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SOME THEOREMS ON C-FUNCTIONS

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The purpose of this note is to announce certain results I have obtained about the behavior of the Harish-Chandra C-function as a meromorphic function. The notation and terminology, if not explained, are that of **[2]**, **[3]**, or **[6]**.

1. **The C-ring.** Let (P, A) be a fixed parabolic pair of a semisimple Lie group G having finite center, $P=MAN$ the corresponding Langlands decomposition, K a fixed maximal compact subgroup. Let $\mathscr{G}$, $\mathscr{K}$, $\mathscr{K}_M$, $\mathscr{M}$ be the universal enveloping algebras of G, K, K_M, and M, respectively ($K_M=K\cap M$)—i.e. of their complexified Lie algebras $\mathfrak{g}_C$, $\mathfrak{k}_C$, $\mathfrak{k}_{M,C}$, $\mathfrak{m}_C$. Let $b\to b^\iota$ ($b \in \mathscr{G}$) denote the unique anti-automorphism of $\mathscr{G}$ such that $X^\iota=-X$ ($X\in \mathfrak{g}$). Consider $\mathscr{K}$ to be a right $\mathscr{K}_M$-module via the multiplication in $\mathscr{K}: b\circ d=bd$ ($b\in\mathscr{K}$, $d\in\mathscr{K}_M$), and consider $\mathscr{M}$ to be a left $\mathscr{K}_M$-module via the operation $d\circ c=cd^\iota$ ($d\in\mathscr{K}_M$, $c\in\mathscr{M}$). We can then form the tensor product $\mathscr{K}\otimes_{\mathscr{K}_M}\mathscr{M}$ of $\mathscr{K}_M$-modules. (We write $b\hat{\otimes}c$ for elements of $\mathscr{K}\otimes_{\mathscr{K}_M}\mathscr{M}$, $b\otimes c$ for elements of $\mathscr{K}\otimes\mathscr{M}$.) The group K_M acts on $\mathscr{K}\otimes_{\mathscr{K}_M}\mathscr{M}$ via the (well-defined) representation $\rho:\rho(m)(b\hat{\otimes}c)=b^m\hat{\otimes}c^m$ ($b\in\mathscr{K}$, $c\in\mathscr{M}$, $m\in K_M$). Let $(\mathscr{K}\otimes_{\mathscr{K}_M}\mathscr{M})^{K_M}$ denote the K_M-invariants.

PROPOSITION 1. *$(\mathscr{K}\otimes_{\mathscr{K}_M}\mathscr{M})^{K_M}$ is a ring (i.e., the "obvious" multiplication is well defined). In fact, it is a left and right Noetherian integral domain (noncommutative, in general), hence has a quotient division algebra.*

We refer to $(\mathscr{K}\otimes_{\mathscr{K}_M}\mathscr{M})^{K_M}$ as the C-ring associated to the pair (P, A).

Let τ be a left or double representation of K on a finite-dimensional Hilbert space V. Then there exists a representation λ_τ of the ring $\mathscr{K}\otimes\mathscr{M}$ on $C^\infty(M:V)$ defined as follows:

$$\lambda_\tau(b\otimes c)\psi(m)=\tau(b)\psi(c_i^\iota m)\qquad (b\in\mathscr{K}, c\in\mathscr{M}, m\in M, \psi\in C^\infty(M:V)).$$

Let $C^\infty(M,\tau_M)$ denote the space of $\psi\in C^\infty(M:V)$ such that

$$\tau(k)\psi(m)=\psi(km)\qquad (k\in K_M, m\in M)$$

if τ is a left representation of K or such that

$$\tau(k_1)\psi(m)\tau(k_2)=\psi(k_1mk_2)\qquad (k_1,k_2\in K_M, m\in M)$$

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if τ is a double representation of K on V. Then the rule

$$\lambda_\tau(\sum b_j \hat{\otimes} c_j)\psi(m) = \sum \tau(b_j)\psi(c_{ji}^{\prime}m) \qquad (b_j \in \mathscr{K}, c_j \in \mathscr{M})$$

defines a representation of the C-ring $(\mathscr{K} \otimes_{\mathscr{K}_M} \mathscr{M})^{K_M}$ on $C^\infty(M, \tau_M)$. Clearly the spaces $\mathscr{C}(M, \tau_M)$ and ${}^\circ\mathscr{C}(M, \tau_M)$ of Schwartz functions and cusp forms in $C^\infty(M, \tau_M)$ respectively are invariant subspaces.

2. **The difference equations satisfied by the C-function.** By a polynomial function on a connected simply connected nilpotent Lie group N, we mean a function $f \in C^\infty(N)$ such that $X \to f(\exp X)$ $(X \in L(N))$ is a polynomial function on the Lie algebra $L(N)$ of N.

By a semilattice L in a real vector space V, we mean an additive semigroup generated by a basis of V.

PROPOSITION 2. *There exists a semilattice $L \subseteq \mathfrak{a}^*$ such that $\mu \in L$ implies that $e^{2\mu(H(\bar{n}))}$ is a polynomial function on $\bar{N}$.*

THEOREM 1 (*The difference equations*). *Let $\mu \in \mathfrak{a}^*$ be such that $e^{2\mu(H(\bar{n}))}$ is a polynomial function on $\bar{N}$. Then there exist polynomials $b^\mu(\nu)$, $c^\mu(\nu)$ with coefficients in the C-ring $(\mathscr{K} \otimes_{\mathscr{K}_M} \mathscr{M})^{K_M}$ such that, for all double unitary representations τ of K,*

$$\lambda_\tau(b^\mu(\nu))C_{\bar{P}|P}(1:\nu) = \lambda_\tau(c^\mu(\nu))C_{\bar{P}|P}(1:\nu - 2i\mu) \qquad (\nu \in \mathfrak{a}_C^*).$$

The polynomials $b^\mu(\nu)$ and $c^\mu(\nu)$ have the same degree and the same leading term, which we may assume lies in $\boldsymbol{C}[\nu]$ (i.e., is a scalar polynomial). The coefficients of $c^\mu(\nu)$, in fact, lie in the subring $\mathfrak{Z}_M$ (the center of $\mathscr{M}$) of the C-ring. The operators $\lambda_\tau(b^\mu(\nu))$, $\lambda_\tau(c^\mu(\nu))$ are never identically zero (as polynomials in ν).

Taking $\mu = \mu_1, \cdots, \mu_l$ to be generators of a semilattice L as in Proposition 2, we get the result that the C-function $C_{\bar{P}|P}(1:\nu)$ satisfies a system of $l = rkP$ linear first order partial difference equations with polynomial coefficients.

3. **The asymptotic development.**

THEOREM 2. *Choose $\lambda \in \mathfrak{a}_C^*$ such that* $\mathrm{Re}\langle\lambda, \alpha\rangle > 0$ *for all roots α of the pair (P, A). Then there exists a formal power series $\sum_{j=0}^\infty t^{-j} b_j^{(\lambda)}(\nu)$ with coefficients in $(\mathscr{K} \otimes_{\mathscr{K}_M} \mathscr{M})^{K_M} \otimes \boldsymbol{C}[\nu]$ (depending analytically on λ) such that*

(1) $b_0^{(\lambda)}(\nu) \in \boldsymbol{C}$;

(2) *$b_j^{(\lambda)}(\nu)$ is of degree at most $2j$ in ν $(j \geq 0)$; and*

(3) *for every double representation τ of K,*

$$C_{\bar{P}|P}(1:\nu + it\lambda) \sim t^{-s/2} \sum_{j=0}^{\infty} t^{-j} \lambda_\tau(b_j^{(\lambda)}(\nu)) \quad \text{as } t \to \infty$$

uniformly for ν in compact subsets of $\mathfrak{a}_C^*$ (both sides being considered as operators on the space ${}^\circ\mathscr{C}(M, \tau_M)$). This means that, for each integer $n \geqq 0$,

$$\lim_{t\to\infty} t^n \left| t^{s/2} C_{\bar{P}|P}(1:\nu + it\lambda) - \sum_{j=0}^{n} t^{-j}\lambda_\tau(b_j^{(\lambda)}(\nu)) \right| = 0.$$

Here $s = \dim N$. Replacing τ by the trivial representation of K, we get the same asymptotic expansion for the integral $\underline{C}(\nu) = \int_{\bar{N}} e^{i\nu - \rho(H(\bar{n}))}\, d\bar{n}$.

COROLLARY 1. *Suppose that λ is as in Theorem 2. Then there exists a constant ζ_λ such that*

$$\lim_{t\to\infty} t^{s/2} C_{\bar{P}|P}(1:\nu + it\lambda) = \zeta_\lambda \times \mathrm{id}$$

as an operator on ${}^\circ\mathscr{C}(M, \tau_M)$, the limit being uniform in ν on compact subsets of $\mathfrak{a}_C^*$.

4. The representation theorems.

THEOREM 3. *Choose $\mu \in \mathfrak{a}^*$ such that $\langle \mu, \alpha \rangle > 0$ for all roots α of (P, A) and $e^{2\mu(H(\bar{n}))}$ is a polynomial function on $\bar{N}$. Let $b(\nu) = b^\mu(\nu)$, $c(\nu) = c^\mu(\nu)$ be as in Theorem 1. Then*

$$C_{\bar{P}|P}(1:\nu) = \mathrm{const} \times \lim_{n\to\infty} n^{-s/2}\lambda_\tau(c(\nu + 2i\mu) \cdots c(\nu + 2in\mu))^{-1} \times \lambda_\tau(b(\nu + 2i\mu) \cdots b(\nu + 2in\mu))$$

(the constant being independent of τ).

THEOREM 4. *Let (P, A) be an arbitrary parabolic subgroup of G; and let τ be a double unitary representation of K. Then there exist $\mu_1, \cdots, \mu_r \in \mathfrak{a}^*$ and constants p_{ij}, q_{ij} $(i = 1, \cdots, r, j = 1, \cdots, j_i)$ depending on τ such that*

$$\det C_{\bar{P}|P}(1:\nu) = \mathrm{const} \times \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma(-i\langle \nu, \alpha_i \rangle / 2\langle \mu_i, \alpha_i \rangle + q_{ij})}{\Gamma(-i\langle \nu, \alpha_i \rangle / 2\langle \mu_i, \alpha_i \rangle + p_{ij})},$$

where $\alpha_1, \cdots, \alpha_r$ are the reduced roots of (P, A).

Open Question. Are the numbers p_{ij}, q_{ij} always rational?

5. **Idea of the proofs.** Theorem 1 is based on the following sequence of results.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$; and let P_+ denote the set of roots β of $(\mathfrak{g}_C, \mathfrak{h}_C)$ such that $\beta|\mathfrak{a} > 0$. Let X_β, $X_{-\beta}$ $(\beta \in P_+)$ be root vectors such that $B(X_\beta, X_{-\beta}) = 1$ and $\theta(X_\beta) = -\bar{X}_{-\beta} = -X_{\theta\beta}$.

Define vector fields $q(X)$ $(X \in \mathfrak{g})$ on $\bar{N} = \theta(N)$ by the following rule:

$$q(X)f(\bar{n}) = -\sum_{\beta\in P_+} B(X, X_\beta^{\bar{n}}) f(\bar{n}; X_{-\beta}) \qquad (f \in C^\infty(\bar{N})).$$

PROPOSITION 3. *$X \to q(X)$ defines a representation of $\mathfrak{g}$ by derivations of $C^\infty(\bar N)$. The ring $\mathscr{R}_{\bar N}$ of polynomial functions on $\bar N$ is a q-invariant subspace of $C^\infty(\bar N)$.*

Let $\sum_0 (P, A)=\{\alpha_1, \cdots, \alpha_l\}$ be the simple roots of (P, A); and choose $H_j \in \mathfrak{a}$ such that $\alpha_i(H_j)=\delta_{ij}$.

PROPOSITION 4. *Suppose that $X \in \mathfrak{g}$. Then*

$$e_\nu(X_i x) = \left\{\sum_{j=1}^{l} \langle i\nu - \rho, \alpha_j\rangle B(X, H_j^{k(x)})\right\} e_\nu(x) \qquad (x \in G).$$

(B is the Killing form on $\mathfrak{g}$; $e_\nu(x)=e^{i\nu-\rho(H(x))}$.)

COROLLARY 2. *Suppose that $Z \in \mathfrak{k}$. Then*

$$q(Z)e_\nu(\bar n) = \left\{\sum_{j=1}^{l} \langle i\nu - \rho, \alpha_j\rangle B(Z, H_j^{\bar n})\right\} e_\nu(\bar n) \qquad (\bar n \in \bar N).$$

Let $V_1, \cdots, V_t$ ($t=\dim \mathfrak{m}$) be an orthonormal basis for $\mathfrak{m}_C$ (with respect to the Killing form). Also, given $\psi \in C^\infty(M, \tau_M)$, define $\hat\Psi: G \to C^\infty(M: V)$ by

$$\hat\Psi(x \mid m) = \psi(xm) \qquad (x \in G, m \in M).$$

PROPOSITION 5. *Suppose that $Z \in \mathfrak{k}$ and $\psi \in C^\infty(M, \tau_M)$. Then*

$$\lambda_\tau(Z \otimes 1)\hat\Psi(\bar n) = -q(Z)\hat\Psi(\bar n) - \sum_j B(Z, V_j^{\bar n})\lambda_\tau(1 \otimes V_j)\hat\Psi(\bar n) \qquad (\bar n \in \bar N).$$

PROPOSITION 6. *There exists a unique $\mathscr{M} \otimes C[\nu]$ module homomorphism*

$$F: \mathscr{K} \otimes_{\mathscr{K}_M} \mathscr{M} \otimes C[\nu] \to \mathscr{M} \otimes \mathscr{R}^- \otimes C[\nu]$$

such that

(1) $F(1) = 1$;

(2) $F(\nu|\bar n)(Z) = \sum \langle i\nu + \rho, \alpha_j\rangle B(Z, H_j^{\bar n}) - \sum B(Z, V_j^{\bar n})V_j \qquad (Z \in \mathfrak{k})$;

(3) $F(Zb) = F(b)F(Z) + q(Z)F(b) \qquad (Z \in \mathfrak{k}, b \in \mathscr{K})$;

(4) $F(b \hat\otimes c) = cF(b) \qquad (b \in \mathscr{K}, c \in \mathscr{M})$.

PROPOSITION 7. *Suppose that $b \in \mathscr{K} \otimes_{\mathscr{K}_M} \mathscr{M}$. Then there exists a constant $C=C(b)>0$ such that if $\operatorname{Im}\langle\nu, \alpha_j\rangle \geq C(b)$ $(j=1, \cdots, l)$, then*

$$\lambda_\tau(b)\int_{\bar N} e_\nu(\bar n)\psi(\bar n m)\, d\bar n = \int_{\bar N} e_\nu(\bar n)\lambda_\tau(1 \otimes F(\nu \mid \bar n)(b))\hat\Psi(\bar n \mid m)\, d\bar n \qquad (m \in M),$$

both integrals being convergent.

PROPOSITION 8. *Given $\phi(\bar n) \in \mathscr{R}_{\bar N}$, we can find $b(\nu) \in \mathscr{K} \otimes_{\mathscr{K}_M} \mathscr{M} \otimes C[\nu]$ and $c(\nu) \in \mathfrak{Z}_M \otimes C[\nu]$ such that $F(b(\nu))=c(\nu)\phi$.*

Proof of theorem 1. First apply Proposition 8 with $\phi(\bar{n})=e^{2\mu(H(\bar{n}))}$. Then apply Proposition 7.

The following is the essential step in the proof of Theorem 2.

Proposition 9. *Suppose that $\nu \in \mathfrak{a}_C^*$ and let $f_\nu(\bar{n})=\nu(H(\bar{n}))$ $(\bar{n} \in \bar{N})$. Then if $\langle \nu, \alpha\rangle \neq 0$ for all $\alpha \in \sum (P, A)$, $\bar{n}=e$ is the only critical point of f_ν, and $\bar{n}=e$ is a nondegenerate critical point. Furthermore if $\nu \in \mathfrak{a}^*$ and $\langle \nu, \alpha\rangle > 0$ for all $\alpha \in \sum (P, A)$, then the critical point of the (real-valued) function $f_\nu(\bar{n})$ has index* 0.

Proposition 9 allows us to apply the method of steepest descent (see [**1**]) to derive the asymptotic expansion of $C_{\bar{P}|P}(1:\nu)$ (Theorem 2).

Theorems 3 and 4 follow fairly easily, given Theorems 1 and 2.

6. **An example: the C-function for the group $SU(1, 2)$.** In this case, the set P_+ consists of three roots β_1, β_2 and β_3, where β_1 and β_2 are simple and $\beta_3=\beta_1+\beta_2$. Also, the parabolic pair (P, A) is minimal; so the C-ring is isomorphic to $\mathscr{H}^M$. If $\mu=\alpha$ (the simple root of (P, A)), $e^{2\mu(H(\bar{n}))}$ is a polynomial function on $\bar{N}$; the corresponding polynomials $b^\mu(\nu)$ and $c^\mu(\nu)$ are then as follows

$$b^\mu(\nu) = b_1^\mu(\nu) b_2^\mu(\nu),$$

where

$$b_1^\mu(\nu) = \{(\langle i\nu + \rho, \alpha\rangle - i(\sqrt{6}/6)Z_{\beta_3})(\langle i\nu, \alpha\rangle + \tfrac{1}{2}V) + \tfrac{1}{3}Z_{\beta_1}Z_{\beta}^{2}\},$$
$$b_2^\mu(\nu) = \{(\langle i\nu + \rho, \alpha\rangle - \tfrac{1}{2}V)(\langle i\nu, \alpha\rangle + i(\sqrt{6}/6)Z_{\beta_3}) + \tfrac{1}{3}Z_{\beta_1}Z_{\beta_2}\},$$

and

$$c^\mu(\nu) = \langle i\nu + \rho, \alpha\rangle \langle i\nu + \alpha, \alpha\rangle(\langle i\nu + \rho, \alpha\rangle + \tfrac{1}{2}V)(\langle i\nu + \rho, \alpha\rangle - \tfrac{1}{2}V).$$

Here $Z_{\beta_i}=\frac{1}{2}(X_{\beta_i}+\theta(X_{\beta_i}))$ (X_{β_i} normalized as above), and V is the element of $\mathfrak{m}_C$ such that $\beta_1(V)=\frac{1}{2}$.

Using the polynomials $b^\mu(\nu)$ and $c^\mu(\nu)$, we obtain the following result.

Proposition 10. *Let $\tau=\tau_{m,n}$ be the $(m+1)$-dimensional representation of $K=U(2)$ such that $\tau(V-i\sqrt{6}Z_{\beta_3})=n\times 1$. ($V-i\sqrt{6}Z_{\beta_3}$ spans the center of $\mathfrak{k}_C$.) Let $\mathscr{V}=\mathscr{V}^{(m,n)}$ denote the space of τ. Then $\mathscr{V}$ has a basis x_j $(j=0, 1, \cdots, m)$ such that $\tau(V)x_j=\frac{1}{4}(m+n-2j)x_j$. Furthermore, the operator*

$$C_{\bar{P}|P}^{(m,n)}(1:\nu) = \int_{\bar{N}} \tau(k(\bar{n}))e^{i\nu-\rho(H(\bar{n}))}\,d\bar{n}$$

on $\mathscr{V}^{(m,n)}$ has each vector x_j as an eigenvector; and the corresponding eigenvalue is

$$\frac{2}{\sqrt{\pi}} \frac{\Gamma(\zeta_1)\Gamma(\zeta_2)\Gamma(\zeta_3)\Gamma(\zeta_4)}{\Gamma(\zeta_5)\Gamma(\zeta_6)\Gamma(\zeta_7)\Gamma(\zeta_8)},$$

where $\zeta_1=\zeta=-i\langle\nu,\alpha\rangle/2\langle\alpha,\alpha\rangle$, $\zeta_2=\zeta+\frac{1}{2}$, $\zeta_3=\zeta+\frac{3}{2}j-\frac{3}{4}m-\frac{3}{4}n$, $\zeta_4=\zeta-\frac{3}{2}j+\frac{3}{4}m+\frac{3}{4}n$, $\zeta_5=\zeta+\frac{1}{2}j-\frac{3}{4}m-\frac{3}{4}n$, $\zeta_6=\zeta+\frac{1}{2}j+\frac{1}{4}m-\frac{3}{4}n+1$, $\zeta_7=\zeta-\frac{1}{2}j-\frac{1}{4}m+\frac{3}{4}n$, *and* $\zeta_8=\zeta-\frac{1}{2}j+\frac{3}{4}m+\frac{3}{4}n+1$.

Detailed proofs of these results and some more examples will appear in a paper in preparation.

References

1. M. V. Fedorjuk, *The stationary phase method and pseudo-differential operators*, Uspehi Mat. Nauk **26** (1971), no. 1 (157), 67–112=Russian Math. Surveys **26** (1971), no. 1, 65–111.

2. Harish-Chandra, *Harmonic analysis on semisimple Lie groups*, Bull. Amer. Math. Soc. **76** (1970), 529–551. MR **41** #1933.

3. ———, *On the theory of the Eisenstein integral*, Lecture Notes in Math., vol. 266, Springer-Verlag, Berlin, 1972, pp. 123–149.

4. N. Wallach, *Harmonic analysis on homogeneous spaces*, Marcel Dekker, New York, 1973.

5. ———, *On Harish-Chandra's generalized C-functions*, Amer. J. Math. (to appear).

6. G. Warner, *Harmonic analysis on semi-simple Lie groups*. I, II, Die Grundlehren der math. Wissenschaften, Bände 188, 189, Springer-Verlag, Berlin and New York, 1972.

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