

G **T** **P** **E**
R **D** **E**

X Y LU*

Faculty of Sciences
Scuola Normale Superiore
Pisa, 56126, Italy

(Communicated by Irena Lasiecka)

Abstract

In this paper we consider the discrete dynamic evolution of a connected network related to an average distance functional minimization problem, with associated dissipation term, under irreversibility assumption. We will analyze its geometric and topological properties, and study whether it inherits regularity properties verified by the static or quasi static case.

AMS Subject Classification: 49Q10, 49J40, 49L15.

Keywords: optimal transport, Euler scheme, dynamic evolution, average distance.

1 Introduction

In a compact, sufficiently regular domain in \mathbb{R}^2 , locally optimal sets for the average distance functional in the static case verify some geometric properties, as the absence of loops or crosses (see [6], [7], [8] and [9] for instance). Passing to the discrete quasi static evolution with small time steps, the absence of loops continues to be verified, while the absence of crosses is not true anymore.

Another type of evolution is the so-called *dynamic evolution* (see [2] for a detailed discussion), significantly different from the quasi static case. Our goal will be checking if these properties pass to the discrete dynamic case, and analyzing if the branching conditions (found in [10]) are still valid. In the following for sake of simplicity we will omit the word *discrete*. Similarly to [10], the main objects of our analysis will be elements in the following set: let Ω be a compact connected subset of \mathbb{R}^2 and define

$$A_l(\Omega) := \left\{ \mathcal{X} \subseteq \Omega : \mathcal{X} \text{ compact, connected by path, } \dim_{\mathcal{H}} \mathcal{X} = 1 \text{ and } \mathcal{H}^1(\mathcal{X}) \leq l \right\}, \quad A(\Omega) := \bigcup_{j \geq 0} A_j. \quad (1.1)$$

*E-mail address: x.lu@sns.it

Moreover, the following sets will be used frequently:

$$A_j^*(\Omega) := A_j(\Omega) \setminus A_0(\Omega), \quad A^*(\Omega) := A(\Omega) \setminus A_0(\Omega).$$

Like in [10], we will omit dependence on the domain if no confusion arises. Differently from the quasi static case in [10], now we need at least two functionals: one is the average distance

$$F : A \longrightarrow (0, \infty), \quad F(S) := \int_{\Omega} \text{dist}(x, S) dx,$$

which will be referred as “*energy*”. Then, as we are studying dynamic evolutions, another functional, a “*dissipation*” is required:

Definition 1.1. Given a domain Ω , a dissipation is a functional

$$D : A \times A \longrightarrow [0, \infty]$$

which verifies:

- $D(S, S) = 0$ for any $S \in A$,
- $D(S_1, S_2) > 0$ for any $S_1, S_2 \in A$ with $\mathcal{H}^1(S_1 \Delta S_2) > 0$

The dissipation gives a “cost” in passing from one configuration to another, not present in the quasi static case. More discussion about evolution schemes will be done in Section 3.

This paper will be structured as follows:

- in Section 2 we will analyze geometric regularity of optimal sets in a dynamic evolution,
- in Section 3 we will find conditions about branching behavior of evolving sets.

Notations

Now, before presenting the results, some word about notations. First, “connected” will mean “connected by path”. The most used symbols in this paper will be:

- Ω to denote the domain,
- $\varepsilon, \eta, \delta, r$ to denote following small positive numbers,
- l, a to denote generic positive numbers,
- S to denote generic elements of A ,
- S_0, X_0 to denote the initial datum of an Euler scheme,
- $w(k, \cdot), w(k)$ ($k \in \mathbb{N}$) to denote the $(k + 1)$ -th set of an Euler scheme.

If a notation is used in two different Definitions/Propositions/Lemma/Theorems, there is no connection between them unless otherwise specified.

Some exceptions are present:

- A_l, A_l^* (with $l \geq 0$), and A, A^* : if there is a given domain Ω , they always denote the sets defined in (1.1),
- F which always stands for the average distance functional,
- $V(\cdot)$ which stands for the Voronoi cell of the point (or set),
- $\text{dist}(\cdot, \cdot)$ which stands for the geodesic distance between two closed sets.

The notion of Voronoi cell used in this paper is defined as follows:

Definition 1.2. Given a domain Ω , a set $S \subseteq \Omega$, a non empty subset $S' \subseteq S$ (S' may consist of one point), the Voronoi cell of S' (with respect to S) is

$$V_{\Omega, S}(S') := \{x \in \Omega : \text{dist}(x, S') = \text{dist}(x, S)\}.$$

While a priori the definition of Voronoi cell is dependent on both Ω and S , when there will be no risk of confusion we will omit both of them, and write $V(S')$ instead of $V_{\Omega, S}(S')$. Moreover, to further simplify notations, when considering Voronoi cells of a single point $X \in S$, we will write $V(X)$ instead of $V(\{X\})$.

In the following, when considering evolutions, given a set Z and a point $Y \in Z$ we will use frequently the expression “a set J is added in the point Y ”: in this case we mean the following condition is satisfied:

- there exists a point $W \in J$ and a path $\gamma : [0, 1] \rightarrow J \cup Z$ such that $\gamma(0) = W$, $\gamma(1) = Y$, $\gamma([0, 1]) \cap Z = \emptyset$.

Moreover, with this expression we will assume implicitly that the set J has \mathcal{H}^1 -negligible intersection with Z .

Finally, we will work only with domains in \mathbb{R}^2 which are closure of an open, connected, bounded set, and the word “domain” will always refer to a similar domain. As we have defined F to have domain in A (in a given domain Ω), when we will write $F(\mathcal{X}_1 \cup \mathcal{X}_2)$ we will assume implicitly that we are considering only cases satisfying $\mathcal{X}_1 \cup \mathcal{X}_2 \in A$.

2 Loops and crosses

Results concerning geometric properties of optimal sets were presented in [6], [7], [8] and [9]: they state that in a domain Ω , with measures f verifying $f \in L^p$, $p > 4/3$, locally optimal sets for the average distance functional in the static case cannot contain loops or crosses (see Definition 2.1), i.e. the set

$$A_l^{opt} := \{S \in A : S \in \text{argmin}_{\mathcal{H}^1(S) \leq l} F\}$$

cannot contain any element S containing a loop or a cross, for any $l > 0$ (the case $l = 0$ is trivial). Before continuing, we define what we mean here with “loop” and “cross”:

Definition 2.1. Given domain $\Omega \subseteq \mathbb{R}^2$, endowed with the Euclidean topology, a set $W \subseteq \Omega$ is:

- a “loop” if W is homeomorphic to S^1 ,
- a cross if there exists distinct points $w^*, w_1, \dots, w_4 \in W$ such that there exists paths $\gamma_1, \dots, \gamma_4 : [0, 1] \rightarrow W$ with $\gamma_i(0) = w^*$, $\gamma_i(1) = w_i$ for any $i = 1, \dots, 4$ and $\gamma_j((0, 1]) \cap \gamma_k((0, 1]) = \emptyset$ for any $j \neq k$.

The absence of loops is inherited by the quasi static evolution when the starting configuration does not contain loops (the other case, when initial configuration already contain a loop, is trivial as due to monotonicity every set in the evolution contains the initial configuration, thus contains a loop), as proven in [11]; on the other side, the same paper shows a counterexample to the absence of crosses, by presenting an example of evolution starting from initial configuration without crosses, but later sets contain crosses.

As we will see in the following, the dynamic evolution is more complicated: if in the quasi static case we could easily determine how much length is added at each step, here this task is much more complicated due to the lack of an explicit constraint on lengths, and the presence of the dissipation term, which may generate qualitatively different behaviors (see [12] for instance).

We first present some estimates for the energy functional.

2.1 Energy estimates

Let us start with some estimates about the energy functional. Most of them can be found on [6], [7], [8] and [9].

Lemma 2.2. *Given a domain Ω , let be $S_1, S_2 \in A$, $S_1 \subset S_2$, then*

$$F(S_1) - F(S_2) \leq O(\max_{y \in S_2} \text{dist}(y, S_1)).$$

Proof. The proof can be found on [9], we limit to report the main idea here. Writing the thesis explicitly we have

$$F(S_1) - F(S_2) = \int_{\Omega} \text{dist}(x, S_1) - \text{dist}(x, S_2) dx$$

and as $S_1 \subset S_2$

$$\int_{\{x \in \Omega : \text{dist}(x, S_1) = \text{dist}(x, S_2)\}} \text{dist}(x, S_1) - \text{dist}(x, S_2) dx = 0.$$

So the thesis is equivalent to estimate

$$\int_{\{x \in \Omega : \text{dist}(x, S_1) > \text{dist}(x, S_2)\}} \text{dist}(x, S_1) - \text{dist}(x, S_2) dx.$$

As for an point $z \in \Omega$ we have

$$\text{dist}(z, S_1) \leq \text{dist}(z, S_2) + \max_{y \in S_2} \text{dist}(y, S_1)$$

so integrating we have

$$\int_{\Omega} \text{dist}(z, S_1) \leq \int_{\Omega} \text{dist}(z, S_2) + |\Omega| \max_{y \in S_2} \text{dist}(y, S_1)$$

or equivalently

$$F(S_1) - F(S_2) \leq |\Omega| \max_{y \in S_2} \text{dist}(y, S_1),$$

and the proof is complete. \square

Now we want a lower bound: we introduce a the notation of *smooth point* as we have done in [10]:

Definition 2.3. Given a domain Ω , $S \in A$ a generic element, a non endpoint $P \in S$ is “smooth” if there exists $r > 0$ such that:

- (1) there exists an homeomorphism $f : B(P, r) \cap S \rightarrow (0, 1)$;
- (2) there exists an unique direction with sign θ (which can be identified with an unit vector in S^1) such that for any sequence $\{P_n\}_{n=0}^{\infty} \rightarrow P$, in $B(P, r)$, $P_n \neq P$ for any n , the directions of vectors $\frac{\overrightarrow{P_n P}}{\text{dist}(P_n, P)}$ converge to θ .

Again, as in [10], we say a subset of S is smooth is all its non endpoints are smooth.

Next we report a lower bound estimate on the gain for the energy functional (the proof can be found on [10]):

Proposition 2.4. Given a domain Ω , let be $S \subset \Omega$ be a connected set, if we add a segment λ_{ε} to a smooth non endpoint of S (with $\mathcal{H}^1(\lambda_{\varepsilon}) = \varepsilon$), then the “gain” $F(S) - F(S_{\varepsilon})$ is at least comparable with $\varepsilon^{3/2}$, where $S_{\varepsilon} := S \cup \lambda_{\varepsilon}$.

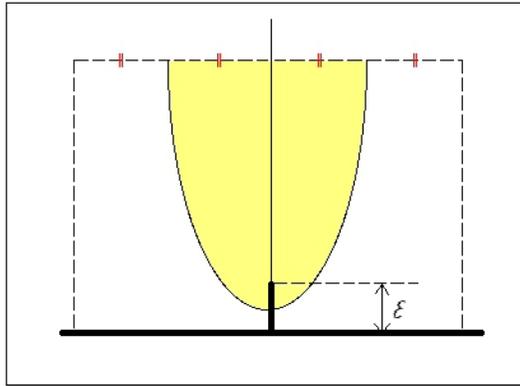


Fig. 1: All the shaded area, whose area is comparable with $\varepsilon^{1/2}$, gains something in path.

Here we limit to give a brief sketch: the shaded parabola (whose area is comparable with $\varepsilon^{1/2}$) yields a gain in energy at least comparable with $\varepsilon^{3/2}$, as the gain in path is comparable with ε on most of the parabola.

The next result is an important one, and provides a sharper lower bound estimate when some geometric properties are verified:

Proposition 2.5. *Given a domain Ω , let $S \in A$, and let it have a point O satisfying:*

- (*) *there exists $\xi > 0$ and $\beta < \pi$ such that $S \cap B(O, \xi)$ is contained in the circular sector with center O and arc $Q'R'$, with $\widehat{Q'OR'} = \beta$.*

Then we have:

- (1) *there exist $\rho > 0$ and $\theta > 0$ and a isosceles triangle $T' \subset V(O)$ with a vertex in O , two sides with length ρ and angle in O measuring θ , that does not intersect S ,*
- (2) *there exists ε_0 such for any $\varepsilon < \varepsilon_0$ adding a segment λ_ε at O , with $\mathcal{H}^1(\lambda_\varepsilon) = \varepsilon$ in O leads to a gain for the energy functional comparable with $O(\varepsilon)$.*

For the proof we refer to [10].

The following argument has been made in [11], but we report it as it is relevant for the next sections. Results of Propositions 2.4 and 2.5 can be summarized as follows: for a configuration X if condition (*) of Proposition 2.5 is verified by some point P (for some $r > 0, \theta < \pi$), then locally near the point the configuration can be figured as

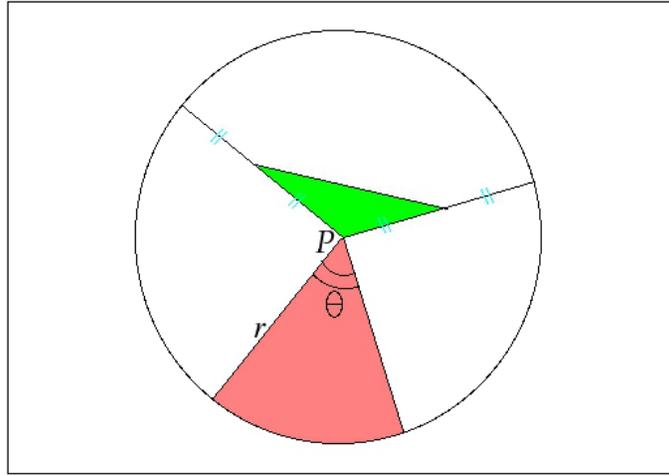


Fig. 2: This is the schematic representation of the configuration near P , with $X \cap B(P, r)$ contained in the shaded circular sector, while the shaded triangle is contained in $V(P)$.

and the gain in energy by adding in a correct manner a segment Seg in P is comparable with $O(\mathcal{H}^1(Seg))$ for $\mathcal{H}^1(Seg)$ sufficiently small.

If condition (*) is not satisfied by the point Q , then the configuration near Q can be figured as

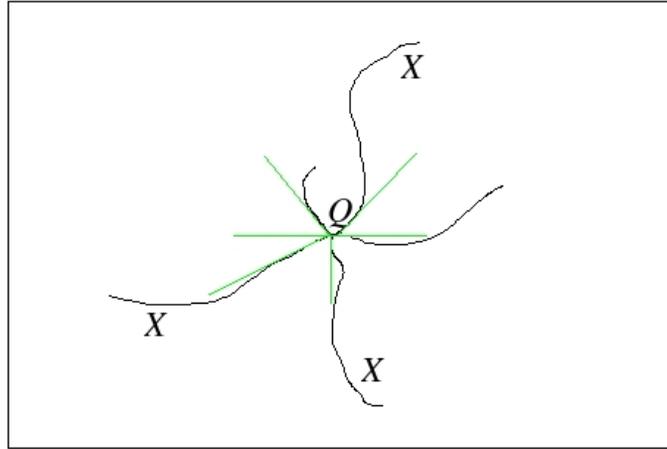


Fig. 3: This is the schematic representation of the configuration near Q , with points of X converging from many directions (denoted by lighter lines).

and the gain in energy by adding a set U in Q is at most comparable with an $o(\mathcal{H}^1(U))$ for $\mathcal{H}^1(U)$ sufficiently small, as it is upper bounded (upon a suitable projection, which changes the order depending on local regularity properties) by the gain in the following configuration (like that in Proposition 2.4)

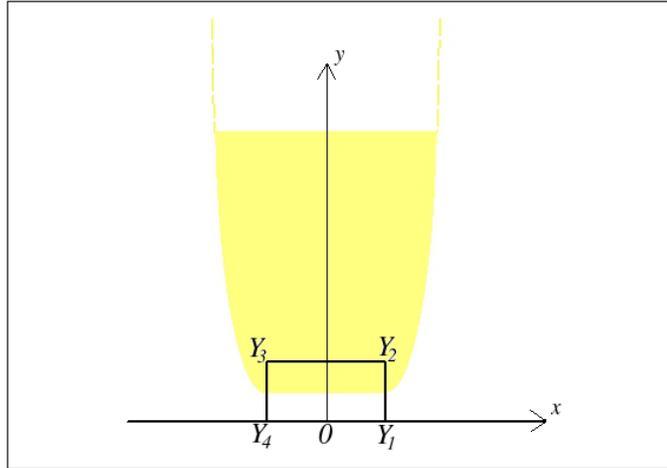


Fig. 4: In this configuration, with $\mathcal{H}^1(Y_1Y_2) = \mathcal{H}^1(Y_3Y_4) = \mathcal{H}^1(Y_2Y_3)/2 = \mathcal{H}^1(U)$, the gain in energy can be comparable with $O(\mathcal{H}^1(U)^{3/2})$ at most. The light shaded region, and that delimited by dashed lines, have a positive gain in path.

which has gain in energy comparable with $O(\mathcal{H}^1(U)^{3/2})$ at most: indeed independently from the exact choice of U , when added in Q (above axis x , if contains parts added below the same argument applies with slight modifications), it must be included in the rectangle $Y_1Y_2Y_3Y_4$ of Figure 4. Then

$$F(X \cup U) \geq F(X \cup Y_1Y_2 \cup Y_2Y_3 \cup Y_3Y_4);$$

for a generic point (x,y) , if it satisfies $\text{dist}((x,y), Y_1Y_2 \cup Y_2Y_3 \cup Y_3Y_4) \leq \text{dist}((x,y), X) = y$, then

$$\text{dist}((x,y), Y_1Y_2) \leq \text{dist}((x,y), X) = y$$

or

$$\text{dist}((x, y), Y_2 Y_3) \leq \text{dist}((x, y), X) = y$$

or

$$\text{dist}((x, y), Y_3 Y_4) \leq \text{dist}((x, y), X) = y$$

and in all these cases a direct computation show (x, y) must belong to a set of area comparable with $O(\mathcal{H}^1(U)^{1/2})$.

2.2 Evolution schemes

Now we are ready to introduce our class of evolutions. We recall some aspects in the abstract case first.

Let be given

- a topological set (X, τ) ,
- a functional

$$\mathcal{F} : [0, T] \times X \times X \longrightarrow \mathbb{R} \cup \{\pm\infty\},$$

- an initial datum $X_0 \in X$.

Then the evolution has form

$$\begin{cases} w(0) = X_0 \\ w(n+1) \in \text{argmin } \mathcal{F}((n+1)\varepsilon, \cdot, w(n)) \end{cases} .$$

an associated function is $\Sigma_\varepsilon : [0, T] \longrightarrow X$ obtained by setting

$$\Sigma_\varepsilon(t) = w\left(\left[\frac{t}{v(\varepsilon)}\right]\right), \tag{2.1}$$

with $[\cdot]$ denoting the integer part mapping, and v a suitable function verifying $\lim_{\varepsilon \rightarrow 0^+} v(\varepsilon) = 0$.

In this paper we are considering the exponent related dynamic evolution. Let us introduce them: given a domain Ω , parameters $\alpha > 1, \varepsilon > 0$ (a similar ε will be sometimes referred as “time step”), we define ,

$$D_{\alpha, \varepsilon} : A \times A \longrightarrow [0, \infty], \quad D_{\alpha, \varepsilon}(X_1, X_2) := \begin{cases} \frac{\mathcal{H}^1(X_1 \setminus X_2)^\alpha}{\varepsilon} & \text{if } X_2 \subseteq X_1 \\ \infty & \text{otherwise} \end{cases} .$$

The functionals we are going to consider in this paper have all the form

$$\mathcal{F}_{\alpha, \varepsilon} : A \times A \longrightarrow [0, \infty], \quad \mathcal{F}_{\alpha, \varepsilon}(X_1, X_2) := F(X_1) + D_{\alpha, \varepsilon}(X_1, X_2).$$

Thus, given parameters $\alpha > 1$, a time step $\varepsilon > 0$, a time $T > 0$, and an initial datum $S_0 \in A$, our Euler scheme is

$$\begin{cases} w(0) = S_0 \\ w(n+1) \in \operatorname{argmin}_{(X \in A_{\mathcal{H}^1(w(n))+1})} \mathcal{F}_{\alpha, \varepsilon}(X, w(n)) \end{cases} \quad (2.2)$$

The main difference between this and the quasi static case is that we have not prescribed how much length is added at each step, and the presence of a dissipation term which partly offsets the gain in energy: but as we are considering small step evolutions, we impose that for any evolution like (2.2), there exists $c_0 > 0$ (not depending neither on ε nor on n) such that at each step

$$\mathcal{H}^1(w(k+1) \setminus w(k)) = \varepsilon^c$$

for some $c \geq c_0 > 0$. In this way we are prohibiting adding too much length, although the exact choice of c is very difficult to determine due to contribution of the dissipation term.

Here we put

$$\Sigma_\varepsilon(t) := w\left(\left[\frac{t}{\varepsilon^{\frac{1}{\alpha-1}}}\right]\right)$$

the associated function, with $[\cdot]$ denoting the integer part mapping. The motivations for exponent $\frac{1}{\alpha-1}$ will be explained later, in the proof of Proposition 3.3, and mainly due to the absence of control on $\mathcal{H}^1(w(k+1) \setminus w(k))$.

Notice that given a random step k , while a priori the new set added $w(k) \setminus w(k-1)$ can be not connected, we can always write $w(k) \setminus w(k-1) = \bigcup_{i \in \mathfrak{I}} C_i$, where C_i are connected components of $w(k) \setminus w(k-1)$ and \mathfrak{I} a suitable set of indexes. In this way we can split the passage of adding $w(k) \setminus w(k-1)$ in many passages, in which we add one C_i at each time: it is obvious that each of these connected components is connected to $w(k-1)$ by a path, and as we have a limitation on $\mathcal{H}^1(w(k) \setminus w(k-1)) \leq 1$, we have that there are at most countable many connected components C_i of positive length (connected components with length 0 can be easily neglected as they do not modify the energy). So this method can allow us to restrict our discussion to situations when the new set added is connected.

The dynamic case is inherently different from the quasi static one due to results in [12], where is shown that for discrete dynamic evolution with sufficiently small time step there exists ‘‘locally stable’’ points, i.e. configurations from which no further evolution is allowed. Here we limit to give a brief sketch, for more details we refer to [12].

Fix parameters $\alpha > 1, \varepsilon > 0, S \in A$, if we add a set $J_{\varepsilon'}$ with length ε' to S :

$$\mathcal{F}_{\alpha, \varepsilon}(S, S) - \mathcal{F}_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = F(S) - F(S \cup J_\varepsilon) - \frac{\mathcal{H}^1(J_{\varepsilon'})^\alpha}{\varepsilon}$$

if Proposition 2.5 is applicable, then the gain for F is has order $O(\varepsilon')$, thus

$$\mathcal{F}_{\alpha, \varepsilon}(S, S) - \mathcal{F}_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon') - \frac{O(\varepsilon'^\alpha)}{\varepsilon}$$

If condition (*) of Proposition 2.5 is not satisfied, then the gain has order $O(\varepsilon'^{3/2})$, thus

$$\mathcal{F}_{\alpha, \varepsilon}(S, S) - \mathcal{F}_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon'^{3/2}) - \frac{O(\varepsilon'^\alpha)}{\varepsilon}.$$

We have imposed $\varepsilon' := \varepsilon^c$, for some $c \geq c_0 > 0$, thus the order equations read

$$\mathcal{F}_{\alpha, \varepsilon}(S, S) - \mathcal{F}_{\alpha, \varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon^c) - O(\varepsilon'^{c\alpha-1})$$

and

$$\mathcal{F}_{\alpha,\varepsilon}(S,S) - \mathcal{F}_{\alpha,\varepsilon}(S \cup J_{\varepsilon'}, S) = O(\varepsilon^{\frac{3}{2}c}) - O(\varepsilon'^{c\alpha-1})$$

respectively.

In the first case $c < c\alpha - 1$ has always solutions for $\alpha > 1$, thus the gain in energy can offset the loss due to the dissipation term; in the second case, $\frac{3}{2}c < c\alpha - 1$ is required, and this has solutions only for $\alpha > 3/2$, and there is no guarantee that the evolution continues if $\alpha \leq 3/2$, as discussed in [12].

2.3 Loops

In [11] we have proven that for the quasi static case, starting from an initial datum not containing loops, every set in the evolution process will not contain loops. Here we prove the analogue result for the dynamic case; the proof is very similar.

Proposition 2.6. *Given a domain Ω , parameters $\alpha > 1$, $\varepsilon > 0$, $S_0 \in A$ an initial datum not containing loops, consider the evolution*

$$\begin{cases} w(0) = S_0 \\ w(n+1) \in \operatorname{argmin}_{X \in A_{\mathcal{H}^1(w(n))+1}} F(X) + D_{\alpha,\varepsilon}(X, w(n)) \end{cases} .$$

Then for any n the set $w(n)$ does not contain loops.

Proof. The proof is done by induction on n . By hypothesis we have that $w(0) := S_0$ does not contain loops.

Suppose that $w(n)$ does not contain loops, and our goal is to prove $w(n+1)$ does not contain loops. This is done by absurdum. Let be $I := \overline{w(n+1) \setminus w(n)}$, the closure of the “new” set added at step $n+1$. Suppose $I \neq \emptyset$, otherwise is trivial.

If $w(n+1)$ contains a loop E , this mean that either $E \subseteq I$ or both $(E \cap I)$ and $(E \cap w(n))$ are non empty, as the third possibility $E \subseteq w(n)$ is excluded by inductive hypothesis. These two cases will be dealt separately.

Case $E \subseteq I$.

In this case we can apply the same argument found in the proof of absence of loops in the static case (see [6], [7], [8] and [9] for instance): by removing a suitable small set $J_\delta \subset E$, with $\mathcal{H}^1(J_\delta) = \delta$, the loss for the energy functional is comparable with $O(\delta^2)$, while adding it elsewhere the gain is at least comparable with $O(\delta^{3/2})$ and the presence of the loop is not optimal.

Case $(E \cap I), (E \cap w(n))$ both non empty.

Let us parameterize E : let be $e : [0, 1] \rightarrow E$, where $e(0) = e(1) = X$ (with $X \in E \cap w(n)$ an arbitrary point, the exact choice is not influent for the proof), and $e_{|(0,1)} : (0, 1) \rightarrow E \setminus \{X\}$ homeomorphism.

As both $(E \cap I), (E \cap w(n))$ are non empty, $V := e^{-1}(E \cap I)$ is non empty.

We show that V has non empty interior. This is done by absurdum: if $(0, 1) \setminus V$ is dense in $(0, 1)$, $e((0, 1) \setminus V) = (E \cap w(n)) \setminus \{X\}$ must be dense in $E \setminus \{X\}$. Combining this with $E \setminus \{X\}$ dense in E , leads to $E \cap w(n)$ dense in E . As both E and $w(n)$ are closed, $E \cap w(n) = E$, which

contradicts $E \cap I \neq \emptyset$. Thus V has non empty interior part, i.e. there exists $(\rho_-, \rho_+) \subseteq V$ with $\rho_- < \rho_+$. Now we can remove a sufficiently small piece $J_\delta \subseteq e((\rho_-, \rho_+))$, and apply the same argument found before. Again the presence of a loop is not optimal, and the proof is complete. \square

This proof is almost identical to the quasi static counterpart in [11]: indeed the main idea is to “remove” some portion and add it elsewhere, obtaining a gain in energy exceeding the loss from the removal. The dissipation plays almost no role here, as this process does not alter the length of the “new” set added (thus the dissipation term), while carrying a gain for the energy. Similarly, the exact value of $\mathcal{H}^1(I)$ is not influent too.

2.4 Crosses

In [11] we have proven that in the quasi static case, even starting from an initial configuration without crosses, later sets in the evolution may exhibit crosses. The same result holds for the dynamic case, and again, independently from parameter α and the same example works again.

The following result holds:

Proposition 2.7. *Given a domain Ω , for any parameter $\alpha > 1$, there exists sets $S_0 \in A$ without crosses such that for any $\varepsilon > 0$ sufficiently small, any set $w(n)$ with $n \geq 1$ contains crosses, where $w(n)$ are defined by*

$$\begin{cases} w(0) := S_0 \\ w(n+1) \in \operatorname{argmin}_{X \in A_{\mathcal{H}^1(w(n))+1}} F(X) + D_{\alpha, \varepsilon}(X, w(n)) \end{cases} .$$

Proof. The same example provided for the quasi static case in [11] also works here. We use the coordinate system in Figure 5.

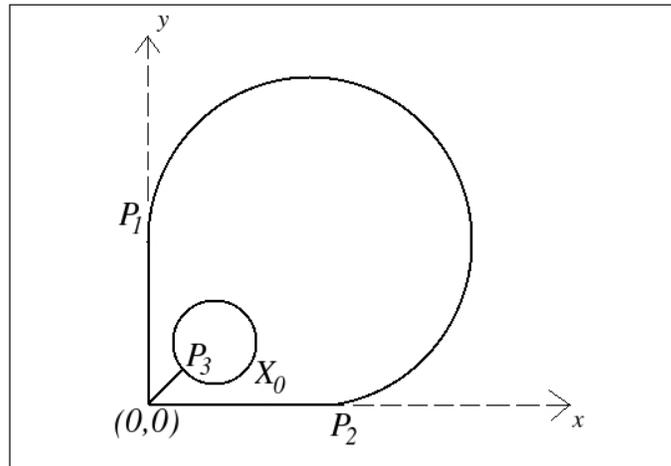


Fig. 5: This is an example of initial configuration without crosses which evolves in sets exhibiting crosses later.

Our domain will be $\Omega := [-5, 5] \times [-5, 5]$; X_0 is union of sets:

- segments connecting $(0,0)$ to P_1, P_2, P_3 respectively, i.e.

$$\{(x,y) \in \Omega : x = 0, y \in [0, 1]\},$$

$$\{(x,y) \in \Omega : y = 0, x \in [0, 1]\},$$

$$\{(x,y) \in \Omega : x = y, y \in [0, 1/4]\},$$

- circle $\{(x,y) \in \Omega : (x - 1/2)^2 + (y - 1/2)^2 = 1/16\}$,
- arc $\{(x,y) \in \Omega : (x - 1)^2 + (y - 1)^2 = 1, x \geq 1 \text{ or } y \geq 1\}$.

Every point apart from $(0,0)$ does not verify condition (*) of Proposition 2.5.

Let be X_0 the initial configuration of the Euler scheme

$$\begin{cases} w(0) := X_0 \\ w(n+1) \in \operatorname{argmin}_{X \in A_{\mathcal{H}^1(w(n))+1}} F(X) + D_{\alpha, \varepsilon}(X, w(n)) \end{cases} .$$

Notice that while there are no crosses in X_0 , there are already points x_1, x_2, x_3 and paths $g_1, g_2, g_3 : [0, 1] \rightarrow X_0$ such that

- $g_1(0) = g_2(0) = g_3(0) = (0,0)$,
- $g_i(1) = x_i$ for any $i = 1, \dots, 3$,
- $g_j((0, 1]) \cap g_k((0, 1]) = \emptyset$ for any $j \neq k$.

Thus, if during the evolution there exists a set $w(n)$ containing a point x_4 and a path $g_4 : [0, 1] \rightarrow w(n)$ with $g_4(0) = (0,0)$, $g_4(1) = x_4$ and $g_4((0, 1]) \cap g_k((0, 1]) = \emptyset$ for any $k = 1, \dots, 3$, then $w(n)$ contains a cross.

As point $(0,0)$ satisfies condition (*) of Proposition 2.5, adding a segment I_{ε^c} (with length ε^c , c will be a free parameter for now) here in a suitable way (as in the proof of Proposition 2.5), the gain for the energy functional is comparable $O(\varepsilon^c)$ while the loss due to the dissipation term is $\frac{\varepsilon^{c\alpha}}{\varepsilon} = \varepsilon^{c\alpha-1}$. Now as $\alpha > 1$, the inequality

$$c < c\alpha - 1$$

has solutions

$$c > \frac{1}{\alpha - 1};$$

these solutions are acceptable, so we can assure that for any $\alpha > 1$, $\varepsilon > 0$, there exists $I_\varepsilon^* \in A_\varepsilon^*$ such that

$$F(X_0 \cup I_\varepsilon^*) + \frac{\mathcal{H}^1(I_\varepsilon^*)}{\varepsilon} < F(X_0)$$

thus $w(1) \neq X_0$. Moreover, from this argument we see that the new set is added in $(0,0)$, the only to satisfy condition (*) of Proposition 2.5 (thus the only to yield a gain in energy comparable with $O(\mathcal{H}^1(I_\varepsilon^*))$) i.e. the closure of $w(1) \setminus X_0$ contains X_0 , thus there exists a fourth point $x_4 \in w(1) \setminus X_0$ connected to $(0,0)$ by a path g_4 not intersecting X_0 , leading to a cross appearing in $w(1)$. □

3 Topology

In [10] we have presented some sufficient results to force a branching behavior in the quasi static case; here we try to extend them to the dynamic case. Results in [12] concerning the presence of locally stable points (i.e. configurations from which no further evolution happens) can potentially compromise this extension.

We first define *branching*, in the general case:

Definition 3.1. Given a domain $\Omega \subseteq \mathbb{R}^2$, endowed with the Euclidean topology, a functional \mathcal{G} , an initial datum $X_0 \in A$, consider a general irreversible evolution

$$\begin{cases} w(0) := X_0 \\ w(n+1) \in \operatorname{argmin}_{\mathbb{P}} \mathcal{G} \\ w(n) \subseteq w(n+1) \end{cases}$$

with \mathbb{P} denoting some constraint. Then we will say that this evolution exhibits a “branching” at step k if there exists

- two points $R \in w(k) \setminus w(k-1)$, $R' \in w(k-1)$ and a path $\gamma : [0, 1] \rightarrow w(k)$ with $\gamma(0) = R$, $\gamma(1) = R'$, $\gamma([0, 1]) \cap \operatorname{ext}(w(k-1)) = \emptyset$,

or

- three points $R_1, R_2 \in w(k) \setminus w(k-1)$, $R' \in \operatorname{ext}(w(k-1))$ and paths $\gamma_1, \gamma_2 : [0, 1] \rightarrow w(k)$ with $\gamma_i(0) = R_i$ ($i = 1, 2$), $\gamma_1(1) = \gamma_2(1) = R'$, $\gamma_i([0, 1]) \cap w(k-1) = \emptyset$ ($i = 1, 2$), $\gamma_1([0, 1]) \not\subseteq \gamma_2([0, 1])$ and $\gamma_2([0, 1]) \not\subseteq \gamma_1([0, 1])$.

Thus, a branching behavior can be roughly imagined as a point increasing its multiplicity by at least 1 for non endpoints, and by at least 2 for endpoints (although this argument suits well only for points with finite multiplicity).

In [10] we have proven that for quasi static case, under special configurations, if the Euler scheme keeps evolving “far” from a particular point, then a branching behavior is exhibited after a critical time. In other words, the following result (for more details we refer to [10]) holds for the quasi static case:

Proposition 3.2. Given a domain Ω , let $S_0^{(1)} \in A$ be a generic element, T a positive time and $\varepsilon > 0$ a (small) positive time step, let us consider the Euler scheme

$$\begin{cases} w(0) := S_0^{(1)} \\ w(k) \in \operatorname{argmin}_{\mathcal{H}^1(\mathcal{X}') \leq \mathcal{H}^1(S_0^{(1)} + k\varepsilon), w(k-1) \subseteq \mathcal{X}'} F(\mathcal{X}') \end{cases}$$

in the time interval $[0, T]$, and the associated function

$$\Sigma_\varepsilon : [0, T] \rightarrow A, \quad \Sigma_\varepsilon(t) := w \left[\left\lfloor \frac{t}{\varepsilon} \right\rfloor \right].$$

Suppose that there exist a non endpoint $P_0 \in S_0^{(1)}$ verifying condition (*) of Proposition 2.5, and $\eta > 0$ such that $B(P_0, \eta) \cap (w(k) \setminus w(0)) = \emptyset$ for any k . Then there is an upper bound T_{\max}^ε such that $T > T_{\max}^\varepsilon$ forces a branching behavior.

For the quasi static case it turns that this upper bound, while dependent on the time step ε , is $T_{\max}^\varepsilon := \varepsilon + \frac{F(S_0^{(1)})}{K}$ with K depending only on η and geometric quantities near P_0 , thus $T_{\max} := 1 + \frac{F(S_0^{(1)})}{K}$ is a valid upper bound for sufficiently small $\varepsilon > 0$.

In this section we aim to adapt the proof to the dynamic case, as the presence of the dissipation term and the absence of accurate estimates on the length added at each step may cause problems.

Proposition 3.3. *Given a domain Ω , parameters $\alpha > 1, \varepsilon > 0$, let $S_0 \in A$ be a generic element, T a positive time and consider the Euler scheme*

$$\begin{cases} w(0) := S_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{X}' \in A_{\mathcal{H}^1(w(k-1))+1}} F(\mathcal{X}') + D_{\alpha, \varepsilon}(\mathcal{X}', w(k-1)) \end{cases}$$

in the time interval $[0, T]$, and the associated function

$$\Sigma_\varepsilon : [0, T] \longrightarrow A, \quad \Sigma_\varepsilon(t) := w \left[\left(t / \varepsilon^{\frac{1}{\alpha-1}} \right) \right].$$

Suppose that there exist a non endpoint $P_0 \in S_0$ verifying condition (*) of Proposition 2.5, and $\eta > 0$ such that $B(P_0, \eta) \cap (w(k) \setminus w(0)) = \emptyset$ for any k . Then there is an upper bound $T_{\max}^{\alpha, \varepsilon}$ such that $T > T_{\max}^{\alpha, \varepsilon}$ forces a branching behavior.

Notice immediately the difference in time scaling: the choice of the exponent $\frac{1}{\alpha-1}$ will be explained in the proof, and is direct result of loss of control on the length added at each step.

Proof. The proof is partly similar to those found in [10] and [11], as they use similar estimates for the energy functional; some caution must be used when treating the dissipation term.

Recalling Proposition 2.5, we see that existence of such P_0 means that the gain in energy, by adding a suitable set $I \in A_1^*$ in P_0 in a suitable manner, is comparable with $O(\mathcal{H}^1(I))$, with a lower bound constant depending only on geometric quantities (for more details see the proof of Proposition 2.5, found in [10]). With this method, the dissipation term is $\frac{\mathcal{H}^1(I)^\alpha}{\varepsilon}$, and as $\mathcal{H}^1(I) = \varepsilon^c$ for some suitable c , we can always have $\varepsilon^c > \varepsilon^{c\alpha-1}$, thus the gain in energy is not offset by the dissipation term.

That is, considering

$$\begin{cases} w(0) := S_0 \\ w(k) \in \operatorname{argmin}_{\mathcal{X}' \in A_{\mathcal{H}^1(w(k-1))+1}} F(\mathcal{X}') + D_{\alpha, \varepsilon}(\mathcal{X}', w(k-1)) \end{cases},$$

for the first step we have

$$F(S_0) - F(w(1)) \geq K^* \varepsilon^{c_{opt}},$$

where K^* is a constant dependent only on geometric quantities, and c_{opt} is such that $\mathcal{H}^1(w(1) \setminus S_0) = \varepsilon^{c_{opt}}$.

Adding sets in P_0 , which is a non endpoint, will cause a branching behavior.

We analyze what happens at subsequent steps. By hypothesis we have there exists $\eta > 0$ such that $B(P_0, \eta) \cap (w(j) \setminus w(0)) = \emptyset$ for any j , and P_0 verifies

- there exists $\rho > 0$ and $\theta < \pi$ such that $B(P_0, \rho) \cap S_0$ is entirely contained in the circular sector (of $B(P_0, \rho)$) with central angle measuring θ .

As these facts combines imply $w(j) \cap B(P_0, \eta) = S_0 \cap B(P_0, \eta)$, for any step the gain in energy by adding correctly a segment $J^* \in A_1^*$ in P_0 will generate a gain in energy at least $K^* \mathcal{H}^1(J^*)$, with K^* a uniform constant valid independently from the step. This exists because P_0 has a positive measure $V(P_0)$ ($|V(P_0)|$ is bounded from below by a constant not dependent from the step, as it contains at least a triangle $Trg \subseteq V(P_0) \cap B(P_0, \eta/3)$), and $w(j) \cap B(P_0, \eta) = S_0 \cap B(P_0, \eta)$ for any j : in S_0 an arbitrary point $y \in Trg$ will satisfy $\text{dist}(y, P_0) \leq \eta/3$ (as $Trg \subseteq B(P_0, \eta/3)$), in $w(j)$ the same point y will satisfy $\text{dist}(y, S_0) = \text{dist}(y, w(j))$ as the opposite, i.e. $\text{dist}(y, w(j)) < \text{dist}(y, S_0) \leq \text{dist}(y, P_0) \leq \eta/3$, forces the existence of a point $z \in w(j) \setminus S_0$ such that $\text{dist}(y, z) = \text{dist}(y, w(j))$, which ultimately leads to $z \in (w(j) \setminus S_0) \cap B(P_0, \eta)$ and contradicts $w(j) \cap B(P_0, \eta) = S_0 \cap B(P_0, \eta)$.

Some estimate on c_{opt} is required in order to estimate the branching time. We have proven that adding a correct segment $J^* \in A_1^*$ in P_0 (which causes a branching) generates a gain for the energy at least $K^* \mathcal{H}^1(J^*)$, and this gain is obviously not greater than $|\Omega| \mathcal{H}^1(J^*)^\alpha$ (from Lemma 2.2; the dissipation term is $\frac{\mathcal{H}^1(J^*)^\alpha}{\varepsilon}$).

Putting $\mathcal{H}^1(J^*) := \varepsilon^c$, the gain for the energy is $d\varepsilon^c$ where d (may depend on several variables, included ε and which particular step we are considering) is a value in $[K^*, |\Omega|]$, and the dissipation is $\varepsilon^{c\alpha-1}$. Maximizing the expression

$$d\varepsilon^c - \varepsilon^{c\alpha-1}$$

the optimal choice is

$$c = \frac{\log \varepsilon + \log(d/\alpha)}{(\alpha - 1) \log \varepsilon}.$$

Thus the optimal exponent c_{opt} is found in the interval

$$\left[\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha - 1) \log \varepsilon}, \frac{\log \varepsilon + \log \frac{|\Omega|}{\alpha}}{(\alpha - 1) \log \varepsilon} \right],$$

and little information is available for the exact value for the optimal exponent c_{opt} apart from the obvious

$$\lim_{\varepsilon \rightarrow 0^+} c_{opt} = \frac{1}{\alpha - 1}.$$

As we are considering small step discrete evolutions, we set the associated function as

$$\Sigma_\varepsilon : [0, T] \longrightarrow A, \quad \Sigma_\varepsilon(t) := w\left(\left\lceil t/\varepsilon^{\frac{1}{\alpha-1}} \right\rceil\right).$$

Now return to the branching time, and consider the first step. As it is possible to add $J^* \in A_1^*$ in P_0 and obtain a gain for the energy at least $K^* \mathcal{H}^1(J^*)$, then $w(1) \neq S_0$ and some

set $J' \in A_1^*$ is added. Recall that it is possible to add a segment Seg' with $\mathcal{H}^1(Seg') = \mathcal{H}^1(J')$ in P_0 (and causing a branching behavior) and having

$$F(S_0 \cup Seg') + \frac{\mathcal{H}^1(Seg')^\alpha}{\varepsilon} < F(S_0).$$

If no branching appears, i.e. J' is added elsewhere, it is forced that adding Seg' in P_0 is either not optimal or not the only optimal choice, thus

$$F(S_0 \cup J') + \frac{\mathcal{H}^1(J')^\alpha}{\varepsilon} \leq F(S_0 \cup Seg') + \frac{\mathcal{H}^1(Seg')^\alpha}{\varepsilon} < F(S_0),$$

and considering $\mathcal{H}^1(Seg') = \mathcal{H}^1(J')$,

$$F(S_0) - F(S_0 \cup J') \geq F(S_0) - F(S_0 \cup Seg') \geq K^* \mathcal{H}^1(J').$$

As $J' = \varepsilon^c$ for some suitable c , and considering the estimates above on the optimal exponent,

$$F(S_0) - F(S_0 \cup J') \geq K^* \mathcal{H}^1(J') \geq K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}}.$$

This argument can be generalized to every step, thus at step h , if no branching behavior has occurred, this implies

$$F(w(l)) - F(w(l+1)) \geq K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}}$$

for any $l = 0, \dots, h-1$, thus

$$F(S_0) - F(w(h)) \geq h K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}},$$

and considering that F takes value in $(0, \infty)$,

$$F(S_0) \geq h K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}}$$

is required, thus

$$h \leq \frac{F(S_0)}{K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}}}.$$

For the associated function, as $\Sigma_\varepsilon(t) := w\left(\left\lceil t/\varepsilon^{\frac{1}{\alpha-1}} \right\rceil\right)$, the above estimate leads to

$$\left\lceil \frac{t}{\varepsilon^{\frac{1}{\alpha-1}}} \right\rceil \leq \frac{F(S_0)}{K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}}},$$

and considering $\frac{t}{\varepsilon^{\frac{1}{\alpha-1}}} - 1 \leq \left\lceil \frac{t}{\varepsilon^{\frac{1}{\alpha-1}}} \right\rceil \leq \frac{t}{\varepsilon^{\frac{1}{\alpha-1}}}$,

$$\frac{t}{\varepsilon^{\frac{1}{\alpha-1}}} \leq 1 + \frac{F(S_0)}{K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}}},$$

and ultimately

$$t \leq \varepsilon^{\frac{1}{\alpha-1}} \left(1 + \frac{F(S_0)}{K^* \varepsilon^{\frac{\log \varepsilon + \log \frac{K^*}{\alpha}}{(\alpha-1) \log \varepsilon}}} \right) = \varepsilon^{\frac{1}{\alpha-1}} + \frac{F(S_0)}{K^*} \varepsilon^{\frac{\log \alpha - \log K^*}{(\alpha-1) \log \varepsilon}}.$$

This finally gives an upper bound estimate for the branching time

$$T_{\max}^{\alpha, \varepsilon} = \varepsilon^{\frac{1}{\alpha-1}} + \frac{F(S_0)}{K^*} \varepsilon^{\frac{\log \alpha - \log K^*}{(\alpha-1) \log \varepsilon}},$$

and the proof is complete. \square

Now analyze $T_{\max}^{\alpha, \varepsilon}$: for a given $\alpha > 1$, as

$$\lim_{\varepsilon \rightarrow 0^+} T_{\max}^{\alpha, \varepsilon} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha-1}} + \frac{F(S_0)}{K^*} \varepsilon^{\frac{\log \alpha - \log K^*}{(\alpha-1) \log \varepsilon}} = \frac{F(S_0)}{K^*},$$

the estimate

$$T_{\max} := \frac{F(S_0)}{K^*} + 1$$

is valid for any ε sufficiently small. Notice that parameter α does not play a very relevant role: indeed its main role is in the exponent $\frac{1}{\alpha-1}$ in the time scaling.

References

- [1] L. Ambrosio, Minimizing movements. *Rend. Accad. Naz. Sci. XL, Mem. Mat. Appl.* **19** (1995), pp. 191-246.
- [2] L. Ambrosio, N. Gigli and G. Savarè, Gradient flows in metric spaces and in the space of probability measures. Second edition.. *Lectures in Mathematics ETH Zürich*, Birkhäuser Verlag, Basel, 2008.
- [3] D. Bucur and G. Buttazzo, Irreversible quasistatic evolutions by minimizing movements. *J. Conv. Anal.*, **15(3)** (2008), pp. 523-534.
- [4] D. Bucur, G. Buttazzo and A. Lux, Quasistatic evolution in debonding problems via capacity methods. *Arch. Rational Mech. Anal.*, **190** (2008), pp. 281–306.
- [5] D. Bucur, G. Buttazzo and P. Trebeschi, An existence result for optimal obstacles. *J. Funct. Anal.*, **162(1)** (1999), pp. 96-119.
- [6] G. Buttazzo, E. Oudet, and E. Stepanov, Optimal transportation problems with free Dirichlet regions. *Progress in Nonlinear Diff. Equations and their Applications* **51** (2002), pp. 41-65.
- [7] G. Buttazzo and E. Stepanov, Minimization problems for average distance functionals. *Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi*, Ed. D. Pallara, Quaderni di Matematica, Seconda Università di Napoli, Caserta **14** (2004), pp. 47-83.

- [8] G. Buttazzo and E. Stepanov, Optimal transportation networks as free Dirichlet regions for the Monge-Kantorovich problem, *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* **2** (2003), pp. 631-678.
- [9] A. Lemenant, A presentation of the average distance minimizing problem. Review paper on CVGMT.
- [10] X.Y. Lu, Branching time estimates in quasi static evolution for the average distance functional. To appear in *Commun. Appl. Anal.*.
- [11] X.Y. Lu, Geometric properties for optimal sets and upper bound branching time under coercive L^p measures. Preprint on CVGMT.
- [12] X.Y. Lu, Locally stable configurations in irreversible dynamic evolution. Preprint on CVGMT.