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FAST RECURSION FORMULA FOR WEIGHT MULTIPLICITIES[1]

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The purpose of this note is to describe and prove a fast recursion formula for computing multiplicities of weights of finite dimensional representations of simple Lie algebras over **C**.

Until now information about weight multiplicities for all but some special cases [1, 2] has had to be found from the recursion formulas of Freudenthal [3] or Racah [4]. Typically these formulas become too laborious to use for hand computations for ranks $\gtrsim 5$ and dimensions $\gtrsim 100$ and for ranks $\simeq 10$ and dimensions $\simeq 10^4$ on a large computer [5, 6]. With the proposed method the multiplicities can routinely be calculated, even by hand, for dimensions far exceeding these. As an example we present a summary of calculations [7] of all multiplicities in the first sixteen irreducible representations of E_8.

Let $\mathfrak{G}$ be a semisimple Lie algebra over **C** with root system Δ and Weyl group W relative to a Cartan subalgebra $\mathfrak{H}$. Let Δ^+ be the positive roots with respect to some ordering and $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots. Let Q and P be the root and weight lattices respectively spanning the real vector space $V \subset \mathfrak{H}^*$. If $X \subset P$ we denote by X^{++} the set of dominant elements of X relative to Π.

Let M be an irreducible $\mathfrak{G}$-module with highest weight Λ and weight system Ω. An important feature of the approach is the direct determination of Ω^{++} without computing outside the dominant chamber. Since every W-orbit is represented by one weight $\lambda \in \Omega^{++}$ of the same multiplicity, it suffices to compute such λ's.

The recursion formula for computing the multiplicities is a modification (Proposition 4) of the Freudenthal formula in which the Weyl group has been exploited to collapse it as much as possible. After describing the procedure, we present the E_8 example. Finally the necessary proofs are given.

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Computation of dominant weights and their multiplicities. Examples.

Determination of Ω^{++}. We define inductively a set of disjoint subsets ('layers') of P^{++}, L_k, $k = 0, 1, 2, \ldots$, by

(1) $L_0 = \{\Lambda\}, \quad L_k = \{\gamma \in P^{++} - L_{k-1} | \gamma = \lambda - \beta, \lambda \in L_{k-1}, \beta \in \Delta^+\}.$

Then (Proposition 1) $\bigcup_{k=0}^{\infty} L_k = \Omega^{++}$. Thus Ω^{++} can be found directly by computing the layers beginning with L_0. After Ω^{++} is computed in this way it is reordered according to level. If $\rho^\vee \in V$ is defined by $(\rho^\vee, \alpha_i) = 1$ for all i, then the new (partial) ordering of Ω^{++} is given by the integers $(\lambda, \rho^\vee)$, $\lambda \in \Omega^{++}$.

Computation of the multiplicity m_λ *of* $\lambda \in \Omega^{++}$. An m_λ of level k is given in terms of the multiplicities $m_{\lambda'}$ of weights λ' of levels above the kth one.

Let $\mathrm{Stab}_W(\lambda)$ be the stabilizer of λ in W. Then $\mathrm{Stab}_W(\lambda) = W_T := \langle r_i | i \in T\rangle$, where $T = \{i | (\lambda, \alpha_i) = 0\}$ [8]. Let $\hat{W}_T = \langle W_T, -1_V\rangle$, where 1_V is the identity transformation on V. $\hat{W}_T$ decomposes Δ into orbits $o_1, \ldots, o_n$. Each orbit o_i contains a unique $\xi_i = \Sigma n_{ij}\omega_j$, $\xi_i \in \Delta^+$ and $n_{ij} \geqslant 0$ for all $j \in T$ (Proposition 3). The modified Freudenthal formula is

$$\sum_{i=1}^{n} |o_i| \sum_{p=1}^{\infty} (\lambda + p\xi_i, \xi_i) m_{\lambda + p\xi_i} = (c_\Lambda - c_\lambda) m_\lambda, \tag{2}$$

where $|o_i|$ is the number of elements of o_i and, for all $\mu \in P$,

$$c_\mu := (\mu + \rho, \mu + \rho) - (\rho, \rho), \qquad \rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha. \tag{3}$$

The sum on p is in reality finite and by standard properties of weight strings $\lambda + p\xi_i \notin \Omega \Rightarrow \lambda + q\xi_i \notin \Omega$ for $q > p$.

It is advantageous to work in the ω-basis of the fundamental weights when computing m_λ. Thus writing $\lambda = \Sigma n_i \omega_i$, $T = \{i | n_i = 0\}$. When the positive roots are expressed in this basis one easily determines ξ_i's. With $S_i := \{j \in T | (\xi_i, \alpha_j) = 0\}$ the orbit sizes $|o_i|$ are given by subgroup indices $[W_T\colon W_{S_i}]$ or $2[W_T\colon W_{S_i}]$ (Proposition 3).

If in the relation (2) some weight $\mu = \lambda + p\xi_i = \Sigma n_j \omega_j$ is not in P^{++}, then some $n_j < 0$ and $r_j\mu = \mu - n_j\alpha_j$ is on a higher level. A finite number ($\leqslant |\Delta^+|$) of reflections r_i transforms μ into $\nu \in P^{++}$ and m_ν is already computed.

If an extensive computation of weight multiplicities is to be undertaken, it is important to notice that for a given $\mathfrak{G}$ there are only finitely many subsets T of $\{1, 2, \ldots, l\}$ and corresponding ξ_i and $|o_i|$. It is natural to compute this information once and for all. We are preparing such a table.

Consider an example of the E_8 representation of dimension 4 096 000. There are only nine weights in Ω^{++}. In the basis of fundamental weights these are (after reordering according to levels) $\lambda_0, \ldots, \lambda_8$ (see Table). Here the layers are $L_0 = \{\lambda_8\}$, $L_1 = \{\lambda_7, \lambda_6, \lambda_3\}$, $L_2 = \{\lambda_5, \lambda_4, \lambda_2, \lambda_1\}$, $L_3 = \{\lambda_0\}$. Given Δ^+ in the ω-basis, even by hand, the computation of (4) can be done in a matter of minutes.

Dim. Weight	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	$[W:W_{T_1}]$
1; $\overset{0}{0000000}$	1	8	35	140	120	370	1 407	2 960	4 480	1 765	6 000	4 104	18 688	43 065	37 680	64 470	1
	248 ; $\overset{0}{0000001}$	1	7	35	29	111	455	1 056	1 624	645	2 296	1 584	7 504	18 081	15 959	28 441	$2^4\cdot 3\cdot 5$
		3 875; $\overset{0}{1000000}$	1	7	6	29	133	350	552	224	826	580	2 864	7 279	6 504	12 103	$2^4\cdot 3^3\cdot 5$
			30 380; $\overset{0}{0000010}$	1	1	6	34	105	174	74	275	198	1 028	2 790	2 535	4 938	$2^6\cdot 3\cdot 5\cdot 7$
				27 000; $\overset{0}{0000002}$	1	-	7	27	56	29	77	63	336	1 008	959	1 897	$2^4\cdot 3\cdot 5$
					147 250; $\overset{1}{0000000}$	1	7	28	48	21	84	62	344	1 007	932	1 925	$2^7\cdot 3^3\cdot 5$
						779 247; $\overset{0}{1000001}$	1	6	12	6	22	17	104	338	323	705	$2^5\cdot 3^3\cdot 5\cdot 7$
							2 450 240; $\overset{0}{0000100}$	1	2	1	5	4	28	103	101	243	$2^6\cdot 3^3\cdot 5\cdot 7$
								4 096 000; $\overset{0}{0000011}$	1	1	-	-	6	27	34	71	$2^7\cdot 3\cdot 5\cdot 7$
									1 763 125; $\overset{0}{0000003}$	1	-	-	-	-	7	-	$2^4\cdot 3\cdot 5$
										6 696 000; $\overset{0}{0100000}$	1	1	6	28	27	77	$2^9\cdot 2^3\cdot 5$
											4 881 384; $\overset{0}{2000000}$	1	-	7	6	21	$2^4\cdot 3^3\cdot 5$
												26 411 008; $\overset{1}{0000001}$	1	6	7	21	$2^{10}\cdot 3^3\cdot 5$
													76 271 625; $\overset{0}{1000010}$	1	1	5	$2^6\cdot 3^4\cdot 5\cdot 7$
														70 680 000; $\overset{0}{1000002}$	1	-	$2^5\cdot 3^3\cdot 5\cdot 7$
															146 325 270; $\overset{0}{0001000}$	1	$2^8\cdot 3^3\cdot 5\cdot 7$

Table of weight multiplicities for the first 16 irreducible representations of E_8 ordered by the levels of their highest weights $\lambda_0, \ldots, \lambda_{15}$. Below the step diagonal the λ_i's are expressed in terms of the fundamental weights.

The determination of the multiplicity m_{λ_2} is representative. Suppose we already know $m_{\lambda_8} = 1$, $m_{\lambda_7} = 2$, $m_{\lambda_6} = 12$, $m_{\lambda_5} = 48$, $m_{\lambda_4} = 56$, $m_{\lambda_3} = 174$. First we find the quantities ξ_i and $|o_i|$ for λ_2. They are $\xi_1 = \overset{0}{0000001}$, $|o_1| = 28$, $\xi_2 = -\overset{1}{1000000}$, $|o_2| = 128$, $\xi_3 = -\overset{0}{1000010}$, $|o_3| = 84$. Next we find the weights $\lambda_2 + p\xi_i$ which either are in Ω^{++} or are transformed there by a sequence of reflections r_j, $j = 1, 2, \ldots, 8$. These weights are $\lambda_2 + \xi_1 = \lambda_6$, $\lambda_2 + \xi_2 = \lambda_5$, $\lambda_2 + \xi_3 = \lambda_3$, and $\lambda_2 + 2\xi_3 = -1\overset{0}{0}00020$ which after 16 reflections r_j is transformed into λ_7. Hence (2) reads

$$
\begin{aligned}
(4) \qquad |o_1|(\lambda_6, \xi_1)m_{\lambda_6} + |o_2|(\lambda_5, \xi_2)m_{\lambda_5} + |o_3|(\lambda_3, \xi_3)m_{\lambda_3} \\
+ |o_3|(\lambda_2 + 2\xi_3, \xi_3)m_{\lambda_7} = (c_{\lambda_8} - c_{\lambda_2})m_{\lambda_2}.
\end{aligned}
$$

Substituting the corresponding values into (4), one has $28 \cdot 4 \cdot 12 + 128 \cdot 3 \cdot 48 + 84 \cdot 2 \cdot 174 + 84 \cdot 4 \cdot 2 = (186 - 96)m_{\lambda_2}$ which gives $m_{\lambda_2} = 552$.

The Table summarizes our results for the 16 irreducible representations of E_8. A useful check of the results is the equality of dimensions

$$
(5) \qquad \dim(M) = \sum_{\lambda_i \in \Omega^{++}} [W\colon W_{T_i}]\, m_{\lambda_i},
$$

where the dimensions $\dim(M)$, the number $[W\colon W_{T_i}]$ of weights on each W-orbit, and the multiplicities m_{λ_i} are given in the Table.

Theory.

PROPOSITION 1. *Let M be an irreducible $\mathfrak{G}$-module with highest weight Λ. Then for $\lambda \in P^{++}$ with $\lambda \neq \Lambda$, $\lambda \in \Omega^{++}$ if and only if $(\lambda + \Delta^+) \cap \Omega^{++} \neq \emptyset$.*

PROOF. Suppose that $\lambda \in P^{++}$, $\alpha \in \Delta^+$ and $\mu := \lambda + \alpha \in \Omega^{++}$. For all $\beta \in \Delta^+$, $(\lambda, \alpha) \geqslant 0$. Then $(\mu, \alpha) = (\lambda + \alpha, \alpha) > 0$. Since the weight string through μ is $\mu + q\alpha, \ldots, \mu, \ldots, \mu - p\alpha$ where $p - q = 2(\mu, \alpha)/(\alpha, \alpha)$, [3], it follows that $p > 0$ and hence $\lambda \in \Omega \cap P^{++} = \Omega^{++}$.

Conversely, suppose that $\lambda \in \Omega^{++}$, $\lambda \neq \Lambda$. We show that there is an $\alpha \in \Delta^+$ with $\lambda + \alpha \in \Omega^{++}$. There is a $\beta \in \Delta^+$ with $\lambda + \beta \in \Omega$. If $\lambda + \beta$ is dominant we are done. If not $(\lambda + \beta, \alpha_j) < 0$ for some j so by the argument above the α_j-weight string through $\lambda + \beta$ contains $\lambda + \beta + \alpha_j$. Also $(\lambda, \alpha_j) \geqslant 0$ since $\lambda \in P^{++}$ so $(\beta, \alpha_j) < 0$. Then $\beta + \alpha_j$ is a root, $\beta + \alpha_j \in \Delta^+$, and $\lambda + \beta + \alpha_j \in \Omega$. We can replace β by $\beta + \alpha_j$ in the above and repeat. The process cannot continue indefinitely, so the required α exists.

An interesting consequence of Proposition 1 is

PROPOSITION 2 (NOTATION OF PROPOSITION 1). *Let k be the largest integer such that $L_k \neq \emptyset$. Then L_k is a singleton $\{\omega\}$. Furthermore, ω depends only on Λ mod 0. In particular $0 \in \Omega$ if and only if $\Lambda \in Q$.*

For $T \subset \{1, 2, \ldots, l\}$, let $V_T := \Sigma_{i \in T} \mathbf{R}\alpha_i$, and let Δ_T be the root system based on sub-Coxeter-Dynkin diagram corresponding to the vertices labelled by T. Then $V_T \cap \Delta = \Delta_T$.

PROPOSITION 3. *Let $T \subset \{1, \ldots, l\}$ be any subset. Then each orbit o of $\hat{W}_T$ in Δ contains a unique element $\xi \in \Delta^+$ of the form $\Sigma n_i \omega_i$ where $n_i \geqslant 0$ for all $i \in T$. Furthermore, if $S = \{i \in T | n_i = 0\}$ then either $|o| = [W_T: W_s]$ if $\xi \in \Delta_T$, or $|o| = 2[W_T: W_s]$ if $\xi \notin \Delta_T$.*

PROOF (EXISTENCE). Let o be an orbit of $\hat{W}_T$ in Δ. Choose $\xi = \Sigma n_i \omega_i = \Sigma c_i \alpha_i \in o$ with $ht\xi := \Sigma c_i$ maximal. Since $-1_V \in \hat{W}_T$, $\xi \in \Delta^+$. Since $ht(r_i \xi) \leqslant ht(\xi)$ for all $i \in T$, we have $(\xi, \alpha_i) \geqslant 0$ hence $n_i \geqslant 0$, if $i \in T$.

(UNIQUENESS). Let π: $V \longrightarrow V_T$ be the orthogonal projection onto $V_T = \Sigma_{j \in T} \mathbf{R}\alpha_j$. Then $i \in T$, $v \in V$, $(\pi(v), \alpha_i) = (v, \alpha_i)$ from which it follows that π is W_T-equivariant. Furthermore, π is injective on any W_T-orbit in V since $V_T \cap \ker \pi = 0$. Thus there is a unique element on each W_T-orbit in V whose projection is in the dominant chamber of V_T under W_T. If $\mu = \Sigma u_j \omega_j$ is such an element then $(\mu, \alpha_i) = (\pi(\mu), \alpha_i) \geqslant 0$ for all $i \in T$, that is $u_i \geqslant 0$ for all $i \in T$.

Now let $\xi \in \Delta^+$. If $\hat{W}_T \xi = W_T \xi$ then $-\xi \in W_T \xi$ so $W_T \xi \cap \Delta^- \neq \varnothing$, from which $\xi \in \Delta_T^+$. Conversely, if $\xi \in \Delta_T^+$ then $-\xi \in W_T \xi$ so $\hat{W}_T \xi = W_T \xi$.

Finally suppose that distinct ξ, ξ' satisfy the hypothesis of the Proposition and define the same $\hat{W}_T$-orbit. Then by the above $W_T \xi \neq W_T \xi'$ so in fact $\hat{W}_T \xi \neq W_T \xi$, $\xi \notin \Delta_T$, $W_T \xi \subset \Delta^+$ and $\xi' \in -W_T \xi \subset \Delta^-$, a contradiction. This proves the uniqueness. The statement on the orbit size is immediate.

PROPOSITION 4 (MODIFIED FREUDENTHAL FORMULA). *Let M be the irreducible $\mathfrak{G}$-module of highest weight Λ and let $\lambda \in \Omega(\Lambda)$. Let $W_T = \mathrm{Stab}_W(\lambda)$ and let $o_1, \ldots, o_n$ be the orbits of $\hat{W}_T$ in Δ. Let $\xi_i \in o_i$ be as in Proposition 3. Then equation (2) holds.*

PROOF. We may suppose the indexing of the orbits is taken so that $o_1 \cup \cdots \cup o_m = \Delta_T$, $0 \leqslant m \leqslant n$. For $i \leqslant m$, $o_i = W_T \xi_i$ whereas for $i > m$, $o_i = W_T \xi_i \cup -W_T \xi_i$ (disjoint). Begin with the form of Freudenthal's formula [3, Equation 48.2]:

$$c_\Lambda m_\lambda = \sum_{\alpha \in \Delta} \sum_{p=0}^{\infty} (\lambda + p\alpha, \alpha) m_{\lambda + p\alpha} + (\lambda, \lambda) m_\lambda. \tag{6}$$

For $w \in W_T$, $(\lambda + pw\alpha, w\alpha) = (w(\lambda + p\alpha), w\alpha) = (\lambda + p\alpha, \alpha)$ and $m_{\lambda + pw\alpha} = m_{\lambda + p\alpha}$, so the double sum of the right-hand side of (6) may be rewritten as

$$\begin{aligned} &\sum_{i=1}^{m} |W_T \xi_i| \sum_{p=0}^{\infty} (\lambda + p\xi_i, \xi_i) m_{\lambda + p\xi_i} \\ &\quad + \sum_{i=m+1}^{n} |W_T \xi_i| \sum_{p=0}^{\infty} \{(\lambda + p\xi_i, \xi_i) m_{\lambda + p\xi_i} + (\lambda - p\xi_i, -\xi_i) m_{\lambda - p\xi_i}\}. \end{aligned} \tag{7}$$

Now for any $\alpha \in \Delta$ there is the relation [3, §48]

$$\sum_{p=-\infty}^{\infty} (\lambda + p\alpha, \alpha) m_{\lambda + p\alpha} = 0 \tag{8}$$

by which (7) becomes

$$\sum_{i=1}^{m} |W_T\xi_i| \sum_{p=0}^{\infty} (\lambda + p\xi_i, \xi_i) m_{\lambda + p\xi_i} + \sum_{i=m+1}^{n} |W_T\xi_i| \left\{ 2 \sum_{p=1}^{\infty} (\lambda + p\xi_i, \xi_i) m_{\lambda + p\xi_i} + (\lambda, \xi_i) m_\lambda \right\}. \tag{9}$$

Collect the coefficients of m_λ appearing in (9). In the first sum occur those ξ_i which are in Δ_T. Since $(\lambda, \alpha_j) = 0$ for $j \in T$, $(\lambda, \xi_i) = 0$ for $j = 1, \ldots, m$. In the second sum we have $\Sigma_{i=m+1}^{n} |W_T\xi_i|(\lambda, \xi_i) = \Sigma_{\alpha \in \Delta^+ - \Delta_T^+}(\lambda, \alpha) = \Sigma_{\alpha \in \Delta^+}(\lambda, \alpha) = 2(\lambda, \rho)$. Taking account of the $(\lambda, \lambda)m_\lambda$ appearing in (6) and $2|W_T\xi_i| = |o_i|$ for $i > m$, we arrive at

$$c_\Lambda m_\lambda = \sum_{i=1}^{n} |o_i| \sum_{p=1}^{\infty} (\lambda + p\xi_i, \xi_i) m_{\lambda + p\xi_i} + (2(\lambda, \rho) + (\lambda, \lambda)) m_\lambda$$

which proves the proposition.

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