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INCIDENCE ALGEBRAS AS ALGEBRAS OF ENDOMORPHISMS

BY WILLIAM GRAVES AND SUZANNE MOLNAR

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1. **Introduction.** The order filters on a locally finite partially ordered set P constitute the open sets for a topology on P. A sheaf of abelian topological groups will be constructed on the topological space P, and the endomorphism ring of this sheaf will be proved isomorphic to the incidence algebra of P (over $\mathbf{Z}$).

2. **A sheaf of abelian groups on** P. Let P be a locally finite (every interval $[x, y]$ of P is finite) partially ordered set. An (order) filter on P is a subset V of P which contains y whenever $x \leqq y$ and $x \in V$. For $x \in P$,

$$V_x = \{y \in P : x \leqq y\}$$

is the principal filter generated by x. The filters on P are easily seen to be the open sets for a topology on P, and the increasing maps from P to another locally finite partially ordered set Q are precisely the continuous functions from P to Q [**5**].

For each filter V on P, let $M(P, V)$, or simply $M(V)$ when reference to P is understood, denote the free abelian group on V. For filters $U \subseteq V$, let $r(V, U): M(V) \to M(U)$ be the group homomorphism determined by

$$\begin{aligned} x &\mapsto x \quad \text{if } x \in U, \\ x &\mapsto 0 \quad \text{if } x \notin U, \end{aligned}$$

for $x \in V$.

PROPOSITION 1. *M (with the restriction maps $r(V, U)$) is a sheaf of abelian groups on P.*

PROOF. M is easily seen to be a presheaf of abelian groups. For any open cover $V = \bigcup V_i$, consider

$$M(V) \xrightarrow{\pi} \prod M(V_i) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \prod M(V_k \cap V_j)$$

where π is induced by the restrictions $r(V, V_i)$, π_1 is induced by the restrictions $r(V_k, V_k \cap V_j)$, and π_2 is induced by the restrictions $r(V_j, V_k \cap V_j)$. That π is injective is clear. Let $\alpha = (\alpha_i) \in \prod M(V_i)$ where $\alpha_i = \sum \alpha_{i,x} x$, and suppose that $\pi_1(\alpha) = \pi_2(\alpha)$. Then $\alpha_{k,x} = \alpha_{j,x}$ for any $x \in V_k \cap V_j$. So

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$\pi(\beta) = \alpha$ where $\beta = \sum \beta_x x$ and $\beta_x = \alpha_{i,x}$ for any i such that $x \in V_i$. Hence, M is a sheaf.

For each filter V of P, let $A(V)$ be the commutative ring with basis V of orthogonal idempotents. So $A(V) = M(V)$ as abelian groups, and $A(P)$ is the Möbius algebra (over $\mathbf{Z}$) studied in [**2**], [**3**], [**4**], [**7**], and [**8**]. It is easy to see that the restriction maps introduced above are ring homomorphisms.

COROLLARY 1. *A (with the restriction maps $r(V, U)$) is a sheaf of rings on P.*

An endomorphism, T, of the sheaf M is a natural transformation of M considered as a contravariant functor. That is, $T = \{T(V)\}$ consists of group homomorphisms $T(V): M(V) \to M(V)$, one for each filter V on P, which commute with the restriction maps $r(V, U)$.

If $x \nleqq y$ in P, then $r(P, V_x)(y) = 0$. So if $T(P)(y) = \sum T(P)(x, y)x$, then it follows from $r(P, V_x) \circ T(P) = T(V_x) \circ r(P, V_x)$ that $T(P)(x, y) = 0$ if $x \nleqq y$. It now follows easily that

THEOREM 1. *The* association $T \mapsto T(P)$ *is an injective ring homomorphism of the endomorphism ring,* $\mathrm{End}(M)$, *of the sheaf M to the incidence algebra, $\mathscr{I}(P)$, of P (over $\mathbf{Z}$).*

P is lower finite if for each $y \in P$, the set $\{x \in P : x \leqq y\}$ is finite.

COROLLARY 2. $\mathrm{End}(M) \simeq \mathscr{I}(P)$ *if and only if P is lower finite.*

3. **A topology on** $M(P)$. Subsets S of P satisfying

$$(*) \qquad \{x \leqq y : x \notin S\} \text{ is finite for every } y \in P$$

will be called $(*)$-sets. In particular, every cofinite subset of P is a $(*)$-set. For each $(*)$-set S, let $N(S)$ be the subgroup of $M(P)$ generated by S. Since $S \cap R$ is a $(*)$-set whenever both of S and R are, $N(S) \cap N(R) = N(S \cap R)$. So the collection $\mathscr{B} = \{N(S) : S \text{ is a } (*)\text{-set}\}$ is a base for a filter of neighborhoods of 0 determining the structure of abelian topological group on $M(P)$ [**1**, III. 1.2].

PROPOSITION 2. *This topology on $M(P)$ is discrete if and only if P is lower finite.*

PROOF. This topology on $M(P)$ is discrete if and only if $\{0\} \in \mathscr{B}$ if and only if $\varnothing$ is a $(*)$-set if and only if P is lower finite.

PROPOSITION 3. *This topology on $M(P)$ is Hausdorff.*

PROOF. For $x \in P$, let $S_x = P - \{x\}$. Then S_x is a $(*)$-set, and $\bigcap N(S_x) = \{0\}$.

A subset R of P is said to be bounded above finitely provided there exists a finite subset F of P such that $x \in R$ implies $x \leqq y$ for some $y \in F$.

PROPOSITION 4. *P is bounded above finitely if and only if every $(*)$-set is cofinite.*

PROOF. Suppose P is bounded above finitely by the finite subset F. For each $(*)$-set S and each $y \in F$, the set $\{x \leqq y : x \notin S\}$ must be finite. Hence, S is the complement of a finite subset of P.

Conversely, assume that every $(*)$-set is cofinite. Then V_x is finite for every $x \in P$ since by local finiteness the complement of V_x is a $(*)$-set. So P is upper finite and admits an antichain U such that for every $x \in P$ there exists $y \in U$ with $x \leqq y$. But the complement of U is a $(*)$-set. Hence U is finite.

Let $\hat{M}(P)$ be the completion of $M(P)$ as abelian topological group. The summability [**1**, III. 5.2] in $\hat{M}(P)$ of families $\{\alpha_x x : x \in P, \alpha_x \in \boldsymbol{Z}\}$ will be discussed.

THEOREM 2. *If $\{x : \alpha_x \neq 0\}$ is bounded above finitely, then $\sum \alpha_x x$ exists in $\hat{M}(P)$.*

PROOF. Let $R = \{x : \alpha_x \neq 0\}$. It suffices [**1**, III. 5.2] to find, for every $(*)$-set S, a finite subset J of R such that $R - J \subseteq S$. But there is a finite subset F of P such that $x \in R$ implies $x \leqq y$ for some $y \in F$. Set

$$J = \{x \in R : x \notin S\}$$

which is finite since, for each $y \in F$, $\{x \leqq y : x \notin S\}$ is finite.

PROPOSITION 5. *If $\sum \alpha_x x$ exists in $\hat{M}(P)$, then for each $y \in P$ there are at most finitely many $x \geqq y$ with $\alpha_x \neq 0$.*

PROOF. For each $y \in P$, the complement of V_y is a $(*)$-set.

The converse to Proposition 5 is not true. Let N denote the negative integers in their natural order. For each positive integer i, let $P_i = N$, and let P be the disjoint union of $\{P_i\}$ with the induced order. That is, P is the coproduct in the category of partially ordered sets and increasing maps of the family $\{P_i\}$. For each $x \in P$, let α_x be 1 or 0 according as $x = -1 \in P_i$ for some i or not. Then for each $y \in P$ there is exactly one $x \in P$ with $x \geqq y$ and $\alpha_x \neq 0$. But $\sum \alpha_x x$ does not converge since

$$\{x \in P : x \leqq -2 \text{ in some } P_i\}$$

is a $(*)$-set which does not meet $\{x \in P : \alpha_x \neq 0\}$.

CONJECTURE. *$\sum \alpha_x x$ exists if and only if $\{x : \alpha_x \neq 0\}$ is bounded above finitely.*

Let $\tilde{M}(P)$ be the subgroup of $\hat{M}(P)$ consisting of all sums of all families $\{\alpha_x x : x \in P, \alpha_x \in \mathbf{Z}\}$ such that $\{x : \alpha_x \neq 0\}$ is bounded above finitely. So $M(P) \subseteq \tilde{M}(P) \subseteq \hat{M}(P)$, and, for example, $\sum_{x \leqq y} x$ is in $\tilde{M}(P)$ for each $y \in P$.

4. **A sheaf of abelian topological groups on** P. For each filter V on P, $M(V)$ is a subgroup of $M(P)$ and is therefore an abelian topological group (with the subspace topology). Let $\hat{M}(V)$, $\tilde{M}(V)$ be the closures of $M(V)$ in $\hat{M}(P)$, $\tilde{M}(P)$ respectively. It is easy to see that the restriction maps $r(V, U)$ are continuous (in fact, they are strict morphisms, [**1**, III. 2.8]). Hence, there are induced restriction maps $\tilde{r}(V, U) : \tilde{M}(V) \to \tilde{M}(U)$ for filters $U \subseteq V$ on P. It follows that

PROPOSITION 6. $\tilde{M}$ *(with the restriction maps* $\tilde{r}(V, U)$*) is a sheaf of abelian topological groups on* P.

Let $T \in \mathscr{I}(P)$. Consider the association $\sum \alpha_y y \to \sum \alpha_y T(x, y)x$. Since $\sum \alpha_y y = 0$ implies $\alpha_y = 0$ for all y, this association defines a map

$$\tilde{T} : \tilde{M}(P) \to \tilde{M}(P).$$

$\tilde{T}$ is a group homomorphism. To show that $\tilde{T}$ is continuous, it suffices to show that for any $(*)$-set S there exists a $(*)$-set S' such that $\tilde{T}(x) \in \tilde{N}(S)$ for every $x \in S'$ ($\tilde{N}(S)$ is the closure of $N(S)$ in $\tilde{M}(P)$). But the set $S' = \{s \in S : x \leqq s \text{ implies } x \in S\}$ is a $(*)$-set (by local finiteness) satisfying this condition. From Theorem 1 and topological considerations it then follows that

THEOREM 2. *The incidence algebra,* $\mathscr{I}(P)$, *of* P *(over* $\mathbf{Z}$*) is isomorphic to* $\operatorname{End}(\tilde{M})$, *the endomorphism ring of the sheaf* $\tilde{M}$ *of abelian topological groups on* P.

5. **Increasing maps and maps of sheaves.** Consider the category whose objects are locally finite partially ordered sets P along with the sheaf $\tilde{M} = \tilde{M}(P, \cdot)$ of abelian topological groups on P and whose morphisms are continuous maps $f : P \to Q$ (continuous = increasing) along with a morphism $\tilde{f}$ of the sheaf $\tilde{M}(Q, \cdot)$ on Q to the sheaf $\tilde{M}(P, \cdot) \circ f^{-1}$ on Q. So, for the increasing map $f : P \to Q$ and each filter V on Q,

$$\tilde{f}(V) : \tilde{M}(Q, V) \to \tilde{M}(P, f^{-1}(V))$$

is a continuous group homomorphism, and these maps commute with the restriction maps $\tilde{r}(V, U)$ for filters $U \subseteq V$ on Q. In particular, for an increasing map $f : P \to Q$ such that, for each $q \in Q$, $\{p \in P : f(p) = q\}$ is bounded above finitely, $\tilde{f}$ can be defined as follows: for each filter V on Q,

$$\tilde{f}(V) : \tilde{M}(Q, V) \to \tilde{M}(P, f^{-1}(V))$$

is determined by $q \mapsto \sum_{f(p)=q} p$ for $q \in V$. Many of the results concerning Möbius inversion admit simple proofs in this setting. For example, the following result due to Rota [**6**] has been elegantly proved by Greene [**4**] in its finite form in the context of the Möbius algebra and here is proved in its most general form using only the continuous linearity of $\tilde{f}$.

THEOREM 3 (ROTA). *Let $\sigma: P \to Q$, $\tau: Q \to P$ be increasing maps of locally finite partially ordered sets satisfying*

(1) $\tau(\sigma(p)) \geqq p$ for all $p \in P$;

(2) $\sigma(\tau(q)) \leqq q$ for all $q \in Q$.

Then for $p \in P$ and $q \in Q$ with $\sigma(p) \leqq q$,

$$\sum_{p \leqq x; \sigma(x)=q} \mu(P)(p, x) = \sum_{y \leqq q; \tau(y)=p} \mu(Q)(y, q) \quad \text{if } \tau(\sigma(p)) = p,$$
$$= 0 \qquad\qquad\qquad\qquad \text{if } \tau(\sigma(p)) > p.$$

PROOF. Here, $\mu(P)$ denotes the Möbius function of $\mathscr{I}(P)$; that is, the (continuous) inverse of the continuous linear automorphism of $\tilde{M}(P)$ determined by $p \mapsto \delta(P, p) = \sum_{x \leqq p} x$ and denoted $\zeta(P)$ (the zeta function of P). Hence,

$$p = \sum_{x \leqq p} \mu(P)(x, p)\delta(P, x). \tag{1}$$

Since $\sigma(x) \leqq y$ if and only if $x \leqq \tau(y)$, it follows that $\tilde{\sigma}(\delta(Q, q)) = \delta(P, \tau(q))$ for any $q \in Q$ where $\tilde{\sigma} = \tilde{\sigma}(P)$. So from (1),

$$\tilde{\sigma}(q) = \sum_{y \leqq q} \mu(Q)(y, q)\delta(P, \tau(y)).$$

But from definition of $\tilde{\sigma}$ and (1),

$$\tilde{\sigma}(q) = \sum_{p \leqq x; \sigma(x)=q} \mu(P)(p, x)\delta(P, p).$$

The theorem follows from comparison of the coefficients of $\delta(P, p)$ in each of these expressions for $\tilde{\sigma}(q)$.

6. **Remarks.** Local finiteness was used only in Proposition 4 and was given only to shed light on the concept of $(*)$-set and the related conjecture on summability. So the construction of the sheaf M does not depend on local finiteness. This suggests letting $\mathrm{End}(\tilde{M})$ be the incidence algebra of an arbitrary partially ordered set. This definition and a more detailed account of the results given here will appear later.

The subgroups $N(S)$, S a $(*)$-set, of $M(P)$ are ideals of the Möbius algebra structure $A(P)$ on $M(P)$. Thus, $\tilde{M}$ can be considered as a sheaf $\tilde{A}$ of topological rings, and $\tilde{A}(P)$ is a continuous generalization of the Möbius algebra construction (over $\boldsymbol{Z}$).

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Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27514