]]

$$\begin{aligned} I(z) &= W_{ij}(1-P_{ij})(z-H_{ij})^{-1}V_{k\ell}(1-P_{ij,k\ell})(z-H_{ij})^{-1}U_{ik} , \\ II(z) &= W_{ij}(1-P_{ij})(z-H_{ij})^{-1}V_{k\ell}(1-P_{ij,k\ell})(z-H_{k\ell})^{-1}U_{ik} , \\ III(z) &= -W_{ij}(1-P_{ij})(z-H_{ij})^{-1}V_{k\ell}(1-P_{ij,k\ell})(z-H_0)^{-1}U_{ik} , \\ IV(z) &= W_{ij}(1-P_{ij})(z-H_{ij})^{-1}V_{k\ell}(1-P_{ij,k\ell})A(z)(1-M(z))^{-1}C(z)U_{ik} . \end{aligned}$$

Our method shows that $I(z)$, $II(z)$, $III(z)$, and $C(z)U_{ik}$ are uniformly bounded and uniformly continuous. However,

$$W_{ij}(1-P_{ij})(z-H_{ij})^{-1}V_{k\ell}(1-P_{ij,k\ell})A(z)(1-M(z))^{-1}$$

is only strongly continuous and uniformly bounded.

Below we show that those terms and factors which are uniformly continuous, actually tend to zero as $\mathrm{Re}\, z\to\infty$. We then show that this implies $\|(M(z))^n\|\to 0$ as $\mathrm{Re}\, z\to\infty$ for $n\geqq 2$.

Step 2. Removal of Local Singularities. Let $X(z)$ be any one of the uniformly continuous operator valued functions, from Step 1, and let $\varepsilon>0$. $X(z)$ is a product of U's, V's, W's, resolvents, and projections. Corollaries 4.2 and 5.6 and Lemma 6.3 show that there exist bounded U's, V's, and W's so that the operator $X_1(z)$ obtained from $X(z)$ by replacing the U's, V's, W's with the bounded U's, V's, and W's satifies $\|X(z)-X_1(z)\|<\varepsilon$ whenever $|\mathrm{Im}\, z|\geqq\eta>0$. Lemma 6.4 then shows that the potentials within the resolvent factors may be replaced by bounded potentials to yield $X_2(z)$, satisfying $\|X_1(z)-X_2(z)\|<\varepsilon$ whenever $|\mathrm{Im}\, z|\geqq\eta$. Here, $\eta>0$ is arbitrary, but fixed.

Step 3. Fall off Away from the Real Axis with Bounded Potentials. Let $X_2(z)$ be constructed as above from a compact function $X(z)$. It has the form $X_2(z) = Y(z)(z-H'_D)^{-1}U'_\beta$ (or is a column matrix with such entries) for some $Y(z)$. We rewrite this as $X_2(z)=Y(z)(z-H_0)^{-1}U'_\beta+Y(z)(z-H'_D)^{-1}\, V'_D(z-H_0)^{-1}U'_\beta$. Then we apply Lemmas 6.5 and 5.2 several times until we obtain

$$\|X_2(z)\|\leqq C_\eta\|(1+r_{D_2}^2)^{-\alpha/2}(z-H_0)^{-1}U'_\beta\|^\nu$$

whenever $|\mathrm{Im}\, z|\geqq\eta$. (Note that the projection factors help us gain configuration space fall off and do not cause trouble.) Here D_2 is some cluster decomposition with at most two clusters. If it has two clusters, then the particles of the pair β belong to different clusters [due to the compactness of $X_2(z)$].

We now apply Lemmas 5.3, 5.2, and 5.1, to see that $\|X_2(z)\|\leqq C_\eta(\mathrm{Re}\, z)^{-\alpha}$ for some α, when $\mathrm{Re}\, z$ is large and $|\mathrm{Im}\, z|\geqq\eta>0$.

Step 4. Combining Steps 1, 2, and 3. Matrix multiplication shows that Theorem 1.3 for $N=4$ will be proved if the compact entries of $M(z)$ go to zero as $\mathrm{Re}\, z\to\infty$. Step 1 shows that each such entry is a sum of various terms, which are products of uniformly bounded factors. At least one factor in each term is uniformly norm continuous and compact. Thus, it suffices to prove that the uniformly norm continuous compact factors vanish as $\mathrm{Re}\, z\to\infty$. Moreover, the uniform continuity implies that we need only show these factors vanish as $\mathrm{Re}\, z\to\infty$ with $|\mathrm{Im}\, z|$ fixed and positive. By Step 2, it suffices to prove this for the case of bounded potentials. This is done in Step 3. □

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