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CYCLIC ELEMENTS IN SOME SPACES OF ANALYTIC FUNCTIONS

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DEFINITIONS. 1. A^{-p} $(p > 0)$ is the Banach space of analytic functions $f(z)$ in $U = \{z \in \mathbf{C} \mid |z| < 1\}$ that satisfy $|f(z)| = o[(1 - |z|)^{-p}]$ $(|z| \longrightarrow 1)$ with the norm $\|f\| = \max\{|f(z)|(1 - z)^p\}$ $(z \in U)$. Note that $f_n \longrightarrow f$ in A^{-s} and $g_n \longrightarrow g$ *in* A^{-t} implies $f_n g_n \longrightarrow fg$ in $A^{-(s+t)}$.

2. B^p $(p > 0)$ is the Bergman space, i.e., the "analytic" subspace of $L^p(rdrd\theta)$ in U.

3. $A^{-\infty} = \bigcup A^{-p} = \bigcup B^p$ $(p > 0)$. $A^{-\infty}$ is a linear topological space [1].

4. P is the set of all algebraic polynomials $P(z)$. P is dense in any of the spaces A^{-p}, B^p, $A^{-\infty}$.

5. Let A be any of the spaces A^{-p}, B^p, $A^{-\infty}$ and let $f \in A$. The *ideal generated by f in* A is defined by

$$I(f; A) = \operatorname{clos}\{fP \mid P \in P\}.$$

If f is bounded, then also $I(f; A) = \operatorname{clos}\{fg \mid g \in A\}$.

6. An $f \in A$ is called *cyclic in* A if $I(f; A) = A$.

7. A closed set $E \subset \partial U$ is called a *Carleson set* if its Lebesgue measure $|E| = 0$ and $\Sigma_n |I_n| \log(2\pi/|I_n|) < \infty$, where I_n are the components of $\partial U \backslash E$.

THEOREM. *A singular inner function*

$$s(z) = \exp\{-\frac{\zeta + z}{\zeta - z} dv(\zeta)\},$$

where v is a nonnegative singular measure on ∂U, is cyclic in any (and hence in all) of the spaces $A^{-\infty}$, A^{-p}, B^p if and only if $v(E) = 0$ for all Carleson sets E.

The "only if" part is due to H. S. Shapiro [2]. The case $A^{-\infty}$ was treated in [3]. Some partial results in a different direction are due to Daniel H. Luecking.

Since every A^{-p} is a dense subset of some $B^{p'}$, and vice versa, it suffices to prove the Theorem for A^{-p}. Now we use the following result from [3]; it is, roughly, equivalent to the above Theorem for $A^{-\infty}$.

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PROPOSITION. *Let v be as described in the Theorem. Then there exists a sequence of functions $\{g_m(z)\}_1^\infty$, each belonging to $A^{-\infty}$, such that*

(a) $g_m(z) \neq 0$ $(z \in U; m = 1, 2, \ldots)$.
(b) $h_m = sg_m$ $(m = 1, 2, \ldots)$ *belong to* A^{-N} *for some fixed* $N > 0$.
(c) $\|1 - h_m\|_{-N} \longrightarrow 0$ $(m \longrightarrow \infty)$.

To use this result for A^{-p} we need the following

LEMMA. *If* $g \in A^{-\infty}$, $g(z) \neq 0$ $(z \in U)$, *and* $sg \in A^{-N}$, *then* $sg \in I(s; A^{-N})$.

PROOF OF THE LEMMA. Since $|s(z)| < 1$, we have for $0 < t < 1$,

$$|s(z)(g(z))^t| \leqslant |(s(z)g(z))^t| \quad \text{and thus} \quad sg^t \in A^{-Nt} \quad (0 < t \leqslant 1).$$

Let $F = \{t \mid 0 \leqslant t \leqslant 1, sg^t \in I(s; A^{-N})\}$. F is closed and $0 \in F$. Let $t_0 = \max F$. If $t_0 < 1$, choose a $\delta > 0$ so that $g^\delta \in A^{-(1-t_0)N}$ and a sequence of polynomials $\{P_m\}_1^\infty$ so that $P_m \longrightarrow g^\delta$ in $A^{-(1-t_0)N}$. We have $sg^{t_0}P_m \longrightarrow sg^{t_0+\delta}$ in A^{-N} and, since $sg^{t_0}P_m \in I(s; A^{-N})$, we obtain $sg^{t_0+\delta} \in I(s; A^{-N})$ and thus $t_0 + \delta \in F$. Therefore $t_0 = 1$. Q.E.D.

PROOF OF THE THEOREM. Fix an arbitrary $p > 0$ and show that $l \in I(s; A^{-p})$. By the Proposition and Lemma, $1 \in I(s; A^{-N})$ for some N, i.e., s is cyclic in *some* A^{-N}. Let $k > 1$ be an arbitrary integer. We have $s^{1/k}g_m^{1/k} \longrightarrow 1$ in $A^{-N/k}$, and hence $sg_m^{1/k} \longrightarrow s^{(k-1)/k}$ in $A^{-N/k}$. By the Lemma this implies $s^{(k-1)/k} \in I(s; A^{-N/k})$. For the same reason $s^{(k-1)/k}g_m^{1/k} \longrightarrow s^{(k-2)k} \in I(s; A^{-N/k})$. After k steps we obtain $1 \in I(s; A^{-N/k})$, and thus s is cyclic in $A^{-N/k}$. Since k is arbitrary, s is cyclic in all A^{-p} $(p > 0)$. Q.E.D.

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