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BOOK REVIEWS

Applied and computational complex analysis, by Peter Henrici. John Wiley & Sons, New York, London, Sydney, Toronto, 1974, xviii+682 pp.

General textbooks (I am not speaking of specialized monographs) on complex analysis fall into two broad classes—systematic treatises that develop the subject from the beginning and go more or less far, and (usually) briefer and more intuitive presentations that are oriented toward physical applications. Complex analysis is so vast a subject, and has such close connections to other parts of mathematics [**1**], that it is possible to write an interesting book, or teach an interesting course, on complex analysis without ever mentioning any of its applications, whether to other parts of mathematics or to the real world. It is equally possible to write a book or teach a course that deals primarily with applications of complex analysis to physics, engineering, and so on, without presenting more than a smattering of rigor, certainly with far less than will satisfy a student who has been trained to expect conclusions to follow logically from carefully stated hypotheses. There is, in principle, nothing morally wrong about either approach. The subject possesses many deep and beautiful results, and an author or a teacher may want to concentrate on these. Equally, engineers do study stability of control systems by means of Nyquist diagrams or invert Laplace transforms by residues in order to solve differential equations, just as they differentiate functions or diagonalize matrices, and their students have to learn these things. Intuitive complex analysis is as legitimate as intuitive calculus [**2**], and is useful for many students. For one thing, the students are going to have to read the books in their own fields, and these books are likely to make statements like "Evaluating this integral by contour integration we obtain $\cdots$", or "Inverting the transform by the residue theorem we have $\cdots$". Students are certainly better off if they have an idea, even a hazy one, of what lies behind such casual remarks.

On the other hand, it is reasonable to ask that even an introductory course intended for mathematicians give the students some idea of how the subject is used. Unfortunately many of the interesting applications of complex analysis to subjects like analytic number theory or functional analysis or integral equations are too intricate and involve too many extraneous ideas to be easily presented. Applications to problems in other sciences are often easier to exhibit, since quite a lot can be done with a fairly superficial vocabulary. The motivation for showing applications is not so much that the complex analysis was invented in order to solve applied problems (much of it wasn't), but just that a mathematician ought to know something about what the subject he or she is studying is good for (beyond being something to write papers about). I find it distressing to realize that in my own undergraduate and graduate training (which seemed fine at the time) I never saw an application; looking at some recent texts I deduce that the same thing may still be going on almost half a century later. Although I have never succeeded in making any significant applications of complex analysis, I do at least try to indicate some of its uses in courses that I teach.

Conversely, a book or a course in complex analysis with words like "applied" in its title ought, unless it is intended only to have the depth of an intuitive calculus course, to provide correct statements even if it has no space for complete proofs; many modern short texts do just this although some texts of an older generation must have left the students feeling like Benjamin Peirce confronted by $e^{i\pi}=-1$: "That is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore, we know it must be the truth." [3] (We have made *some* progress in a hundred years.) In addition, a textbook that is to have any excuse for existence ought to offer some variation on the hackneyed problems that are repeated from textbook to textbook and can even be looked up in tables; they are not hackneyed for the student who is seeing them for the first time, of course. However, these problems are not the final word. Strange integrals that are not in the tables do in fact arise in physics and have to be evaluated, new special functions force themselves on scientists and have to be studied, and so on; some students need to learn methods that they can eventually use to solve their own problems. Furthermore, improved versions of various classical techniques have appeared in journals in recent years, although they seem not to have had any noticeable influence on the textbooks [4]. (This is understandable in the light of the survey, a few years ago, that found that more mathematicians were writing than were reading.) It isn't just that students don't learn the best methods; they don't necessarily learn why methods work. It is not uncommon to find a graduate engineer trying to close a contour on the wrong side, or to sum a divergent series by adding up residues, or to neglect an integral that fails to tend to zero, because he has no real idea of why or under what circumstances the techniques work.

Henrici's book is that rare phenomenon, a really original textbook, and one that achieves originality honestly, by choice of topics and attitude, not merely by introducing neologisms. It is not a theoretical presentation in depth; your or my favorite profundity may well be missing; yet it does not at all slight the theoretical subtleties of the topics that it covers.

It is not a recipe book, but it has more details on many technical applications than we will find anywhere else outside the specialized journals (and more are promised for later volumes). Finally, as the word "computational" should suggest, it differs in attitude from even the most "applied" of its predecessors in adhering whenever possible to the principle "not to consider a problem solved unless an algorithm for constructing the solution has been found." By "algorithm" we are to understand, not the word in its broad sense, but an algorithm that produces an answer in a reasonable time. An ordinary classical analyst would (I suppose) be happy to find a solution of a physical problem in the form of a convergent infinite series, without worrying about how fast it converges; Henrici, however, would clearly be unhappy with a soi-disant algorithm that involved so many steps (terms, in this case) that no computer now in existence could produce an acceptably accurate result in a year of computing time. Indeed, throughout the book he points out computational difficulties and gives error estimates. It is this algorithmic principle that gives this book its distinctive character, which I find quite intriguing.

The theory also influences the algorithms, by telling the practitioner to avoid algorithms that theory says won't work, but to pursue those for which theory promises success. Another interesting feature of the algorithmic principle is that it brings back topics that had almost, if not quite, vanished from the mathematical curriculum, such as formulas for calculating the inverse of a function defined by a power series, or the expansion of a rational function in partial fractions. That this should be so is less surprising if we reflect that the old masters who discovered the formulas were interested in actually calculating things. Their algorithms may have fallen into disrepute because they were not practical before the advent of modern calculating machines, but they can be quite practical today.

To summarize briefly: this first volume covers the most basic parts of the theory, the stuff that no book on complex analysis can decently omit; and in addition, conformal mapping and its applications, and (in great detail) the calculation of zeros of polynomials. The second volume is to cover topics connected with ordinary differential equations: special functions, integral transforms, asymptotic formulas, continued fractions. The third volume is to cover connections with partial differential equations: harmonic functions, more conformal mapping, elliptic equations, three-dimensional potential problems. Let us now look at the first volume in more detail.

Pólya [5] once enunciated two rules of style that apply neatly to books on complex analysis. The first rule is to have something to say—in this context, this means to have something to say that has not been said many times before, and the present book easily satisfies this criterion. The second rule is that if you have two things to say you must say one first, and the other only afterwards. This rule forces every author to choose between beginning with Riemann's approach (differentiability) and Weierstrass' (power series). Each approach has its own impassioned advocates. Interestingly enough, Bourbaki and Henrici both come out squarely for power series, presumably for different reasons. Henrici's reason is quite simply that since power series are in effect just sequences of numbers, they are ideal material for being manipulated by digital computers. The emphasis on power series is in any case reasonable enough since (as Hadamard emphasized) the sequence of coefficients contains all conceivable information about the function—it is just a matter of getting it out. Nevertheless, one should not forget that Riemann's stated motive for his approach was precisely to avoid explicit formulas in the interest of generality and unification [6]—and look what he got—but then, one can expect only one Riemann.

Henrici begins by going as far as he can with formal power series and formal Laurent series, purely algebraically, and this is pretty far: he is able to include, for example, some fairly deep results on hypergeometric series as well as the Lagrange-Bürmann theorem (which expands one function in powers of another, and appears here in a particularly transparent form instead of the traditional tangle of symbols). Henrici then discusses functions that are analytic at a point, i.e., represented by a convergent power series in a neighborhood (to begin with, in a general Banach algebra). This gets us as far as the existence of local inverses, the local maximum principle, elementary transcendental functions, and the Weierstrass double series theorem, besides a discussion of matrix-valued functions (to be used in a later volume). Next we have analytic continuation, with (naturally) a workable algorithm for performing analytic continuation (by means of power series) numerically. The existence of a continuously varying $\arg z$ is established, the winding number of a curve is defined, and then the Jordan curve theorem can be proved for piecewise C' curves. Here we have to face up to a problem that does not arise in other books: given that the winding number is defined by an integral, it is nevertheless a number, and we ought to be able to calculate it exactly, since it is an integer. The actual computation is not as simple as one would naïvely suppose, but Henrici is able to provide an algorithm that is guaranteed to work. We then have the residue theorem, Cauchy's integral formula, the evaluation of definite integrals (which Henrici rather slights), and the summation of series (where he goes further than usual, including the now nearly forgotten Plana formula that expresses $\sum_0^\infty f(n)$ by $\int_0^\infty f(x)dx$ plus a rapidly convergent integral). This chapter ends with the argument principle, Rouché's theorem, Hurwitz' theorem, and the continuity of the zeros of a polynomial as functions of its coefficients. From Henrici's point of view the last theorem is rather unsatisfactory, since it does not tell us how continuous the functions are: they can in fact be alarmingly bad (Henrici quotes an example of Wilkinson's where changing one coefficient by about 10^{-7} replaces 10 real zeros by 5 conjugate pairs with rather large imaginary parts). This suggests (although Henrici does not make the point explicitly) that intuition nourished only on the standard fare of undergraduate mathematics is quite likely to generate misleading ideas about the quantitative aspects of continuity and convergence.

The next chapter is about conformal mapping; it is applied to problems in electrostatics, hydrodynamics, and torsional rigidity. The theory is carried up to the statement (without proof) of the Riemann mapping theorem. Schwarz-Christoffel maps are discussed in detail, including the modifications that can be made in order to round off the corners and make the solutions more physically satisfying (a refinement that seems to have been neglected by even the most encyclopaedic of Henrici's predecessors).

Up to this point anyone with previous experience in complex analysis will have been on familiar ground. From here on, however, the scenery changes. The final third of the book is, in fact, a substantial monograph on zeros of polynomials and poles of rational functions, a subject that Henrici obviously loves and one to which he has made many contributions. Here everything is algorithmic, the general theory serving as an essential guide to what is possible. We begin with that discredited bit of college algebra, Horner's method. Although unsatisfactory as an algorithm for locating zeros, it turns out to be efficient for expanding a polynomial in z in powers of $z-a$. We go on to methods, mostly classical, for locating or counting the zeros in particular regions, and then turn to the really hard part, methods for determining one or more zeros numerically with arbitrary accuracy. Several classes of methods are examined for their possession (or not) of desirable properties: Insensitivity to the choice of starting value; independence of special properties of the polynomial; availability of error estimates; uniformity of convergence with respect to polynomials of given degree; rapidity of convergence; simultaneous determination of all zeros; insensitivity to clusters of zeros; insensitivity to idiosyncracies of machine computation. The last chapter is about partial fractions, with initial emphasis on methods for finding them. They are shown to have applications not only to the integration of rational functions (for which, indeed, they may not provide the best method if a numerical answer is wanted), but to combinatorial analysis, difference equations, and interpolation; but they are here principally because they generate further methods of finding zeros of a polynomial $p(z)$: one has "only" to write the partial fraction expansion for $1/p(z)$. This idea can be made useful in various ways, particularly by Rutishauser's *qd* algorithm, which is developed in full detail and applied to entire functions as well as to polynomials. Anyone seeking information on how to calculate zeros will be well advised to consult these two chapters.

Some years ago there was a college president who said that he could accept functional architecture as long as it was "functional for use." This superficially fatuous remark makes, on reflection and considering the vogue use of the adjective, a good deal of sense. In the same vein, Henrici's book is about applied complex analysis for use. By using topics from it, any course that is oriented toward applications can be made more realistic and more useful; an abstract course that nevertheless acknowledges the mundane utility of the subject by including some applications can do so more intelligently. There is surely satisfaction in knowing whether one has to do with a pure existence result or with one that makes it possible to calculate something reasonably accurately in a reasonable amount of time. The algorithmic approach has also led to the formulation of a number of results in simpler or more elegant forms than are usually given. I think that Henrici has shown that his approach has a good deal to contribute to our understanding of complex analysis.

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Buildings of spherical type and finite BN-pairs, by Jacques Tits, Lecture Notes in Mathematics, no. 386, Springer-Verlag, Berlin, Heidelberg, New York, 1974, 299+x pp., $9.90

The relationships between certain algebraic, analytic and geometric structures and root systems in Euclidean spaces have been a source of methods and ideas that have had a profound impact on various parts of mathematics. Some particularly fruitful instances of this interaction are E. Cartan's classification of semisimple Lie algebras over the complex