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SURGERY AND BORDISM INVARIANTS

BY MICHAEL WEISS

Introduction. The approach used here to relate the two subjects in the title is best explained in terms of three "machines".

Machine (1) is the "L-theory machine ", or "surgery machine"; on being fed a discrete group G and homomorphism $w\colon G \to Z_2$, it produces a spectrum $\underline{L}_{\cdot}(G,w)$ whose homotopy groups are the surgery obstruction groups (choose your favourite version),

$$\pi_n(\underline{L}_{\cdot}(G,w)) = L_n(G,w) \quad \text{for } n \in \mathbf{Z}.$$

Machine (2) is the "bordism theory machine": on being fed a CW-space B and vector bundle γ on B, it produces a bordism spectrum (or Thom spectrum) $M(B,\gamma)$. The homotopy groups $\pi_n(M(B,\gamma))$ are the bordism groups of closed smooth manifolds N^n equipped with a bundle map from the normal bundle ν_N to γ.

This note will describe a third machine, obtained by welding together the previous two. (The aim is to extend the theory of the "generalized Kervaire invariant": cf. [**1**, **2**].)

Description of Machine (3).

Input. The following input data are required:

— a group G and homomorphism $w\colon G \to Z_2$, as for Machine (1);

— a CW-space B and bundle γ on B, as for Machine (2);

— a principal G-bundle α on B and an identification j of the two double covers of B arising from these data. (They are the orientation cover associated with γ, and the double cover induced from α via w.)

Output. Machine (3) produces a spectrum $\underline{L}^{\cdot}(G,w;B,\gamma;\alpha,j)$ (informally: $\underline{L}^{\cdot}(B,\gamma)$) and maps of spectra

$$\underline{L}_{\cdot}(G,w) \to \underline{L}^{\cdot}(B,\gamma) \leftarrow M(B,\gamma).$$

Like Machines (1) and (2), Machine (3) is functorial: Given two input strings $(G,w;B,\gamma;\alpha,j)$ and $(G',w';B',\gamma';\alpha',j')$, and

— a map $f\colon B \to B'$ covered by a bundle map $\gamma \to \gamma'$;

— a homomorphism $h\colon G \to G'$ so that $w' \cdot h = w$;

— an identification of principal G'-bundles on B,

$$h_*(\alpha) \cong f^*(\alpha'),$$

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compatible with the bundle map as regards j and j', Machine (3) yields a commutative diagram

$$\begin{array}{ccccc} \underline{\mathbf{L}}_{:}(G,w) & \to & \underline{\mathbf{L}}^{\cdot}(B,\gamma) & \leftarrow & M(B,\gamma) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathbf{L}}_{:}(G',w') & \to & \underline{\mathbf{L}}^{\cdot}(B',\gamma') & \leftarrow & M(B',\gamma'). \end{array}$$

More surprising is the following property. Write (informally) $\hat{L}^n(B,\gamma)$ for the nth homotopy group of the map of spectra

$$\underline{\mathbf{L}}_{:}(G,w) \to \underline{\mathbf{L}}^{\cdot}(G,w;B,\gamma;\alpha,j).$$

THEOREM. *There is a functorial long exact sequence*

$$\cdots \to \hat{Q}^{n+1}(C(\tilde{B})) \to \hat{L}^n(B,\gamma) \to Q^n(C(\tilde{B})) \to \hat{Q}^n(C(\tilde{B})) \to \hat{L}^{n-1}(B,\gamma) \cdots$$
$$(n \in \mathbf{Z}).$$

(*Explanation*: $C(\tilde{B})$ is the cellular chain complex of the total space of α — regarded as a chain complex of left projective modules over the ring with involution $A := \mathbf{Z}[G]$. The chain homotopy invariant functors $Q^n(-)$, $\hat{Q}^n(-)$ are defined on the category of such chain complexes, as follows.

Let C be any chain complex of left projective A-modules; then, using the involution on A, C can be regarded as a chain complex of right A-modules, written C^t. So $C^t \otimes_A C$ is defined, and is a chain complex of $\mathbf{Z}[Z_2]$-modules; the generator $\tau \in Z_2$ acts by switching factors, with the usual sign rules.

Define the $\mathbf{Z}[Z_2]$-module chain complexes W, $\hat{W}$ as follows:

$$W_r := \mathbf{Z}[Z_2] \quad \text{for } r \geq 0, \qquad W_r = 0 \quad \text{for } r < 0,$$
$$d\colon W_r \to W_{r-1}; \quad x \mapsto (1 + (-)^r\tau)\cdot x \quad \text{for } r > 0,$$

and $\hat{W}_r := \mathbf{Z}[Z_2]$ for $r \in \mathbf{Z}$,

$$d\colon \hat{W}_r \to \hat{W}_{r-1}; \quad x \mapsto (1 + (-)^r\tau)\cdot x \quad \text{for } r \in \mathbf{Z}.$$

Now let $Q^n(C)$ and $\hat{Q}^n(C)$ be the nth homology groups of the chain complexes $\mathrm{Hom}_{\mathbf{Z}[Z_2]}(W, C^t \otimes_A C)$ and $\mathrm{Hom}_{\mathbf{Z}[Z_2]}(\hat{W}, C^t \otimes_A C)$, respectively. (See [**4**] for more details.)

Ingredients of the construction. The spectra $\underline{\mathbf{L}}^{\cdot}(G,w;B,\gamma;\alpha,j)$ are algebraic bordism spectra. Their construction is inspired by the following "dictionary" (A is a ring with involution):

chain complex D of projective A-modules — space;

symmetric algebraic Poincaré complex over A — closed manifold;

chain bundle on D — vector bundle on a space.

Here a *chain bundle* on a chain complex D is a 0-dimensional cycle in

$$\mathrm{Hom}_{\mathbf{Z}[Z_2]}(\hat{W}, (D^{-*})^t \otimes_A D^{-*})$$

(and D^{-*} is the "dual" of D, cf. [**4**]). It represents, but should not be confused with, an element in $\hat{Q}^0(D^{-*})$.

Symmetric algebraic Poincaré complexes (C, φ) (of dimension n) are defined in [**4**] (or [**3**]); I insist that φ be an n-dimensional cycle in $\mathrm{Hom}_{\mathbf{Z}[Z_2]}(W, C^t \otimes_A C)$ (whereas [**4**] only requires a class in $Q^n(C)$). On the other hand, C is allowed to be nontrivial in negative dimensions (less restrictive than [**4**]).

Only the third entry of the dictionary is new, but even so it is strongly suggested by [**4**, Part II, Proposition 9.3].

$\underline{\mathbf{L}}^{\cdot}(G, w; B, \gamma; \alpha, j)$ is obtained in two steps: first, passage from B, γ to a chain complex $C(\tilde{B})$ with chain bundle $c(\gamma)$; second, construction of an algebraic bordism spectrum associated with $C(\tilde{B})$ and $c(\gamma)$, using the dictionary above. So the homotopy groups of $\underline{\mathbf{L}}^{\cdot}(G, w; B, \gamma; \alpha, j)$, informally written $L^n(B, \gamma)$, are the bordism groups of symmetric algebraic Poincaré complexes (C, φ) equipped with a classifying chain map $C \to C(\tilde{B})$ which is covered by a "chain bundle map" from the "normal chain bundle of (C, φ)" to $c(\gamma)$.

REMARKS. (i) If (C, φ) is a symmetric algebraic Poincaré complex, the chain complex C carries a "normal chain bundle" (easy to construct). It is well defined up to an infinity of higher homologies (resembling the normal bundle of a geometric manifold, which is well defined up to an infinity of higher concordances).

(ii) Given a string of data $(G, w; B, \gamma; \alpha, j)$, we are first of all faced with the problem of constructing a chain bundle on $C(\tilde{B})$ (the image $c(\gamma)$ of γ in the chain complex world). The previous remark shows how to proceed if B happens to be a closed manifold, γ its normal bundle; functoriality dictates the rest.

(iii) The theorem is obtained using algebraic surgery in the style of [**4**].

(iv) If B is empty, $\underline{\mathbf{L}}^{\cdot}(G, w; B, \gamma; \alpha, j) \simeq \underline{\mathbf{L}}_{\cdot}(G, w)$ as follows from the theorem.

By-products. (i) The theory also gives a homological description of the homotopy groups of the forgetful map

$$J\colon (\text{quadratic } L\text{-theory}) \to (\text{symmetric } L\text{-theory}).$$

(For a fixed ring with involution A, the homotopy groups of the quadratic L-theory spectrum of A are the Wall groups $L_n(A)$. Those of the symmetric L-theory spectrum are the groups $L^n(A)$ of [**4**] (or [**3**]); their main application is to problems of the following kind. "The product of an n-dimensional surgery problem with an m-dimensional closed manifold is an $(n+m)$-dimensional surgery problem; how are the surgery obstructions of the two surgery problems related?" See [**4**] for details.)

Here is the philosophy behind the homological description: If the notion of chain bundle (on a chain complex of left projective A-modules) is any good, there ought to exist a suitable chain complex D and a *universal chain bundle* μ on D (even though D, μ may not correspond to any geometric reality). The associated algebraic bordism theory should be symmetric L-theory, and the long exact sequence of the theorem above should remain valid (with $\hat{L}^n(B, \gamma)$ replaced by $\pi_n(J)$, and $C(\tilde{B})$ by D). All this is true if carefully interpreted.

(ii) There is a version of the theory where orientations are ignored (because Z_2 is used as coefficient ring instead of $\mathbf{Z}$); a typical "input string" for Machine (3) would then have the form $(G; B, \gamma; \alpha)$.

Investigating this modified Machine (3) with $G = \{1\}$, one finds that it reproduces more or less the "generalized Kervaire invariants" of [**1**, **2**]. More precisely:

Suppose that the $(k+1)$st Wu class of γ is zero; then there is a commutative diagram

$$\pi_{2k}(M(B,\gamma)) \qquad \rightarrow \qquad Z_8$$

$$L^{2k}(B,\gamma)$$

(with $L^{2k}(B,\gamma) = \pi_{2k}(\underline{L}^{\cdot}(\{1\}; B, \gamma; \mathrm{id})))$ in which the horizontal arrow is the invariant of [**2**]. The homomorphism from $L^{2k}(B,\gamma)$ to Z_8 is obtained by imitating [**2**]: the elements of $L^{2k}(B,\gamma)$ are represented by $2k$-dimensional symmetric algebraic Poincaré complexes (C,φ) (over $A = Z_2$) with a certain structure, and the said structure permits to refine the nondegenerate symmetric bilinear form on $H^k(C; Z_2)$ to a *quadratic form* with values in Z_4.

(Choices are necessary to make this work; but [**2**] also uses certain choices, and there is a canonical one-one correspondence between the two kinds of choices.)

To summarize, there is a good case for regarding the homomorphism

$$\pi_{2k}(M(B,\gamma)) \rightarrow L^{2k}(B,\gamma)$$

itself as "the" generalized Kervaire invariant: it requires no choices or restrictions of any kind and, more important, it gives very slick product formulae. The computation of the groups $L^{2k}(B,\gamma)$ is easy in the case at hand (use the theorem, and bear in mind that any chain complex over Z_2 is homotopy equivalent to its homology).

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INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, 91440 BURES-SUR-YVETTE, FRANCE