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TOPOLOGICAL SCHUR LEMMA AND RELATED RESULTS

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We announce here some results of a paper to appear elsewhere [1].

Let a torus T act continuously on a topological space X. Let $X \to X_T \to^{\pi} B_T$ be the fibre bundle with fibre X associated (by means of the action of T on X) to the universal principal T bundle $T \to E_T \to B_T$. We define the equivariant cohomology ring $H_T^*(X) = H^*(X_T)$ where H^* denotes Čech cohomology with rational coefficients. When Y is an invariant subspace of X, we define $H_T^*(X, Y) = H^*(X_T, Y_T)$. Then $R = H^*(B_T)$ is a polynomial ring and $H_T^*(X, Y)$ is a module over R by means of π^*.

For each subtorus L of T let PL be the kernel of $H^*(B_T) \to H^*(B_L)$. Let $X^L = F(L, X)$ be the set of points fixed by L. We will assume that X is compact. Given a closed invariant subspace $Y \subset X$ and an element $x \in H_T^*(Y)$, we define

$$I_x = \{a \in R \mid ax \text{ lies in the image of } H_T^*(X) \to H_T^*(Y)\}, \quad \text{and}$$

$$I_x^L = \{a \in R \mid ax \text{ lies in the image of } H_T^*(X^L \cup Y) \to H_T^*(Y)\}.$$

When $L \subset K$ are subtori, $I_x \subset I_x^L \subset I_x^K$. We say that K *belongs to* x if K is maximal with respect to the property $I_x^K \neq R$.

1. THEOREM. *The isolated primary components of the ideal I_x are the ideals I_x^K where K belongs to x. The radical of I_x^K is PK, hence $\sqrt{I_x} = \bigcap PK$ where K ranges over the subtori belonging to x.*

2. COROLLARY. *If I_x is principal, the subtori belonging to x are all of corank 1 and $I_x = \bigcap I_x^K$ where K ranges over the subtori belonging to x. For each such K, $I_x^K = (\omega^d)$ where $d \geqq 1$ and $\omega \in H^2(B_T)$ generates PK.*

Assume that the fixed point set F of the T action on X is not connected. Let $F = F^1 + \cdots + F^s$ be the connected components of the fixed point set, $s \geqq 2$. We say that a subtorus L connects F^1 and F^2 if they lie in the same component of X^L. We assume that dim $H^*(X)$ is finite.

3. THEOREM. *Let $N \subset H_T^*(X)$ be the ideal generated by odd degree and R torsion elements. Assume that $H_T^*(X)/N$ is generated by k elements as an R algebra. Then for every maximal subtorus K connecting F^1 and F^2,* rank $K \geqq$ rank $T - k$.

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4. REMARK. This generalizes a result of Hsiang [3] that F is connected whenever $H_T^*(X)$ is generated as an R algebra by odd degree and R torsion elements.

The following proposition is a technical result related to a theorem of Golber [2].

5. PROPOSITION. *Assume that* $\dim H^*(X) = \dim H^*(F) < \infty$. *Let* $S = \{x \in X \mid \operatorname{rank} T_x \geqq \operatorname{rank} T - 1\}$. *Then the homomorphism* $H_T^*(X, F) \to H_T^*(S, F)$ *is injective.*

We use the notation $X \sim Y$ to indicate that there is an isomorphism of rational cohomology rings $H^*(X) = H^*(Y)$. When $X \sim S^{k_1} \times \cdots \times S^{k_n}$ where the k_i are odd integers, we define $e(X)$ to be the second symmetric polynomial $\sum_{i<j} (k_i + 1)(k_j + 1)$. If $\dim H^*(X) = \dim H^*(F)$, we know that $X^L \sim S^{d_1} \times \cdots \times S^{d_n}$ where the d_i are odd integers, for every subtorus L of T [3]. Hence $e(X^L)$ is defined. Further we define $g(X) = e(X) - e(F) - \sum_L [e(X^L) - e(F)]$ where L ranges over the corank 1 subtori. For each subtorus H of corank 2, we define $g(X^H)$ by using the induced T/H action on X^H.

6. PROPOSITION. $g(X) = \sum_H g(X^H)$ *where H ranges over the corank* 2 *subtori.*

7. REMARK. Golber [2] has proved that $g(X) = \sum g(X^H)$ when $X \sim S^{k_1} \times S^{k_2}$ where the k_i are odd, and $F = \varnothing$.

When X is a compact rational cohomology manifold and $F = F^1 + \cdots + F^s$ are the components of the fixed point set, let f_i be a generator of the top dimensional cohomology group of F^i. After including $f_i \in H^*(F^i) \subset H^*(F) \subset H_T^*(F)$, we can define the ideal I_{f_i}. The following result was conjectured by Hsiang. It is a kind of splitting principle or Schur lemma for torus actions.

8. THEOREM. *The ideal I_{f_i} is principal with a generator of degree* $\dim X - \dim F^i$. *This generator splits as a product of linear factors in R corresponding to the subtori belonging to f_i.*

Here $n = \dim X$ means that $H^n(X)$ is the top dimensional nonzero cohomology group of X. We do an explicit computation of I_{f_i} when $X \sim$ quaternionic projective n space [1].

9. REMARK. Theorem 8 holds for torus actions on Poincaré duality spaces. It also holds for actions of p-tori on Poincaré duality spaces over Z_p. The Borel formula (see [3]) also holds for such actions [5].

Theorem 8 yields the following result of Hsiang and Su [4].

10. THEOREM. *When X is a compact rational cohomology manifold and $X \sim QP^n$, quaternionic projective n space, and a torus of rank $\geqq 2$ acts*

effectively on X, the fixed point set has at most one component $\sim QP^k$ with $k \geqq 1$.

The results announced here also hold for actions of p-tori using Z_p cohomology.

References

1. T. Chang and T. Skjelbred, *The topological Schur lemma and related results*, Ann. of Math. (to appear).

2. D. Golber, *Torus actions on a product of two odd spheres*, Topology **10** (1971), 313–326. MR **44** #1024.

3. Wu-yi Hsiang, *On some fundamental theorems in cohomology theory of topological transformation groups*, Taita J. Math. **2** (1970), 66–87.

4. Wu-yi Hsiang and J. C. Su, *On the geometric weights system of topological actions on cohomology quaternionic projective spaces*, Taipei-Berkeley-Amherst (mimeographed).

5. T. Skjelbred, Thesis, University of California, Berkeley, Calif., 1972.

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