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ON THE EXISTENCE OF SOLUTIONS FOR NONCONVEX FRACTIONAL HYPERBOLIC DIFFERENTIAL INCLUSIONS

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Abstract

We establish some Filippov type existence theorems for solutions of certain nonconvex fractional hyperbolic differential inclusions involving Caputo's fractional derivative.

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1 Introduction

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([4, 16, 17, 20] etc.). The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([12]). Very recently several qualitative results for fractional differential inclusions were obtained in [3, 5, 9, 10, 14, 15, 18] etc.. Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced in [3] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks and examples we refer to [3]. At the same time, since fractional differential inclusions represent a special class of integral inclusions, other related results may be found in [19].

In this paper we study fractional hyperbolic differential inclusions of the form

$$D_c^r u(x,y) \in F(x,y,u(x,y)) \quad a.e.\ (x,y) \in \Pi, \tag{1.1}$$

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$$u(x,0)=\varphi(x), \quad u(0,y)=\psi(y) \quad (x,y)\in\Pi, \tag{1.2}$$

where $\Pi=[0,T_1]\times[0,T_2]$, $\varphi(.):[0,T_1]\to\mathbb{R}^n$, $\psi(.):[0,T_2]\to\mathbb{R}^n$ with $\varphi(0)=\psi(0)$, $F(.,.):\Pi\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is a set-valued map and D_c^r is the Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$.

Very recently in [1,2] problem (1.1)-(1.2) with $F(.,.)$ single valued is studied and several existence results are obtained using fixed point techniques.

The aim of the present paper is twofold. On one hand, we show that Filippov's ideas ([13]) can be suitably adapted in order to obtain the existence of a solution of problem (1.1)-(1.2). We recall that for a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([13]) consists in proving the existence of o solution starting from a given "almost" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. On the other hand, we prove the existence of solutions continuously depending on a parameter for problem (1.1)-(1.2). This result may be interpreted as a continuous variant of Filippov's theorem for problem (1.1)-(1.2). The key tool in the proof of this theorem is a result of Bressan and Colombo ([7]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. This result allows to obtain a continuous selection of the solution set of the problem considered.

Our results may be interpreted as extensions of previous results of Staicu ([22]) and Tuan ([23,24]) obtained for "classical" hyperbolic differential inclusions. In fact, in the proof of our theorems we essentially use several technical results due to Staicu ([22]) and Tuan ([23,24]).

The paper is organized as follows: in Section 2 we briefly recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by $d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\}$, $d^*(A,B)=\sup\{d(a,B);a\in A\}$, where $d(x,B)=\inf\{d(x,y);y\in B\}$. With $cl(A)$ we denote the closure of the set $A\subset X$.

Consider $I_1=[0,T_1]$, $I_2=[0,T_2]$ and $\Pi=[0,T_1]\times[0,T_2]$. Denote by $\mathcal{L}(\Pi)$ the σ- algebra of the Lebesgue measurable subsets of Π and by $\mathcal{B}(\mathbb{R}^n)$ the family of all Borel subsets of $\mathbb{R}^n$.

Let $C(\Pi,\mathbb{R}^n)$ be the Banach space of all continuous functions from Π to $\mathbb{R}^n$ with the norm $||u||_C=\sup\{\|u(x,y)\|;(x,y)\in\Pi\}$ where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, and $L^1(\Pi,\mathbb{R}^n)$ be the Banach space of functions $u(\cdot,\cdot):\Pi\to\mathbb{R}^n$ which are integrable, normed by $\|u\|_{L^1}=\int_0^{T_1}\int_0^{T_2}\|u(x,y)\|dxdy$.

Recall that a subset $D\subset L^1(\Pi,\mathbb{R}^n)$ is said to be *decomposable* if for any $u(\cdot),v(\cdot)\in D$ and any subset $A\in\mathcal{L}(\Pi)$ one has $u\chi_A+v\chi_B\in D$, where $B=I\backslash A$. We denote by $\mathcal{D}$ the family of all decomposable closed subsets of $L^1(\Pi,\mathbb{R}^n)$.

Let $F(.,.):\Pi\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ be a set-valued map. Recall that $F(.,.)$ is called $\mathcal{L}(\Pi)\otimes\mathcal{B}(\mathbb{R}^n)$ measurable if for any closed subset $C\subset\mathbb{R}^n$ we have $\{(x,y,z)\in\Pi\times\mathbb{R}^n;F(x,y,z)\cap$

$C\} \neq \emptyset\} \in \mathcal{L}(\Pi) \otimes \mathcal{B}(\mathbb{R}^n)$.

Definition 2.1. ([21]) a) The left-sided mixed Riemann-Liouville integral of order $r = (r_1, r_2) \in (0,1] \times (0,1]$ of $f(.,.) \in L^1(\Pi, \mathbb{R}^n)$ is defined by

$$(I_0^r f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t) ds dt,$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t$, $\alpha > 0$.

b) The Caputo fractional-order derivative of order r of $f(.,.) \in L^1(\Pi, \mathbb{R}^n)$ is defined by

$$(D_c^r f)(x,y) = (I_0^{1-r} \frac{\partial^2 f}{\partial x \partial y})(x,y).$$

In the definition above by $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1] \times (0,1]$.

Definition 2.2. A function $u(.,.) \in C(\Pi, \mathbb{R}^n)$ is said to be a solution of problem (1.1)-(1.2) if there exists $f(.,.) \in L^1(\Pi, \mathbb{R}^n)$ such that

$$f(x,y) \in F(x,y,u(x,y)) \quad a.e.\ (\Pi), \tag{2.1}$$

$$D_c^r u(x,y) = f(x,y,u(x,y)) \quad (x,y) \in \Pi, \tag{2.2}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y) \quad (x,y) \in \Pi, \tag{2.3}$$

The pair $(u(.,.), f(.,.)$ is called a *trajectory-selection* pair of problem (1.1)-(1.2).

Lemma 2.3. *([1]) $u(.,.) \in C(\Pi, \mathbb{R}^n)$ is a solution of problem (2.2)-(2.3) if and only if $u(.,.)$ satisfies*

$$u(x,y) = \mu(x,y) + (I_0^r f)(x,y), \quad (x,y) \in \Pi,$$

where $\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0)$.

Consider the Banach space $\mathbb{S} := \{(\varphi, \psi) \in C(I_1, \mathbb{R}^n) \times C(I_2, \mathbb{R}^n); \varphi(0) = \psi(0)\}$ endowed with the norm $||(\varphi, \psi)|| = ||\varphi||_C + ||\psi||_C$ and for $(\varphi, \psi) \in \mathbb{S}$ denote $\mathcal{S}(\varphi, \psi)$ the set of all solutions of problem (1.1)-(1.2).

We recall now some results that we are going to use in the next section.

Lemma 2.4. *([23]) Let $H(\cdot,\cdot) : \Pi \to \mathcal{P}(\mathbb{R}^n)$ be a compact valued measurable multifunction and $v(\cdot,\cdot) : \Pi \to \mathbb{R}^n$ a measurable function.*

Then there exists a measurable selection $h(\cdot,\cdot)$ of $H(\cdot,\cdot)$ such that

$$\|v(x,y) - h(x,y)\| = d(v(x,y), H(x,y)), \quad a.e.\ (\Pi).$$

Next (S, d) is a separable metric space and X is a Banach space. We recall that a multifunction $G(\cdot) : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed in S.

Lemma 2.5. *([22]) Let $F^*(.,.) : \Pi \times S \to \mathcal{P}(\mathbb{R}^n)$ be a closed valued $\mathcal{L}(\Pi) \otimes \mathcal{B}(S)$ measurable multifunction such that $F^*((x,y),.)$ is l.s.c. for any $(x,y) \in \Pi$.*

Then the set-valued map $G(.)$ defined by

$$G(s) = \{v \in L^1(\Pi, \mathbb{R}^n); \quad v(x,y) \in F^*(x,y,s) \quad a.e.\ (\Pi)\}$$

is l.s.c. with nonempty decomposable closed values if and only if there exists a continuous mapping $p(.) : S \to L^1(\Pi, \mathbb{R}^n)$ such that

$$d(0, F^*(x,y,s)) \le p(s)(x,y) \quad a.e.\ (\Pi),\ \forall s \in S.$$

Lemma 2.6. *([22]) Let $G(.) : S \to \mathcal{D}$ be a l.s.c. set-valued map with closed decomposable values and let $f(.) : S \to L^1(\Pi, \mathbb{R}^n)$, $q(.) : S \to L^1(\Pi, \mathbb{R})$ be continuous such that the multifunction $H(.) : S \to \mathcal{D}$ defined by*

$$H(s) = cl\{v(.) \in G(s); \quad ||v(x,y) - f(s)(x,y)|| < q(s)(x,y) \quad a.e.\ (\Pi)\}$$

has nonempty values.

Then $H(.)$ has a continuous selection, i.e. there exists a continuous mapping $h(.) : S \to L^1(\Pi, \mathbb{R}^n)$ such that$h(s) \in H(s) \quad \forall s \in S$.

3 The main results

In order to obtain a Filippov type existence result for problem (1.1)-(1.2) one need the following assumptions on $F(.,.)$.

Hypothesis 3.1. *$F(.,.) : \Pi \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map with non-empty, compact values that verifies:*
i) For all $u \in \mathbb{R}^n$, $F(.,.,u)$ is measurable.
ii) There exists $l(.,.) \in L^1(\Pi, \mathbb{R}_+)$ such that there exists $L := \sup_{(x,y)\in\Pi}(I_0^r l)(x,y)$, $L < 1$ and for almost all $(x,y) \in \Pi$, $F(x,y,\cdot)$ is $l(x,y)$ - Lipschitz in the sense that

$$d_H(F(x,y,u_1), F(x,y,u_2)) \le l(x,y)\|u_1 - u_2\|, \quad \forall u_1, u_2 \in \mathbb{R}^n.$$

In what follows $g(.,.) \in L^1(\Pi, \mathbb{R}^n)$ is given such that there exists $\lambda(.,.) \in L^1(\Pi, \mathbb{R}_+)$ with $\Lambda := \sup_{(x,y)\in\Pi}(I_0^r \lambda)(x,y) < +\infty$ which satisfies

$$d(g(x,y), F(x,y,w(x,y))) \le \lambda(x,y) \quad a.e.\ (\Pi),$$

where $w(.,.)$ is a solution of the fractional hyperbolic differential equation

$$D_c^r w(x,y) = g(x,y) \quad (x,y) \in \Pi, \tag{3.1}$$

$$w(x,0) = \varphi_1(x), \quad w(0,y) = \psi_1(y) \quad (x,y) \in \Pi, \tag{3.1}$$

with $(\varphi_1, \psi_1) \in \mathbb{S}$. Set $\mu_1(x,y) = \varphi_1(x) + \psi_1(y) - \varphi_1(0)$, $(x,y) \in \Pi$.

Theorem 3.2. *Let Hypothesis 3.1 be satisfied and consider $g(.,.)$, $\lambda(.,.)$, $w(.,.)$ as above, $(\varphi,\psi)\in\mathbb{S}$ and $\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0)$, $(x,y)\in\Pi$.*

Then there exists $(u(.,.),f(.,.))$ a trajectory-selection pair of problem (1.1)-(1.2) such that

$$||u(x,y)-w(x,y)||\leq\frac{||\mu-\mu_1||_C+\Lambda}{1-L},\quad\forall(x,y)\in\Pi,\tag{3.3}$$

$$||f(x,y)-g(x,y)||\leq\frac{l(x,y)(||\mu-\mu_1||_C+\Lambda)}{1-L}+\lambda(x,y),\quad a.e.\ (\Pi).\tag{3.4}$$

Proof. We define $f_0(.,.)=g(.,.)$, $u_0(.,.)=w(.,.)$. It follows from Lemma 2.4 and Hypothesis 3.1 that there exists a measurable function $f_1(.,.)$ such that $f_1(x,y)\in F(x,y,u_0(x,y))$ a.e. (Π) and for almost all $(x,y)\in\Pi$

$$\|f_0(x,y)-f_1(x,y)\|=d(g(x,y),F(x,y,u_0(x,y)))\leq\lambda(x,y).$$

Define, for $(x,y)\in\Pi$

$$u_1(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f_1(s,t)dsdt.$$

Since

$$w(x,y)=\mu_1(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f_0(s,t)dsdt$$

one has

$$||u_1(x,y)-u_0(x,y)||\leq||\mu(x,y)-\mu_1(x,y)||+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}.$$

$$.(y-t)^{r_2-1}||f_1(s,t)-f_0(s,t)||dsdt\leq||\mu-\mu_1||_C+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}.$$

$$.(y-t)^{r_2-1}\lambda(s,t)dsdt\leq||\mu-\mu_1||_C+\Lambda.$$

From Lemma 2.4 and Hypothesis 3.1 we deduce the existence of a measurable function $f_2(.,.)$ such that $f_2(x,y)\in F(x,y,u_1(x,y))$ a.e. (Π) and for almost all $(x,y)\in\Pi$

$$\|f_2(x,y)-f_1(x,y)\|\leq d(f_1(x,y),F(x,y,u_1(x,y)))\leq d_H(F(x,y,u_0(x,y)),$$

$$F(x,y,u_1(x,y)))\leq l(x,y)||u_1(x,y)-u_0(x,y)||\leq l(x,y)(||\mu-\mu_1||_C+\Lambda).$$

Define, for $(x,y)\in\Pi$

$$u_2(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f_2(s,t)dsdt$$

and one has

$$||u_2(x,y)-u_1(x,y)||\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}||f_2(s,t)-$$

$$-f_1(s,t)||dsdt \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l(s,t)(||\mu-\mu_1||_C+$$
$$+\Lambda)dsdt \le L(||\mu-\mu_1||_C+\Lambda).$$

Assuming that for some $p \ge 2$ we have already constructed the sequences $(u_i(.,.))_{i=1}^p$, $(f_i(.,.))_{i=1}^p$ satisfying

$$\|u_p(x,y)-u_{p-1}(x,y)\| \le L^{p-1}(||\mu-\mu_1||_C+\Lambda) \quad (x,y)\in \Pi, \tag{3.5}$$

$$\|f_p(x,y)-f_{p-1}(x,y)\| \le l(x,y)L^{p-2}(||\mu-\mu_1||_C+\Lambda) \quad a.e. \ (\Pi). \tag{3.6}$$

We apply Lemma 2.4 and we find a measurable function $f_{p+1}(.,.)$ such that $f_{p+1}(x,y) \in F(x,y,u_p(x,y))$ a.e. (Π) and for almost all $(x,y)\in \Pi$

$$\|f_{p+1}(x,y)-f_p(x,y)\| \le d(f_{p+1}(x,y),F(x,y,u_p(x,y))) \le d_H(F(x,y,u_p(x,y)),$$

$$F(x,y,u_{p-1}(x,y))) \le l(x,y)||u_p(x,y)-u_{p-1}(x,y)|| \le l(x,y)L^{p-1}(||\mu-\mu_1||_C+\Lambda).$$

Define, for $(x,y)\in \Pi$

$$u_{p+1}(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f_{p+1}(s,t)dsdt. \tag{3.7}$$

We have

$$||u_{p+1}(x,y)-u_p(x,y)|| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}||f_{p+1}(s,t)-$$

$$-f_p(s,t)||dsdt \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l(s,t)||u_p(s,t)-$$

$$-u_{p-1}(s,t)||dsdt \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l(s,t)L^{p-1}(||\mu-$$

$$-\mu_1||_C+\Lambda)dsdt \le L^p(||\mu-\mu_1||_C+\Lambda).$$

Therefore from (3.5) it follows that the sequence $(u_p(.,.))_{p\ge 0}$ is a Cauchy sequence in the space $C(\Pi,\mathbb{R}^n)$, so it converges to $u(.,.)\in C(\Pi,\mathbb{R}^n)$. From (3.6) it follows that the sequence $(f_p(.,.))_{p\ge 0}$ is a Cauchy sequence in the space $L^1(\Pi,\mathbb{R}^n)$, thus it converges to $f(.,.)\in L^1(\Pi,\mathbb{R}^n)$.

Using the fact that the values of $F(.,.)$ are closed we get that $f(x,y)\in F(x,y,u(x,y))$ a.e. (Π).

One may write successively,

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}|| \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f_p(s,t)dsdt - \int_0^x \int_0^y (x-s)^{r_1-1}.$$

$$.(y-t)^{r_2-1} f(s,t)dsdt|| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}||f_p(s,t)-$$

$$-f(s,t)||dsdt \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l(s,t)||u_{p-1}(s,t)-$$

$$-u(s,t)||dsdt \leq L||u_{p-1}(.,.) - u(.,.)||_C.$$

Therefore, we may pass to the limit in (3.2) and we obtain, via Lemma 2.3, that $u(.,.)$ is a solution of problem (1.1)-(1.2). On the other hand, by adding inequalities (3.5) for any $(x,y) \in \Pi$ we have

$$\begin{array}{l} ||u_p(x,y) - w(x,y)|| \leq ||u_p(x,y) - u_{p-1}(x,y)|| + ||u_{p-1}(x,y) - u_{p-2}(x,y)|| \\ + \ldots + ||u_2(x,y) - u_1(x,y)|| + ||u_1(x,y) - u_0(x,y)|| \leq \\ \leq (L^{p-1} + L^{p-2} + \ldots + L + 1)(||\mu - \mu_1||_C + \Lambda) \leq \frac{||\mu-\mu_1||_C+\Lambda}{1-L}. \end{array} \tag{3.8}$$

Similarly, by adding inequalities (3.6) for almost all $(x,y) \in \Pi$ we have

$$\begin{array}{l} ||f_p(x,y) - g(x,y)|| \leq ||f_p(x,y) - f_{p-1}(x,y)|| + ||f_{p-1}(x,y) - f_{p-2}(x,y)|| \\ + \ldots + ||f_2(x,y) - f_1(x,y)|| + ||f_1(x,y) - f_0(x,y)|| \leq l(x,y)(L^{p-2} + \ldots + \\ + L + 1)(||\mu - \mu_1||_C + \Lambda) + \lambda(x,y) \leq l(x,y)\frac{||\mu-\mu_1||_C+\Lambda}{1-L} + \lambda(x,y). \end{array} \tag{3.9}$$

It remains to pass to the limit with $p \to \infty$ in (3.8) and (3.9) in order to obtain (3.3) and (3.4), respectively and the proof is complete. □

If in Theorem 3.2 we take $g = 0$, $w = 0$, $\varphi_1 = 0$, $\psi_1 = 0$ and $\lambda = l$ then we obtain the following existence result for solutions of problem (1.1)-(1.2).

Corollary 3.3. *Let Hypothesis 3.1 be satisfied and assume that* $d(0, F(x,y,0)) \leq l(x,y)$ $\forall (x,y) \in \Pi$.

Then there exists $u(.,.) \in C(\Pi, \mathbb{R}^n)$ *a solution of problem (1.1)-(1.2) such that*

$$||u(x,y)|| \leq \frac{||\mu||_C + L}{1 - L}, \quad \forall (x,y) \in \Pi.$$

We note that the proof of Corollary 3.3 can be performed also by using the Covitz-Nadler set-valued contraction principle.

Example 3.4. Consider the following problem which is a slight modification of an example in [1]

$$D_c^r u(x,y) = \frac{1}{3e^{x+y+2}(1 + |u(x,y)|)} \quad a.e.\ (x,y) \in [0,1] \times [0,1],$$

$$u(x,0) = x, \quad u(0,y) = y^2 \quad (x,y) \in [0,1] \times [0,1].$$

In this case $\varphi(x) = x$, $\psi(y) = y^2$, $F(x,y,u) = \{\frac{1}{3e^{x+y+2}(1+|u|)}\}$, $T_1 = T_2 = 1$. A straightforward computation shows that $l(x,y) \equiv \frac{1}{3e^2}$, $L = \sup_{(x,y)\in\Pi}(I_0^r l)(x,y) = \frac{1}{3e^2\Gamma(r_1+1)\Gamma(r_2+1)} < 1$ if $r_1, r_2 \in (0,1]$ and $d(0, F(x,y,0)) = \frac{1}{3e^{x+y+2}} \leq \frac{1}{3e^2}$.

Therefore, we can apply Corollary 3.3 and we obtain the existence of a solution which satisfies

$$||u(x,y)|| \leq \frac{6e^2\Gamma(r_1+1)\Gamma(r_2+1) + 1}{3e^2\Gamma(r_1+1)\Gamma(r_2+1) - 1}, \quad \forall (x,y) \in [0,1] \times [0,1].$$

Next we obtain a continuous version of Theorem 3.1. This result allows to provide a continuous selection of the solution set of problem (1.1)-(1.2).

Hypothesis 3.5. *$F(.,.) : \Pi \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ has nonempty compact values, $F(.,.)$ is $\mathcal{L}(\Pi) \otimes \mathcal{B}(\mathbb{R}^n)$ measurable and there exists $l(.,.) \in L^1(\Pi, \mathbb{R}_+)$ such that there exists $L := \sup_{(x,y)\in\Pi} (I_0^r l)(x,y)$, $L < 1$ and for almost all $(x,y) \in \Pi$, $F(x,y,\cdot)$ is $l(x,y)$ - Lipschitz.*

Hypothesis 3.6. *i) S is a separable metric space, $\varphi(.) \to C(I_1,\mathbb{R}^n), \psi(.) : S \to C(I_2,\mathbb{R}^n)$ and $\varepsilon(.) : S \to (0,\infty)$ are continuous mappings.*

ii) There exists the continuous mappings $\varphi_1(.) \to C(I_1,\mathbb{R}^n), \psi_1(.) : S \to C(I_2,\mathbb{R}^n)$ $g(.) : S \to L^1(\Pi,\mathbb{R}^n)$, $\lambda(.) : S \to L^1(\Pi,\mathbb{R})$ and $w(.) : S \to C(\Pi,\mathbb{R}^n)$ such that

$$(Dw(s))_c^r(x,y) = g(s)(x,y) \quad a.e.\ (\Pi), \quad \forall s \in S,$$

$$w(s)(x,0) = \varphi_1(s)(x), \quad w(s)(0,y) = \psi_1(s)(y) \quad (x,y) \in \Pi, \quad \forall s \in S,$$

$$d(g(s)(x,y), F(x,y,w(s)(x,y))) \le \lambda(s)(x,y) \quad a.e.\ (\Pi),\ \forall\, s \in S$$

and the mapping $s \to \Lambda(s) := \sup_{(x,y)\in\Pi}(I_0^r\lambda(s))(x,y)$ is continuous.

We use next the following notations $\mu(s)(x,y) = \varphi(s)(x) + \psi(s)(y) - \varphi(s)(0)$, $\mu_1(s)(x,y) = \varphi_1(s)(x) + \psi_1(s)(y) - \varphi_1(s)(0)$ $(x,y) \in \Pi$, $a(s) = \sup_{(x,y)\in\Pi} ||\mu(s)(x,y) - \mu_1(s)(x,y)||$ $s \in S$.

Theorem 3.7. *Assume that Hypotheses 3.5 and 3.6 are satisfied.*

Then there exist a continuous mapping $u(.) : S \to C(\Pi,\mathbb{R}^n)$ such that for any $s \in S$, $u(s)(.,.)$ is a solution of problem (1.1) which satisfies $u(s)(x,0) = \varphi(s)(x)$, $u(s)(0,y) = \psi(s)(y)$ $(x,y) \in \Pi, s \in S$ and

$$||u(s)(x,y) - w(s)(x,y)|| \le \frac{a(s) + \varepsilon(s) + \Lambda(s)}{1-L} \quad \forall (x,y) \in \Pi, \forall s \in S.$$

Proof. We make the following notations $u_0(.,.) = w(.,.)$, $\lambda_p(s) := L^{p-1}(a(s) + \varepsilon(s) + \Lambda(s))$, $p \ge 1$.

We consider the set-valued maps $G_0(.), H_0(.)$ defined, respectively, by

$$G_0(s) = \{v \in L^1(\Pi,\mathbb{R}^n); \quad v(x,y) \in F(x,y,w(s)(x,y)) \quad a.e.\ (\Pi)\},$$

$$H_0(s) = cl\{v \in G_0(s); ||v(x,y) - g(s)(x,y)|| < \lambda(s)(x,y) + \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{T_1^{r_1}T_2^{r_2}}\varepsilon(s)\}.$$

Since $d(g(s)(x,y), F(x,y,w(s)(x,y))) \le \lambda(s)(x,y) < \lambda(s)(x,y) + \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{T_1^{r_1}T_2^{r_2}}\varepsilon(s)$ the set $H_0(s)$ is not empty.

Set $F_0^*(x,y,s) = F(x,y,w(s)(x,y))$ and note that

$$d(0, F_0^*(x,y,s)) \le ||g(s)(x,y)|| + \lambda(s)(x,y) =: \lambda^*(s)(x,y)$$

and $\lambda^*(.) : S \to L^1(I,\mathbb{R})$ is continuous.

Applying now Lemmas 2.5 and 2.6 we obtain the existence of a continuous selection f_0 of H_0 such that $\forall s \in S,\ (x,y) \in \Pi$,

$$f_0(s)(x,y) \in F(x,y,w(s)(x,y)) \quad a.e.\ (\Pi),\ \forall s \in S,$$

$$||f_0(s)(x,y) - g(s)(x,y)|| \leq \lambda_0(s)(x,y) = \lambda(s)(x,y) + \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{T_1^{r_1}T_2^{r_2}}\varepsilon(s).$$

We define

$$u_1(s)(x,y) = \mu(s)(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1} f_0(s)(z,t)dzdt.$$

and one has

$$||u_1(s)(x,y) - u_0(s)(x,y)|| \leq a(s) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-z)^{r_1-1}.$$

$$.(y-t)^{r_2-1}||f_0(s)(z,t) - g(s)(z,t)||dzdt \leq a(s) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}$$

$$\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1}[\lambda(s)(z,t) + \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{T_1^{r_1}T_2^{r_2}}\varepsilon(s)]dzdt \leq$$

$$\leq a(s) + \Lambda(s) + \varepsilon(s) =: \lambda_1(s), \quad (x,y) \in \Pi, s \in S.$$

We shall construct, using the same idea as in [11], two sequences of approximations $f_p(.,.) : S \to L^1(\Pi,\mathbb{R}^n)$, $u_p(.,.) : S \to C(\Pi,\mathbb{R}^n)$ with the following properties

a) $f_p(.,.) : S \to L^1(\Pi,\mathbb{R}^n)$, $u_p(.,.) : S \to C(\Pi,\mathbb{R}^n)$ are continuous,

b) $f_p(s)(x,y) \in F(x,y,u_p(s)(x,y))$, a.e. (Π), $s \in S$,

c) $||f_p(s)(x,y) - f_{p-1}(s)(x,y)|| \leq l(x,y)\lambda_p(s)$, a.e. (Π), $s \in S$.

d) $u_{p+1}(s)(x,y) = \mu(s)(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1} f_p(s)(z,t)dzdt$, $(x,y) \in \Pi, s \in S$.

Suppose we have already constructed $f_i(.), u_i(.)$ satisfying a)-c) and define $u_{p+1}(.)$ as in d). From c) and d) one has

$$\begin{aligned}&||u_{p+1}(s)(x,y) - u_p(s)(x,y)|| \leq \tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\textstyle\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1}\\ &||f_p(s)(z,t) - f_{p-1}(s)(z,t)||dzdt \leq \tfrac{1}{\Gamma(r_1)\Gamma(r_2)}\textstyle\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1}.\\ &.l(z,t)\lambda_p(s)dzdt < L\lambda_p(s) =: \lambda_{p+1}(s).\end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned}&d(f_p(s)(x,y), F(x,y,u_{p+1}(s)(x,y)) \leq l(x,y)||u_{p+1}(s)(x,y) - u_p(s)(x,y)|| <\\ &< l(x,y)\lambda_{p+1}(s).\end{aligned} \tag{3.11}$$

For any $s \in S$ we define the set-valued maps

$$G_{p+1}(s) = \{v \in L^1(\Pi,\mathbb{R}^n); \quad v(x,y) \in F(x,y,u_{p+1}(s)(x,y)) \quad a.e. (\Pi)\},$$

$$H_{p+1}(s) = cl\{v \in G_{p+1}(s); \quad ||v(x,y) - f_p(s)(x,y)|| < l(x,y)\lambda_{p+1}(s)\}.$$

We note that from (3.11) the set $H_{p+1}(s)$ is not empty.

Set $F^*_{p+1}(x,y,s) = F(x,y,u_{p+1}(s)(x,y))$ and note that

$$d(0, F^*_{p+1}(x,y,s)) \leq ||f_p(s)(x,y)|| + l(x,y)\lambda_{p+1}(s) =: \lambda^*_{p+1}(s)(x,y)$$

and $\lambda^*_{p+1}(.) : S \to L^1(I,\mathbb{R})$ is continuous.

By Lemmas 2.5 and 2.6 we obtain the existence of a continuous function $f_{p+1}(.) : S \to L^1(\Pi, \mathbb{R}^n)$ such that

$$f_{p+1}(s)(x,y) \in F(x,y,u_{p+1}(s)(x,y)) \quad a.e.\,(\Pi),\ \forall s \in S,$$

$$||f_{p+1}(s)(x,y) - f_p(s)(x,y)|| \le l(x,y)\lambda_{p+1}(s) \quad \forall s \in S,\ (x,y) \in \Pi.$$

From (3.10), c) and d) we obtain

$$||u_{p+1}(s)(.,.) - u_p(s)(.,.)||_C \le \lambda_{p+1}(s) = L^p(a(s) + \varepsilon(s) + \Lambda(s)), \tag{3.12}$$

$$||f_{p+1}(s)(.,.) - f_p(s)(.,.)||_1 \le ||l||_1 \lambda_p(s) = L^{p-1}||l||_1(a(s) + \varepsilon(s) + \Lambda(s)). \tag{3.13}$$

Therefore $f_p(s)(.,.)$, $u_p(s)(.,.)$ are Cauchy sequences in the Banach space $L^1(\Pi, \mathbb{R}^n)$ and $C(\Pi, \mathbb{R}^n)$, respectively. Let $f(.) : S \to L^1(\Pi, \mathbb{R}^n)$, $x(.) : S \to C(\Pi, \mathbb{R}^n)$ be their limits. The function $s \to a(s) + \varepsilon(s) + \Lambda(s)$ is continuous, hence locally bounded. Therefore (3.13) implies that for every $s' \in S$ the sequence $f_p(s')(.,.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighborhood of s. Hence, $s \to f(s)(.,.)$ is continuous from S into $L^1(\Pi, \mathbb{R}^n)$.

From (3.12), as before, $u_p(s)(.,.)$ is Cauchy in $C(\Pi, \mathbb{R}^n)$ locally uniformly with respect to s. So, $s \to u(s)(.,.)$ is continuous from S into $C(\Pi, \mathbb{R}^n)$. On the other hand, since $u_p(s)(.,.)$ converges uniformly to $u(s)(.,.)$ and

$$d(f_p(s)(x,y), F(x,y,u(s)(x,y)) \le l(x,y)||u_p(s)(x,y) - u(s)(x,y)|| \quad a.e.\,(\Pi),$$

$\forall s \in S$ passing to the limit along a subsequence of $f_p(s)(.,.)$ converging pointwise to $f(s)(.,.)$ we obtain

$$f(s)(x,y) \in F(x,y,u(s)(x,y)) \quad a.e.\ (\Pi),\ \forall s \in S.$$

One may write successively,

$$\begin{aligned}
&\frac{1}{\Gamma(r_1)\Gamma(r_2)}||\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1} f_p(s)(z,t)dzdt - \int_0^x\int_0^y (x-z)^{r_1-1}. \\
&.(y-t)^{r_2-1} f(s)(z,t)dzdt|| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1}||f_p(s)(z,t) \\
&-f(s)(z,t)||dzdt \le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1} l(z,t)||u_{p-1}(s)(z,t) \\
&-u(s)(z,t)||dzdt \le L||u_{p-1}(s)(.,.) - u(s)(.,.)||_C.
\end{aligned}$$

Therefore one may pass to the limit in d) and we get $\forall (x,y) \in \Pi, s \in S$

$$u(s)(x,y) = \mu(s)(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-z)^{r_1-1}(y-t)^{r_2-1} f(s)(z,t)dzdt,$$

i.e., $u(s)(.,.)$ is the desired solution.

Moreover, by adding inequalities (3.10) for all $p \ge 1$ we get

$$||u_{p+1}(s)(x,y) - w(s)(x,y)|| \le \sum_{l=1}^{p+1} \lambda_l(s) \le \frac{a(s) + \varepsilon(s) + \Lambda(s)}{1-L}. \tag{3.14}$$

Passing to the limit in (3.14) we obtain the conclusion of the theorem. □

Hypothesis 3.8. *Hypothesis 3.5 is satisfied and there exists* $q(.,.) \in L^1(\Pi, \mathbb{R}_+)$ *with* $\sup_{(x,y)\in\Pi}(I_0^r q)(x,y) < \infty$ *such that* $d(0, F(x,y,0)) \leq q(x,y)$ *a.e.* (Π).

Corollary 3.9. *Assume that Hypothesis 3.8 is satisfied.*
Then there exists a function $u(.,.) : \Pi \times \mathbb{S} \to \mathbb{R}^n$ *such that*
a) $x(.,(\xi,\eta)) \in \mathcal{S}(\xi,\eta)$, $\forall(\xi,\eta) \in \mathbb{S}$.
b) $(\xi,\eta) \to x(.,(\xi,\eta))$ *is continuous from* $\mathbb{S}$ *into* $C(\Pi, \mathbb{R}^n)$.

Proof. We take $S = \mathbb{S}$, $\varphi(\xi,\eta) = \xi$, $\psi(\xi,\eta) = \eta$ $\forall(\xi,\eta) \in \mathbb{S}$, $\varepsilon(.) : \mathbb{S} \to (0,\infty)$ an arbitrary continuous function, $g(.) = 0$, $w(.) = 0$, $\lambda(s)(x,y) \equiv q(x,y)$ $\forall s = (\xi,\eta) \in \mathbb{S}$, $(x,y) \in \Pi$ and we apply Theorem 3.7 in order to obtain the conclusion of the corollary. □

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