Commun. math. Phys. 37, 19—28 (1974)

Construction of a Strictly Renormalizable Effective Lagrangian for the Massive Abelian Higgs Model

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Received January 25, 1974

Abstract. It is shown, using the BPHZ renormalization program and Zimmermann's normal product algorithm, that a strictly renormalizable effective Lagrangian for the Abelian massive Higgs model does exist: Ward identities are fulfilled, and normalization conditions, defining a theory in an indefinite metric Fock space, may be implemented.

1. Introduction

A number of examples of renormalizable Lagrangian models involving symmetry breaking [1–3] have been recently formulated, in versions which do not make use of any symmetric regularization procedure [4–7]. The basic tool is the so-called "normal product algorithm" (NPA) due to Zimmermann [8], which provides a cut-off free formulation of the BPH renormalization procedure.

For models in which symmetric mass parameters do not vanish, there are two alternative ways of using the NPA: one which respects the super-renormalizibility of the non-symmetric couplings [6, 9] and which we shall call, according to Schroer's terminology, "soft quantization", and another one, the "hard quantization", which treats all couplings as having power index 4 [4, 5]. These two methods yield identical Green's functions, according to an equivalence theorem [10, 6]. The first approach meets, however, difficulties in cases where some symmetric mass parameters have to vanish, whereas the second method is applicable to all cases – and only meets difficulties in principle when some renormalized masses vanish.

Recently, Lowenstein, Weinstein and Zimmermann [6] have formulated the soft renormalization method for the massive Abelian Higgs-Kibble model in the Stueckelberg gauge [11] (massive QED of the σ model). In this case, the equivalence theorem [6, 10] ensures that the hard renormalization procedure exists. It turns out, however, that a direct formulation of this hard renormalization is not completely trivial, which is the motivation of this paper.

2. Construction of the Effective Lagrangian

The Abelian massive Higgs model is characterized, in the *tree approximation*, by the Lagrangian obtained by performing a σ field translation v on the Lagrangian for the electrodynamics of the σ model:

$$\begin{aligned}
\mathscr{L}_{tree} &= \mathscr{L}_0 + \mathscr{L}_{I,tree} \\
\mathscr{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(m^2+e^2v^2)A_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \\
&\quad -\tfrac{1}{2}(\mu_\pi^2+8\lambda v^2)\sigma^2 - \tfrac{1}{2}\mu_\pi^2\pi^2 + \tfrac{1}{2}(\partial_\mu\pi\partial^\mu\pi + \partial_\mu\sigma\partial^\mu\sigma) \\
&\quad - evA^\mu\partial_\mu\pi \\
\mathscr{L}_{I,tree} &= e^2vA_\mu A^\mu\sigma + eA^\mu(\pi\partial_\mu\sigma - \sigma\partial_\mu\pi) \\
&\quad + \tfrac{1}{2}e^2A_\mu A^\mu(\sigma^2+\pi^2) - 4\lambda v\sigma(\sigma^2+\pi^2) - \lambda(\sigma^2+\pi^2)^2 .
\end{aligned} \tag{1}$$

The notations are the following. The diagonal elements of the metric are $(1,-1,-1,-1)$. The spontaneous breakdown parameter v is the vacuum expectation value $\langle\varphi\rangle$ of the complex scalar Higgs field $\varphi \equiv v + \sigma + i\pi$, so that $\langle\sigma\rangle = \langle\pi\rangle = 0$. A_μ is a vector field, with mass m in the symmetric limit $v=0$, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$; ∂ means the gradient operator ∂_μ: e.g., $\partial A \equiv \partial_\mu A^\mu$.

At the symmetric limit $v=0$, $\mathscr{L}_{tree}$ is left invariant by gauge transformations, with the exception of the mass term A^2 and the gauge term $(\partial A)^2$. Symmetry breaking ($v \neq 0$) does not alter the charge conjugation symmetry, under which the σ field is even, and the π and A fields odd.

An alternative but equivalent characterization of this model in the tree approximation is provided directly in terms of Green functions. This is done by requiring the Ward identities (notation will be explained in Appendix A)

$$\begin{aligned}
\left(m^2 + \frac{1}{\alpha}\partial^2\right)\partial A(x) &= -ev\mu_\pi^2\pi(x) + e(\sigma(x)+v)\,\delta_\pi(x) \\
&\quad - e\pi(x)\,\delta_\sigma(x) - \partial_\mu\delta_{A_\mu}(x)
\end{aligned} \tag{2}$$

(in fact, only *the equality of the $\sigma\delta_\pi$ and $-\pi\delta_\sigma$ terms is required)* and the 8 normalization conditions for the vertices (one-particle-irreducible or "proper" Green's functions; the definitions are given in Appendix A)

$$\begin{aligned}
&\Gamma_{\sigma\sigma}(p^2=m_\sigma^2)=0, \qquad \frac{d}{dp^2}\Gamma_{\sigma\sigma}(p^2=m_\sigma^2)=i \\
&\Gamma_{AA}^T(p^2=m_A^2)=0, \qquad \frac{d}{dp^2}\Gamma_{AA}^T(p^2=m_A^2)=i \\
&\Gamma_{\sigma^4}(p_1,p_2,p_3,p_4)\Big|_{\substack{p_i^2=m_\sigma^2\\ sym.\,pt.}} = -24i\lambda \\
&\Gamma_{\sigma^3}(p_1,p_2,p_3)\big|_{p_2^2=p_1^2=m_\sigma^2} = -24i\lambda v \\
&D(p^2=\varkappa^2)=D(p^2=\chi^2)=0
\end{aligned} \tag{3}$$

where

$$D(p^2) \equiv \Gamma_{\pi\pi} \Gamma^L_{AA} + \Gamma_{\pi A_\mu} \Gamma_{\pi A^\mu} \tag{4}$$

and $\varkappa^2$, χ^2 are the zeros of

$$\frac{1}{\alpha}(p^2 - \mu_\pi^2)(p^2 - \alpha m^2 - \alpha e^2 v^2) + e^2 v^2 p^2 = 0 \tag{5}$$

i.e., the zeros of Eq. (4) calculated from the tree Lagrangian (1). Γ^T_{AA} and Γ^L_{AA} are the transverse and longitudinal parts of the two-vertex for the A field:

$$\Gamma_{\mu\nu}(p, -p) \equiv \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \Gamma^T_{AA}(p^2) + \frac{p_\mu p_\nu}{p^2} \Gamma^L_{AA}(p^2) .$$

The first four normalization conditions define the masses and scales for the physical fields σ and A^T; the fifth defines the $4-\sigma$ coupling constant, the sixth the symmetry breaking parameter v. The last two define the masses $\varkappa^2$ and χ^2 of the ghosts [12]: recall that the matrix propagator for the two coupled fields π and A^L, A^L being a ghost, is given by minus the inverse of the matrix

$$\begin{bmatrix} \Gamma_{\pi\pi}(p^2) & \Gamma_{\pi A_\nu}(p) \\ -\Gamma_{\pi A_\mu}(p) & \dfrac{p_\mu p_\nu}{p^2} \Gamma^L_{AA}(p^2) \end{bmatrix} .$$

The six physical parameters of our theory will thus be m_A^2, m_σ^2, λ, v, $\varkappa^2$, χ^2 or, equivalently, those appearing in the tree Lagrangian (1), with the relations

$$m_A^2 = m^2 + e^2 v^2$$
$$m_\sigma^2 = \mu_\pi^2 + 8\lambda v^2 ;$$

$\varkappa^2$, χ^2 are the roots of Eq. (5).

We now turn to the "hard" quantization of this model: we shall construct an effective Lagrangian $\mathscr{L}_4$ (all couplings are treated as having power index 4), such that Ward identities with the structure of Eq. (2) and normalization conditions (3) are fulfilled in all orders of perturbation theory. For this purpose, we shall make extensive use of Zimmermann's effective equations of motion and Zimmermann's identities relating normal products of different degrees [8]. The procedure applied here was first designed for the σ model with nucleons [5].

Let us write the most general effective Lagrangian $\mathscr{L}_4$ such that charge conjugation symmetry holds. No linear term will be present, because of the conditions: $\langle A\rangle = \langle\sigma\rangle = \langle\pi\rangle = 0$;

$$\begin{aligned}\mathscr{L}_4 = & -\tfrac{1}{4} d_A F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2} M^2 A^2 + v(A^2)^2 - \tfrac{1}{2}\gamma(\partial A)^2 \\ & - \tfrac{1}{2}\varrho_\sigma \sigma^2 - \tfrac{1}{2}\varrho_\pi \pi^2 + \tfrac{1}{2} d_\sigma(\partial\sigma)^2 + \tfrac{1}{2} d_\pi(\partial\pi)^2 \\ & - tA\cdot\partial\pi + u\sigma A^2 + A\cdot(\varepsilon_\pi \pi\partial\sigma - \varepsilon_\sigma\sigma\partial\pi) \\ & + A^2(h_\sigma\sigma^2 + h_\pi\pi^2) - (f_\sigma\sigma^2 + f_\pi\pi^2)\sigma \\ & - g_\sigma\sigma^4 - g_{\sigma\pi}\sigma^2\pi^2 - g_\pi\pi^4 .\end{aligned} \tag{6}$$

This Lagrangian contains 19 parameters, which will be reduced to 8 free parameters by requiring the proper Ward identities. The effective equations of motion for the σ, π and A fields are, in the symbolic notation (A.6) of Appendix A:

$$\begin{aligned} d_\sigma\partial^2\sigma &= -\varrho_\sigma\sigma^2 + S_\sigma + \delta_\sigma \\ d_\pi\partial^2\pi &= -\varrho_\pi\pi^2 + S_\pi + \delta_\pi \end{aligned} \tag{7}$$

$$d_A(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu + M^2 A_\mu + \gamma\partial_\mu\partial_\nu A^\nu = -j_\mu - \delta_{A^\mu}$$

with

$$\begin{aligned} S_\sigma \equiv\; & uA^2 - \varepsilon_\pi\pi\partial A - (\varepsilon_\pi + \varepsilon_\sigma)A\cdot\partial\pi + 2h_\sigma A^2\sigma \\ & - 3f_\sigma\sigma^2 - f_\pi\pi^2 - 4g_\sigma\sigma^3 - 2g_{\sigma\pi}\sigma\pi^2 \\ S_\pi \equiv\; & \varepsilon_\sigma\sigma\partial A + (\varepsilon_\sigma + \varepsilon_\pi)A\cdot\partial\sigma + 2h_\pi A^2\pi \\ & - 2f_\pi\sigma\pi - 2g_{\sigma\pi}\sigma^2\pi - 4g_\pi\pi^3 \\ j_\mu \equiv\; & 4vA^2 A_\mu - t\partial_\mu\pi + 2u\sigma A_\mu + \varepsilon_\pi\pi\partial_\mu\sigma \\ & - \varepsilon_\sigma\sigma\partial_\mu\pi + 2(h_\sigma\sigma^2 + h_\pi\pi^2)A_\mu . \end{aligned} \tag{8}$$

Now, taking the divergence of the equation of motion for A, eliminating terms such as $\partial^2\pi$, $\pi\partial^2\sigma$ and $\sigma\partial^2\pi$ by use of the proper, possibly non-linear, versions of the effective equations of motion for σ and π with the correct degree's assignments (Appendix A), and applying the rule [13]:

$$\partial_\mu^x \langle T N_\delta[\mathcal{O}](x) X\rangle = \langle T N_{\delta+1}[\partial_\mu\mathcal{O}](x) X\rangle$$

we obtain

$$\begin{aligned}(M^2 + \gamma\partial^2)\partial A = & \frac{t^2}{d_\pi}\partial A - \frac{t}{d_\pi}\varrho_\pi\pi + \frac{t}{d_\pi}\delta_\pi + \frac{\varepsilon_\sigma}{d_\pi}\sigma\delta_\pi - \frac{\varepsilon_\pi}{d_\sigma}\pi\delta_\sigma - \partial^\mu\delta_{A_\mu} \\ & + \frac{t}{d_\pi}N_3[S_\pi] + N_4\left[\frac{\varepsilon_\sigma}{d_\pi}\sigma(S_\pi - \varrho_\pi\pi + t\partial A) - \frac{\varepsilon_\pi}{d_\sigma}\pi(S_\sigma - \varrho_\sigma\sigma)\right. \\ & \left. - 4v\partial\cdot(A^2 A) - 2u\partial(\sigma A) + (\varepsilon_\sigma - \varepsilon_\pi)\partial\sigma\cdot\partial\pi - 2\partial\cdot\left(A(h_\sigma\sigma^2 + h_\pi\pi^2)\right)\right] .\end{aligned} \tag{9}$$

Using Zimmermann's identity, we may reduce the N_3 product to an N_4 product:

$$\begin{aligned}\frac{t}{d_\pi}(N_3-N_4)[S_\pi] = & -N_4[r_1\pi\partial^2\sigma + r_2\sigma\partial^2\pi + r_3\partial\sigma\cdot\partial\pi + r_4\sigma\pi^3 \\ & + r_5\sigma^3\pi + r_6\sigma^2\partial A + r_7\pi^2\partial A + r_8 A\cdot\sigma\partial\sigma + r_9 A\cdot\pi\partial\pi \\ & + r_{10}\sigma\pi A^2 + r_{11}A^2\partial A + r_{12}A^\mu A^\nu\partial_\mu A_\nu + r_{13}\partial^2\partial A] \,.\end{aligned} \tag{10}$$

The 13 reduction coefficients r_i are expressed in Appendix B as proper Green's functions evaluated at the origin of momentum space (in fact, due to Bose statistics and Lorentz covariance, $r_2 = r_3$ and $r_9 = 0$). Eliminating the $\pi\partial^2\sigma$ and $\sigma\partial^2\pi$ terms in Eq. (10) by use of the equation of motion, and substituting in (9), we obtain

$$\begin{aligned}\left[M^2 - \frac{t^2}{d_\pi} + (\gamma + r_{13})\partial^2\right]\partial A = & -\frac{t}{d_\pi}\varrho_\pi\pi + \frac{t}{d_\pi}\delta_\pi \\ & + \frac{\varepsilon_\sigma - r_2}{d_\pi}\sigma\delta_\pi - \frac{\varepsilon_\pi + r_1}{d_\sigma}\pi\delta_\sigma - \partial_\mu\delta_{A_\mu} \\ & + N_4[\ldots] \,.\end{aligned} \tag{11}$$

The last term is a sum of 14 independent products. In order to obtain a Ward identity with the same structure as in (2), we have to cancel them and, moreover, to require the equality of the $\sigma\delta_\pi$ and $-\pi\delta_\sigma$ terms:

$$\frac{\varepsilon_\sigma - r_2}{d_\pi} = \frac{\varepsilon_\pi + r_1}{d_\sigma} \,. \tag{12}$$

This leads to 15 equations between the 19 coefficients of $\mathscr{L}_4$. 11 equations allow us to express implicitly these coefficients in terms of eight free parameters a, b, c, d, g, w, ε, and ϱ, according to:

$$\begin{aligned} & d_A = 1 - b\,; \quad M^2 = a + w^2(1+d)\,; \quad v = -\frac{r_{12}}{8}\,; \quad \gamma = g - r_{13}\,; \\ & \varrho_\sigma = c + 8\varrho\frac{w^2}{\varepsilon^2}\,; \quad \varrho_\pi = c\,; \quad d_\sigma = 1 + d + \frac{r_1}{\varepsilon}\,; \quad d_\pi = 1 + d\,; \\ & t = w(1+d)\,; \quad u = \varepsilon w(1+d) + \frac{w}{2}r_2\,; \quad \varepsilon_\sigma = \varepsilon(1+d) + r_2\,; \\ & h_\sigma = \tfrac{1}{2}\varepsilon^2(1+d) + \tfrac{1}{4}(\varepsilon r_2 - r_8)\,; \quad h_\pi = \tfrac{1}{2}\varepsilon^2(1+d) + \tfrac{1}{4}\varepsilon r_2\,; \\ & f_\sigma = 4\varrho\frac{w}{\varepsilon} + \frac{w}{3\varepsilon^2}r_4\,; \quad f_\pi = 4\varrho\frac{w}{\varepsilon}\,; \quad \varepsilon_\pi = \varepsilon(1+d)\,; \\ & g_\sigma = \varrho + \frac{1}{4\varepsilon}(r_4 + r_5)\,; \quad g_{\sigma\pi} = 2\varrho + \frac{r_4}{2\varepsilon}\,; \quad g_\pi = \varrho\,. \end{aligned} \tag{13}$$

As we shall see later, the remaining four equations are then identically fulfilled.

The eight free parameters can be recursively determined as formal power series by the normalization conditions (3), for instance. The presence of couplings involving the reduction coefficients r_i does not make any difficulty. Indeed, let us take $\hbar$ (the number of loops) [14, 10] as the expansion parameter; the r_i's are given by superficially convergent one-particle irreducible graphs [see their expressions (B.3)], so they are calculable in order n in terms of the $(n-1)^{st}$ approximations of the parameters $a, b, c \ldots$. (For instance, in the tree approximation, $r_i = 0$; $b = d = 0$, $a = m^2$, $c = \mu^2$, $g = 1/\alpha$, $w = ev$, $\varepsilon = e$, $\varrho = \lambda$, so that the recursive procedure can start.)

It now remains to prove the Ward identities, the effective Lagrangian being (6) with assignments (13). Identities (11) have now the form:

$$\begin{aligned}(a+g\partial^2)\,\partial A + \partial_\mu \delta_{A_\mu} + cw\pi - w\delta_\pi - \varepsilon(\sigma\delta_\pi - \pi\delta_\sigma) \\ = N_4[\theta]\end{aligned} \tag{14}$$

where

$$\begin{aligned}\theta = R_1\sigma^2\partial A + R_2\pi^2\partial A + R_3\sigma\pi A^2 + R_4 A^2\partial A \\ R_1 = \frac{\varepsilon}{2}r_2 - r_6 + \frac{1}{2}r_8 \\ R_2 = -\frac{\varepsilon}{2}r_2 - r_7 \\ R_3 = \frac{\varepsilon}{2}r_8 - r_{10} \\ R_4 = -r_{11} + \tfrac{1}{2}r_{12}\,.\end{aligned} \tag{15}$$

The four equations left over during the reduction of the N_4 products in (11) are precisely the equations $R_i = 0$, $i = 1, \ldots, 4$. What remains to prove is that they are fulfilled to all orders in $\hbar$.

By standard methods [see, for instance, Ref. 1], we can express identities (14) in terms of proper Green's functions; in momentum space:

$$\begin{aligned}&i p^\mu \Gamma_{\mu\mu_1\ldots\mu_S\sigma^M\pi^N}(p, K; P; Q) + w\Gamma_{\mu_1\ldots\mu_S\sigma^M\pi^{N+1}}(K; P; Q, p) \\ &\quad - \varepsilon \sum_{j=1}^{N} \Gamma_{\mu_1\ldots\mu_S\sigma^{M+1}\pi^{N-1}}(K; P, p+q_j; Q_{\hat{q}_j}) \\ &\quad + \varepsilon \sum_{j=1}^{M} \Gamma_{\mu_1\ldots\mu_S\sigma^{M-1}\pi^{N+1}}(K; P_{\hat{p}_j}; Q, p+p_j) \\ &= -i\Gamma_{\theta\mu_1\ldots\mu_S\sigma^M\pi^N}(p; K; P; Q);\ (S, M, N) \neq (1,0,0) \quad \text{or} \quad (0,0,1)\,.\end{aligned} \tag{16}$$

Here $P \equiv (p_1 \ldots p_N)$, $Q \equiv (q_1 \ldots q_M)$, $K \equiv (k_1 \ldots k_S)$; $p + \Sigma k_i + \Sigma p_j + \Sigma q_k = 0$ $Q_{\hat{q}_j}$ means that element q_j has to be omitted from Q. The right-hand side is a proper Green's function involving the normal product $N_4[\theta]$.

Among identities (16), let us select those whose right-hand sides *do not possess a Born approximation*, and write them symbolically as

$$\begin{gathered}W_{\mu_1\ldots\mu_S\sigma^M\pi^N}(p;K;P;Q)=-i\Gamma_{\theta\mu_1\ldots\mu_S\sigma^M\pi^N}(p;K;P;Q)\\(S,M,N)\neq(1,0,0),(0,0,1);(1,2,0),(1,0,2),(2,1,1)\quad\text{or}\quad(3,0,0)\,.\end{gathered}\tag{17}$$

One remarks that the R_i's may be expressed as linear combinations of derivatives of left-hand sides of identities (17) taken at zero momenta [use expressions (B.3) of Appendix B]:

$$\begin{aligned}\frac{16i}{w}R_1&=\frac{\partial^2}{(\partial p)^2}[W_{\sigma^2\pi}(p;-p,0;0)-W_{\sigma^2\pi}(0;-p,0;p)]_{p=0}\\&\quad+2\frac{\partial^2}{\partial p\cdot\partial q}[W_{\sigma^2\pi}(p;q,0;-p-q)-W_{\sigma^2\pi}(-p-q;p,0;q)]_{p=q=0}\\\frac{16i}{w}R_2&=\frac{\partial^2}{(\partial p)^2}[W_{\pi^3}(0;p,-p,0)-W_{\pi^3}(p;-p,0,0)]_{p=0}\\\frac{8}{w}R_3&=\frac{\partial}{\partial p_\mu}W_{\mu\sigma\pi^2}(-p;0;0;p,0)|_{p=0}\\\frac{48i}{w}R_4&=g^{\mu\nu}\frac{\partial^2}{(\partial p)^2}[W_{\mu\nu\pi}(0;0,-p;p)-W_{\mu\nu\pi}(p;0,-p;0)]_{p=0}\,.\end{aligned}$$

It is easy, now, to prove by induction the desired result. In the tree approximation (zeroth order in $\hbar$) $R_i=0$. Suppose now that $R_i=0$ up to and including order $n-1$. Since the right-hand sides of identities (17) have no Born approximation and are linearly homogeneous in the R_i's, the left-hand sides are zero in order n. Hence, by the remark hereabove, $R_i=0$ in order n.

Acknowledgements. I am very grateful to Professor R. Stora for having introduced me in this field and for his constant help, suggestions and encouragements. I also enjoyed interesting discussions with Drs. C. Becchi, P. Butera, and K. Yoshida.

Appendix A – Some Notations

Let X denote any string of fields

$$X\equiv A_{\mu_1}(z_1)\ldots A_{\mu_S}(z_S)\,\sigma(x_1)\ldots\sigma(x_M)\,\pi(y_1)\ldots\pi(y_N)\,.\tag{A.1}$$

Disconnected, connected and proper (one-particle irreducible) *Green's functions* are respectively:

$$\begin{aligned}\langle TX\rangle&\equiv G_{\mu_1\ldots\mu_S\sigma^M\pi^N}(z_1\ldots z_S;x_1\ldots x_M;y_1\ldots y_N)\\\langle TX\rangle^C&\equiv G^C_{\mu_1\ldots\mu_S\sigma^M\pi^N}(\ldots)\\\langle TX\rangle^P&\equiv\Gamma_{\mu_1\ldots\mu_S\sigma^M\pi^N}(\ldots)\,.\end{aligned}\tag{A.2}$$

Fourier transforms are defined, e.g., by

$$\begin{aligned}\langle T\tilde{X}\rangle^{P} &\equiv (2\pi)^4\,\delta(\Sigma\,k_i + \Sigma\,p_i + \Sigma\,q_i)\\ &\quad\cdot \Gamma_{\mu_1\ldots\mu_S\sigma^M\pi^N}(k_1\ldots k_S;\,p_1\ldots p_M;\,q_1\ldots q_N)\\ &\equiv \int\Pi\,dz_i\int\Pi\,dx_i\int\Pi\,dy_i\,e^{i(\Sigma k_i z_i+\Sigma p_i x_i+\Sigma q_i y_i)}\,\langle TX\rangle^{P}\,.\end{aligned}\tag{A.3}$$

Green's functions involving a normal product $N_\delta[\theta](x)$ of degree δ associated to a polynomial of fields $\theta(x)$ [8] are noted

$$\langle TN_\delta[\theta](x)X\rangle^{(C)(P)}\,.$$

Classical equations of motion are the Euler-Lagrange equations $\delta\mathcal{L}(x)/\delta\Phi(x)=0$, $\Phi(x)$ being any field A_μ, σ or π. For the case in which all couplings in the effective Lagrangians are taken with power index 4, the quantized *effective equations of motion* [8, 12] are the following relations between Green's functions (linear and bilinear equations only are considered here):

$$\begin{aligned}&\left\langle TN_3\left[\frac{\delta\mathcal{L}}{\delta\Phi}\right](x)X\right\rangle + \langle T\delta_\Phi(x)X\rangle = 0\\ &\left\langle TN_4\left[\Phi_1\frac{\delta\mathcal{L}}{\delta\Phi_2}\right](x)X\right\rangle + \langle T\Phi_1(x)\,\delta_{\Phi_2}(x)X\rangle = 0.\end{aligned}\tag{A.4}$$

The "contact terms" δ_Φ are defined by (e.g., for $\Phi\equiv A_\mu$):

$$\langle T\delta_{A_\mu}(x)X\rangle \equiv -i\sum_{j=1}^{S} g^{\mu}_{\mu_j}\,\delta(x-z_j)\,\langle TX_{\widehat{\mu j}}\rangle\tag{A.5}$$

where $X_{\widehat{\mu j}}$ means that field $A_{\mu_j}(z_j)$ has to be omitted in string (A.1).

Identities like (A.4) will be written symbolically as

$$\frac{\delta\mathcal{L}}{\delta\Phi} + \delta_\Phi = 0\tag{A.6}$$

keeping in own one's mind the proper degree's assignments explicitly displayed in (A.4).

Appendix B

We give here closed expressions for the reduction coefficients r_i of the Zimmermann identity (10). They are calculated as usual by use of the normalization conditions for normal products [8]. We obtain, for instance [we replace t/d_π by w, anticipating results (13)]:

$$\begin{aligned}r_1 &= \frac{w}{8}\,\frac{\partial^2}{(\partial p)^2}\,\langle TN_3[S_\pi](0)\,\tilde{\sigma}(p)\,\tilde{\pi}(0)\rangle^{P}|_{p=0}\\ &\;\vdots\\ r_{13} &= -\frac{iw}{48}\,\frac{\partial^3}{(\partial k)^2\,\partial k_\mu}\,\langle TN_3[S_\pi](0)\,\tilde{A}_\mu(k)\rangle^{P}|_{k=0}\,.\end{aligned}\tag{B.1}$$

By use of the equation of motion (7) for the π field, we easily verify that, for proper Green's functions

$$\begin{gathered}\langle T N_3[S_\pi](x) X\rangle^P = -i\langle T\pi(x) X\rangle^P \\ (\text{if} \quad X \neq \pi \quad \text{or} \quad A_\mu)\end{gathered} \tag{B.2}$$

and also

$$\langle T N_3[S_\pi](x)\, A_\mu(y)\rangle^P = -i\langle T\pi(x)\, A_\mu(y)\rangle^P - t\partial_\mu\delta(x-y)\,.$$

The expressions for the r_i's are thus:

$$\begin{aligned}
r_1 &= -i\frac{w}{8}\,\frac{\partial^2}{(\partial p)^2}\,\Gamma_{\sigma\pi^2}(p;0,-p)|_{p=0}\\
r_2 &= -i\frac{w}{8}\,\frac{\partial^2}{(\partial q)^2}\,\Gamma_{\sigma\pi^2}(0;q,-q)|_{q=0}\\
r_3 &= -i\frac{w}{4}\,\frac{\partial^2}{\partial p\,\partial q}\,\Gamma_{\sigma\pi^2}(p;q,-p-q)|_{p=q=0}\\
r_4 &= i\frac{w}{6}\,\Gamma_{\sigma\pi^4}(0;0,0,0,0)\\
r_5 &= i\frac{w}{6}\,\Gamma_{\sigma^3\pi^2}(0,0,0;0,0)\\
r_6 &= \frac{w}{8}\,\frac{\partial}{\partial k_\mu}\,\Gamma_{\mu\sigma^2\pi}(k;0,0;-k)|_{k=0}\\
r_7 &= \frac{w}{8}\,\frac{\partial}{\partial k_\mu}\,\Gamma_{\mu\pi^3}(k;0,0,-k)|_{k=0}\\
r_8 &= \frac{w}{4}\,\frac{\partial}{\partial p_\mu}\,\Gamma_{\mu\sigma^2\pi}(0;p,0;-p)|_{p=0}\\
r_9 &= \frac{w}{4}\,\frac{\partial}{\partial q_\mu}\,\Gamma_{\mu\pi^3}(0;q,-q,0)|_{q=0}\\
r_{10} &= i\frac{w}{8}\,g^{\mu\nu}\Gamma_{\mu\nu\sigma\pi^2}(0,0;0;0,0)\\
4r_{11}+r_{12} &= \frac{w}{8}\,g^{\nu\lambda}\frac{\partial}{\partial k_\mu}\,\Gamma_{\mu\nu\lambda\pi}(k,0,0;-k)|_{k=0}\\
2r_{11}-5r_{12} &= \frac{w}{4}\,g^{\nu\lambda}\frac{\partial}{\partial k_\mu}\,\Gamma_{\mu\nu\lambda\pi}(0,k,0;-k)|_{k=0}\\
r_{13} &= -\frac{w}{48}\,\frac{\partial^3}{(\partial k)^2\,\partial k_\mu}\,\Gamma_{\mu\pi}(k;-k)|_{k=0}\,.
\end{aligned} \tag{B.3}$$

One also has

$$2r_{11} - r_{12} = \frac{w}{12} g^{\nu\lambda} \frac{\partial}{\partial k_\mu} \Gamma_{\mu\nu\lambda\pi}(k, 0, -k; 0)|_{k=0}.$$

The following two identities are true:

$$r_2 = r_3; \qquad r_9 = 0. \tag{B.4}$$

Indeed, $\Gamma_{\sigma\pi^2}(p; q, -q-p)$ is symmetric under the interchange $q \rightleftarrows -p-q$ at p fixed (Bose statistics); moreover, this function is a scalar: then

$$\Gamma_{\sigma\pi^2}(p; q, -p-q) = f(x, y)$$

with $x = p^2$, $y = q(p+q)$; hence the result $r_2 = r_3$ is easily derived. To show that $r_9 = 0$, one remarks that $\Gamma_{\mu\pi^3}(0; q, -q, 0)$ must be even in q due to Bose statistics; on the other hand, it is a vector: therefore it has to vanish.

We finally note that the r_i's (B.3) are given by one-particle irreducible graphs which have at least one loop: they vanish identically in the tree approximation.

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Communicated by K. Symanzik

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