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An extension of P. Lévy's distributional properties to the case of a Brownian motion with drift

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We extend the well-known P. Lévy theorem on the distributional identity $(M_t - B_t, M_t) \simeq (|B_t|, L(B)_t)$, where (B_t) is a standard Brownian motion and $(M_t) = (\sup_{0 \leq s \leq t} B_s)$ to the case of Brownian motion with drift λ. Processes of the type

$$\mathrm{d}X_t^\lambda = -\lambda \operatorname{sgn}(X_t^\lambda)\,\mathrm{d}t + \mathrm{d}B_t$$

appear naturally in the generalization.

Keywords: Brownian motion; local time; Markov processes

1. Introduction

A classical result of Paul Lévy states that if $B = (B_t)_{0 \leq t \leq 1}$ is a standard *Brownian motion* ($B_0 = 0$, $\mathrm{E}B_t = 0$, $\mathrm{E}B_t^2 = t$) then

$$(M - B, M) \overset{\text{law}}{=} (|B|, L(B)), \tag{1}$$

i.e. $((M_t - B_t, M_t);\ 0 \leq t \leq 1) \overset{\text{law}}{=} (|B_t|, L(B)_t;\ 0 \leq t \leq 1)$, where $M = (M_t)_{0 \leq t \leq 1}$, $M_t = \max_{0 \leq s \leq t} B_s$, and $L(B) = (L(B)_t)_{0 \leq t \leq 1}$ is the local time of B at zero:

$$L(B)_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(|B_s| \leq \epsilon)}\,\mathrm{d}s \tag{2}$$

(see, for example, Revuz and Yor, 1994, Chapter VI).

The main aim of this note is to give an extension of the distributional property (1) to the case of a Brownian motion *with drift* B^λ, where $B^\lambda = (B_t^\lambda)_{0 \leq t \leq 1}$, $B_t^\lambda = B_t + \lambda t$. Let us denote $M^\lambda = (M_t^\lambda)_{0 \leq t \leq 1}$, $M_t^\lambda = \max_{0 \leq s \leq t} B_s^\lambda$.

For our presentation the process $X^\lambda = (X_t^\lambda)_{0 \leq t \leq 1}$, defined as the unique strong solution of the stochastic differential equation

$$\mathrm{d}X_t^\lambda = -\lambda \operatorname{sgn} X_t^\lambda\,\mathrm{d}t + \mathrm{d}B_t, \qquad X_0^\lambda = 0, \tag{3}$$

plays a key role. (Here $\operatorname{sgn} x$ is defined to be 1 on $\mathbb{R}_+$, -1 on $\mathbb{R}_-$ and 0 at 0.) In particular, we shall see that the process $|X^\lambda| = (|X_t^\lambda|)_{0\leqslant t\leqslant 1}$ realizes an explicit construction of the process RBM$(-\lambda)$, i.e. a *reflecting Brownian motion with drift* $(-\lambda t)$.

2. Main result

Theorem 1. *For any* $\lambda \in \mathbb{R}$

$$(M^\lambda - B^\lambda,\, M^\lambda) \overset{\text{law}}{=} (|X^\lambda|,\, L(X^\lambda)), \tag{4}$$

i.e. $((M_t^\lambda - B_t^\lambda,\, M_t^\lambda);\, 0 \leqslant t \leqslant 1) \overset{\text{law}}{=} (|X_t^\lambda|,\, L(X^\lambda)_t;\, 0 \leqslant t \leqslant 1)$, *where*

$$L(X^\lambda)_t = \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon}\int_0^t \mathbf{1}_{(|X_s^\lambda|\leqslant\epsilon)}\,\mathrm{d}s.$$

Proof. Denote by $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leqslant t\leqslant 1}, P)$ a filtered probability space and let $B = (B_t)_{0\leqslant t\leqslant 1}$ be a standard Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leqslant t\leqslant 1}, P)$. Define on $(\Omega, \mathscr{F})$ a new probability measure P^λ:

$$\mathrm{d}P^\lambda = \mathrm{e}^{-\lambda B_1 - \lambda^2/2}\,\mathrm{d}P\,(= \mathrm{e}^{-\lambda B_1^\lambda + \lambda^2/2}\,\mathrm{d}P). \tag{5}$$

By Girsanov's theorem (Revuz and Yor 1994; Liptser and Shirayev 1977),

$$\mathrm{Law}(B^\lambda | P^\lambda) = \mathrm{Law}(B|P). \tag{6}$$

Denoting by $C^+[0, 1]$ the space of non-negative continuous functions on $[0, 1]$ we obtain, using (5), (6) and (1), that for any non-negative measurable functional $G = G(x, y)$, $(x, y) \in C^+[0, 1] \times C^+[0, 1]$:

$$\begin{aligned}\mathrm{E}[G(M^\lambda - B^\lambda, M^\lambda)] &= \mathrm{E}^\lambda[\mathrm{e}^{\lambda B_1^\lambda - \lambda^2/2} G(M^\lambda - B^\lambda, M^\lambda)] \\ &= \mathrm{E}[\mathrm{e}^{\lambda B_1 - \lambda^2/2} G(M - B, M)] = \mathrm{E}[\mathrm{e}^{\lambda(L(B)_1 - |B_1|) - \lambda^2/2} G(|B|, L(B))].\end{aligned} \tag{7}$$

From another angle, let us introduce a new measure $\tilde{P}^\lambda$:

$$\mathrm{d}\tilde{P}^\lambda = e^{\lambda\int_0^1 \operatorname{sgn} X_s^\lambda \mathrm{d}B_s - \lambda^2/2}\,\mathrm{d}P\left(= \mathrm{e}^{\lambda\int_0^1 \operatorname{sgn} X_s^\lambda \mathrm{d}X_s^\lambda + \lambda^2/2}\,\mathrm{d}P\right). \tag{8}$$

Again by Girsanov's theorem,

$$\mathrm{Law}(X^\lambda | \tilde{P}^\lambda) = \mathrm{Law}(B|P). \tag{9}$$

From (8) and (9) we find that (with $\tilde{\mathrm{E}}^\lambda$ denoting expectation with respect to $\tilde{P}^\lambda$)

$$\begin{aligned}\mathrm{E}[G(|X^\lambda|, L(X^\lambda))] &= \tilde{\mathrm{E}}^\lambda\left[\mathrm{e}^{-\lambda\int_0^1 \operatorname{sgn} X_s^\lambda\,\mathrm{d}X_s^\lambda - \lambda^2/2} G(|X^\lambda|, L(X^\lambda))\right] \\ &= \mathrm{E}\left[\mathrm{e}^{-\lambda\int_0^1 \operatorname{sgn} B_s\,\mathrm{d}B_s - \lambda^2/2} G(|B|, L)\right].\end{aligned} \tag{10}$$

Now we note that by Tanaka's formula (Revuz and Yor 1994, Chapter VI)

$$|B_t| = \int_0^1 \operatorname{sgn} B_s \,\mathrm{d}B_s + L(B)_t.$$

So, from (10)

$$\mathrm{E}[G(|X^\lambda|,\, L(X^\lambda))] = \mathrm{E}[\mathrm{e}^{\lambda(L(B)_1 - |B_1|) - \lambda^2/2} G(|B|,\, L(B))]. \tag{11}$$

Comparing (7) and (11), we obtain (4). □

3. Study of X^λ

In this section we consider some properties of the processes X^λ and $|X^\lambda|$. If $\lambda = 0$ then $X^0 = B$, $|X^0| = |B|$ and, as is well known, $\mathrm{Law}(|B|) = \mathrm{Law}(\mathrm{RBM}(0))$, where RBM(0) is a *Brownian motion reflecting at zero* (Revuz and Yor 1994, Chapter III; Ikeda and Watanabe 1981, Chapter IV). In this sense the process $|B|$ gives an *explicit* construction of the reflecting Brownian motion. We shall see below that for reflecting Brownian with drift the process $|X^\lambda|$ plays the corresponding role.

Let us describe first of all some properties of X^λ and $|X^\lambda|$ from the point of view of the general theory of Markov processes.

On a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, P)$ for given $\lambda \in \mathbb{R}$ and every $x \in \mathbb{R}$ we consider the stochastic process $X^{x,\lambda} = (X^{x,\lambda}_t)_{t\geqslant 0}$ which satisfies the stochastic differential equation

$$\mathrm{d}X^{x,\lambda}_t = -\lambda \operatorname{sgn} X^{x,\lambda}_t \,\mathrm{d}t + \mathrm{d}B_t, \qquad X^{x,\lambda}_0 = x. \tag{12}$$

This equation has a unique strong solution and, as a corollary (see Revuz and Yor 1994; Chapter IX, Theorem 1.11), we also have uniqueness in law. Denote the corresponding distribution of $X^{x,\lambda}$ on the space $(C, \mathscr{C})$ of continuous functions by $P^{x,\lambda}$:

$$\mathrm{Law}(X^{x,\lambda} |\, P) = P^{x,\lambda}. \tag{13}$$

Denote also by $(T^\lambda_t,\, t \geqslant 0)$ the set of operators given by

$$T^\lambda_t f(x) = \int f(c_t) P^{x,\lambda}(\mathrm{d}c), \tag{14}$$

where $f \in \mathscr{B}_b(\mathbb{R})$ (the set of bounded Borel measurable real-valued functions defined on $\mathbb{R}$) and $c = (c_t)_{t\geqslant 0}$ denotes the coordinate process, $c \in C$.

If τ is a finite $(\mathscr{F}_t)_{t\geqslant 0}$-stopping time and $A \in \mathscr{F}_\tau$ then

$$\mathrm{E}[f(X^{x,\lambda}_{\tau+t}) \cdot \mathbf{1}_A] = \mathrm{E}[T_t f(X^{x,\lambda}_\tau) \cdot \mathbf{1}_A]. \tag{15}$$

Indeed, from (12),

$$X^{x,\lambda}_{\tau+t} = X^{x,\lambda}_\tau - \lambda \int_0^t \operatorname{sgn}(X^{x,\lambda}_{\tau+u}) \,\mathrm{d}u + (B_{\tau+t} - B_\tau). \tag{16}$$

But $\mathrm{Law}(B_{\tau+t} - B_\tau,\, t \geqslant 0 | P) = \mathrm{Law}(B_t,\, t \geqslant 0 |\, P)$ and $(B_{\tau+t} - B_\tau)_{t\geqslant 0}$ is independent of $\mathscr{F}_\tau$ and so by the uniqueness in law of equation (12) we obtain (15).

Thus the process $X^{x,\lambda} = (X_t^{x,\lambda})_{t\geq 0}$ is a time-homogeneous *Markov process* with transition function $(T_t^\lambda(x,\cdot),\ t\geq 0)$ defined above. From Karatzas and Shreve (1988), Chapter 6, Result 6.5] it is known that $T_t^\lambda(x, \mathrm{d}y)$ for all x and λ admits a density $p_t^\lambda(y|x)$, i.e.

$$T_t^\lambda(x, \mathrm{d}y) = p_t^\lambda(y|x)\,\mathrm{d}y,$$

and, for example, for $x \geq 0$, $\lambda \geq 0$, the following formula holds:

$$\begin{aligned} p_t^\lambda(y|x) &= \frac{1}{\sqrt{2\pi t}}\left(\mathrm{e}^{-(x-y-\lambda t)^2/2t} + \lambda \mathrm{e}^{-2\lambda y}\int_{x+y}^{\infty} \mathrm{e}^{-(v-\lambda t)^2/2t}\,\mathrm{d}v\right), \qquad y \geq 0, \\ &= \frac{1}{\sqrt{2\pi t}}\left(\mathrm{e}^{-(2\lambda x-(x-y+\lambda t)^2/2t)} + \lambda \mathrm{e}^{2\lambda y}\int_{x-y}^{\infty} \mathrm{e}^{-(v-\lambda t)^2/2t}\,\mathrm{d}v\right), \qquad y<0. \end{aligned} \tag{17}$$

This explicit form of the transition density can be used to show that $X^{x,\lambda}$ is a *Feller process* – indeed this can also be deduced using Zvonkin's method (Revuz and Yor 1994, Chapter IX, (2.11)).

Now we show that $|X^{x,\lambda}|$ is also a time-homogeneous Markov process. Indeed, $\operatorname{sgn} x$ is an odd function and $\{t|X_t^{x,\lambda} = 0\}$ is P-a.s. a Lebesgue null set (it is clearly true for $\lambda = 0$, that is, for $(x + B_t)_{t\geq 0}$, but the measures $P^{x,0}$ and $P^{x,\lambda}$ are locally equivalent so it holds, in fact, for any $\lambda \in \mathbb{R}$). Thus it follows that P-a.s.

$$-X_t^{x,\lambda} = -x - \lambda\int_0^t \operatorname{sgn}(-X_s^{x,\lambda})\,\mathrm{d}s - B_t, \tag{18}$$

and by the uniqueness in law we then obtain

$$\mathrm{Law}(-X^{x,\lambda}|P) = \mathrm{Law}(X^{-x,\lambda}|P). \tag{19}$$

Using the Markov property of $X^{x,\lambda}$-processes this implies that for all s, $t \geq 0$, $x \in [0,\infty)$ and all bounded real-valued Borel functions f on $[0,\infty)$ we have, for any $A^x \in \sigma(|X_u^{x,\lambda}|\,|\,u \leq s)$,

$$\mathrm{E}[f(|X_{s+t}^{x,\lambda}|), A^x] = \mathrm{E}[\tilde{f}(X_{s+t}^{x,\lambda}), A^x] = \mathrm{E}[T_t\tilde{f}(X_s^{x,\lambda}), A^x]$$

and

$$\begin{aligned} \mathrm{E}[f(|X_{s+t}^{x,\lambda}|), A^x] &= \mathrm{E}[f(|-X_{s+t}^{x,\lambda}|), A^x] = \mathrm{E}[f(|X_{s+t}^{-x,\lambda}|), A^{-x}] \\ &= \mathrm{E}[\tilde{f}(X_{s+t}^{-x,\lambda}), A^{-x}] = \mathrm{E}[T_t\tilde{f}(X_s^{-x,\lambda}), A^{-x}] \\ &= \mathrm{E}[T_t\tilde{f}(-X_s^{x,\lambda}), A^x]. \end{aligned} \tag{20}$$

Here we have used the notation $\tilde{f}(x)$ for $f(|x|)$, $x \in \mathbb{R}$. We have thus shown that $|X^{x,\lambda}|$ is indeed a Feller–Markov process.

Theorem 2. *For each* $x \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$,

$$\mathrm{Law}(|X^{x,\lambda}|) = \mathrm{Law}(\mathrm{RBM}^x(-\lambda)). \tag{21}$$

Proof. In Markov theory, as is well known (see, for example, Ikeda and Watanabe 1981, Chapter IV, §5), the process $\mathrm{RBM}^x(-\lambda)$, called a *Brownian motion with drift* $(-\lambda t)$ *started at* $x \geqslant 0$ *and reflected at zero*, is a diffusion Markov process with infinitesimal operator $\mathscr{A}^\lambda$ acting on functions

$$\mathscr{D}(\mathscr{A}^\lambda) = \left\{ f \in C_\mathrm{b}^2([0, \infty)), \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x\downarrow 0} = 0 \right\}$$

by the formula

$$\mathscr{A}^\lambda f(x) = \tfrac{1}{2} f''(x) - \lambda f'(x). \tag{22}$$

(It is well known that the operator $\mathscr{A}^\lambda$ generates a unique (diffusion) family of measures $Q^{x,\lambda}$, $x \geqslant 0$, and the corresponding Markov process is by definition the process $\mathrm{RBM}^x(-\lambda)$ (Ikeda and Watanabe 1981).)

Now let us consider our process $X^{x,\lambda}$. By the Itô–Tanaka formula (Revuz and Yor 1994, Chapter VI),

$$\begin{aligned} \mathrm{d}|X_t^{x,\lambda}| &= \mathrm{sgn}_t^{x,\lambda}\, \mathrm{d}X_t^{x,\lambda} + \mathrm{d}L(X^{x,\lambda})_t \\ &= -\lambda\, \mathrm{d}t + \mathrm{sgn}\, X_t^{x,\lambda}\, \mathrm{d}B_t + \mathrm{d}L(X^{x,\lambda})_t, \end{aligned} \tag{23}$$

where $L(X^{x,\lambda})_t$ is a local time at zero on the time interval $[0, t]$ for the process $X^{x,\lambda}$. Suppose that $f \in C_\mathrm{b}^2([0, \infty))$ with $f'(0+) = \mathrm{d}f/\mathrm{d}x|_{x\downarrow 0} = 0$. Then by Itô's formula,

$$\begin{aligned} f(|X_t^{x,\lambda}|) - f(|X_0^{x,\lambda}) &= \int_0^t f'(|X_s^{x,\lambda}|)\, \mathrm{d}|X_s^{x,\lambda}| + \frac{1}{2}\int_0^t f''(|X_s^{x,\lambda}|)\, \mathrm{d}s \\ &= \int_0^t f'(|X_s^{x,\lambda}|)(-\lambda\, \mathrm{d}s + \mathrm{sgn}\, X_s^{x,\lambda}\, \mathrm{d}B_s + \mathrm{d}L(X^{x,\lambda})_s) + \frac{1}{2}\int_0^t f''(|X_s^{x,\lambda}|)\, \mathrm{d}s \\ &= \int_0^t \left(-\lambda f'(|X_s^{x,\lambda}|) + \frac{1}{2} f''(|X_s^{x,\lambda}|)\right) \mathrm{d}s + M_t + \int_0^t f'(|X_s^{x,\lambda}|)\, \mathrm{d}L(X^{x,\lambda})_s, \end{aligned} \tag{24}$$

where $M_t = \int_0^t f'(|X_s^{x,\lambda}|) \mathrm{sgn}\, X_s^{x,\lambda}\, \mathrm{d}B_s$ is a local martingale and

$$\int_0^t f'(|X_s^{x,\lambda}|)\, \mathrm{d}L(X^{x,\lambda})_s = 0$$

because $f'(0+) = 0$ and $L(X^{x,\lambda})$ increases only on the time set $\{t \mid X_t^{x,\lambda} = 0\}$. From (24) we see that

$$f(|X_t^{x,\lambda}|) - f(|X_0^{x,\lambda}|) - \int_0^t \mathscr{A}^\lambda f(|X_s^{x,\lambda}|)\, \mathrm{d}s \tag{25}$$

is a local martingale and thus the infinitesimal operators for the two processes $|X^{x,\lambda}|$ and $\mathrm{RBM}^x(-\lambda)$ are the same (acting on $\mathscr{D}(\mathscr{A}^\lambda)$). Therefore (21) is proved. □

4. Some remarks

The theorem of P. Lévy (1) and its extension (4) given above both have 'two-dimensional' character in the sense that they are statements for pairs of processes $((M^\lambda - B^\lambda), M^\lambda)$ and $(|X^\lambda|, L(X^\lambda))$.

M. Yor has pointed out the connection between Theorem 1 and 2 above and the results in Kinkladze (1982) and Fitzsimmons (1987). From Kinkladze (1982) one may obtain easily the corresponding 'one-dimensional' result saying that $M^\lambda - B^\lambda \overset{\text{law}}{=} \text{RBM}(-\lambda)$. (For the notion of $\text{RBM}(-\lambda)$ see Definition 1 in Kinkladze (1982).) Indeed by Theorem 1 and 2 in Kinkladze (1982) the process $Y^\lambda \equiv \text{RBM}(-\lambda)$ can be realized with some Brownian motion B in the form

$$Y_t^\lambda = \sup_{0 \leqslant s \leqslant t} (-\lambda(t-s) - (B_t - B_s)), \qquad t \geqslant 0.$$

So $Y_t^\lambda = \sup_{0 \leqslant s \leqslant t}((\lambda s + B_s) - (\lambda t + B_t))$ and as a corollary $Y^\lambda \equiv M^\lambda - B^\lambda$ with $B_t^\lambda = \lambda t + B_t$. Together with formula (21) of Theorem 2 we obtain that $M^\lambda - B^\lambda \overset{\text{law}}{=} |X^\lambda|$. In connection with this formula it is useful to remark that the process X^λ has appeared in many different problems; however, the very natural property $\text{RBM}(-\lambda) \overset{\text{law}}{=} |X^\lambda|$ apparently has not been noted before.

It is very reasonable to ask about possible extensions of the result $M^\lambda - B^\lambda \overset{\text{law}}{=} |X^\lambda|$ for the more general class of processes $Z = (Z_t)_{t \geqslant 0}$ besides the processes $B^\lambda = (B_t^\lambda)_{t \geqslant 0}$, $\lambda \in \mathbb{R}$. According to Fitzsimmons (1987), if $Z = (Z_t)_{t \geqslant 0}$ is a conservative real-valued diffusion process and the process $\max Z - Z$ is a time-homogeneous strong Markov process then necessarily $Z = B^{\lambda,\sigma}$, where $B_t^{\lambda,\sigma} = \lambda t + \sigma B_t$ with $\lambda \in \mathbb{R}$, $\sigma > 0$. So, this result shows that in some sense a direct extension of the P. Lévy's result is possible only for Brownian motion with drift. This is exactly the framework of Theorem 1 above.

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