BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

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SOME INEQUALITIES FOR UNIFORMLY BOUNDED DEPENDENT VARIABLES[1]

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Communicated by Jack Feldman, July 7, 1972

1. **Introduction.** In this note, I would like to state some inequalities, with an indication of proof. I hope to publish a more detailed treatment elsewhere.

A sum of uniformly bounded variables tends to be near the sum of the conditional expectations given the past; large deviations are exponentially unlikely, as noted in §2. The inequalities give Lévy's conditional Borel-Cantelli lemmas and his strong law as corollaries. They extend inequalities of Bernstein, Chernoff, and Hoeffding [**3**] to the dependent case; Hoeffding has a review of the literature.

If you study a sum of uniformly bounded variables, such that each has conditional expectation 0 given the past, and the sum of the conditional variances given the past is bounded, then large deviations are exponentially unlikely, as noted in §3. This inequality can be used to prove Lévy's law of the iterated logarithm for dependent variables. It makes explicit

AMS (*MOS*) *subject classifications* (1970). Primary 60F10, 60F15, 60F05, 60G45.

Key words and phrases. Martingales, crossing times, tail probabilities, Borel-Cantelli lemmas, strong law of large numbers, law of the iterated logarithm, central limit theorem, normal approximation, Poisson approximation.

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, USAF, under Grant AFOSR-71-2100. The United States Government is authorized to reproduce and distribute reprints for Govermental purposes notwithstanding any copyright notation hereon.

a bound in [**1**], and extends an inequality of Kolmogorov to the dependent case.

§5 concerns the Poisson approximation for dependent events. Suppose you study a sequence of dependent events, such that each has uniformly small conditional probability given the past. Suppose you stop when the sum of the conditional probabilities is near a. Then the number of events which occur is approximately Poisson with parameter a. An explicit bound for the variation distance is given, which extends an inequality of Hodges and LeCam [**2**] to the dependent case.

Throughout this note, $(\Omega, \mathfrak{F}, P)$ is a probability triple, $\mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots$ are sub-σ-fields of $\mathfrak{F}$, and τ is a stopping time: τ takes the values $0, 1, \ldots, \infty$; and $\{\tau = n\} \in \mathfrak{F}_n$ for $n = 0, 1, \cdots$. As a final convention, $\exp x = e^x$.

2. **On conditional means.**
that $0 \leqq X_i \leqq 1$ and X_i is $\mathfrak{F}_i$-measurable. Let $M_i = E(X_i | \mathfrak{F}_{i-1})$.

(1) For $0 \leqq a \leqq b$,

$$P\left\{\sum_{i=1}^{\tau} X_i \leqq a \text{ and } \sum_{i=1}^{\tau} M_i \geqq b\right\} \leqq \left(\frac{b}{a}\right)^a e^{a-b} \leqq \exp\left[-\frac{(a-b)^2}{2b}\right].$$

(2) For $0 \leqq b \leqq a$,

$$P\left\{\sum_{i=1}^{\tau} X_i \geqq a \text{ and } \sum_{i=1}^{\tau} M_i \leqq b\right\} \leqq \left(\frac{b}{a}\right)^a e^{a-b} \leqq \exp\left[-\frac{(b-a)^2}{2a}\right].$$

These two inequalities have, as a corollary, Lévy's conditional form of the Borel-Cantelli lemmas and strong law:

(3a) $$\sum_{i=1}^{\infty} X_i < \infty \quad \text{a.e. on} \left\{\sum_{i=1}^{\infty} M_i < \infty\right\},$$

(3b) $$\sum_{i=1}^{\infty} X_i = \infty \quad \text{a.e. on} \left\{\sum_{i=1}^{\infty} M_i = \infty\right\},$$

(4) $$\lim_{n\to\infty} \left(\sum_{i=1}^{n} X_i\right) \Big/ \left(\sum_{i=1}^{n} M_i\right) = 1 \quad \text{a.e. on} \left\{\sum_{i=1}^{\infty} M_i = \infty\right\}.$$

To prove (1) and (2), let $-\infty < h < \infty$. Confirm that $R_h(m, x) = \exp[hx - (e^h - 1)m]$ is excessive: $R_h(m, x) \geqq E[R_h(m + M, x + X)]$ for $0 \leqq m, x < \infty$, and variables X with $0 \leqq X \leqq 1$ and $E(X) = M$.

As a by-product,

(5) If $\sum_{i=1}^{\tau} M_i \leqq b$ a.e., and $h \geqq 0$, then

$$E\left\{\exp\left[h \sum_{i=1}^{\tau} X_i\right]\right\} \leqq \exp[b(e^h - 1)],$$

the Poisson generating function. This bound is sharp.

3. **On the conditional variance.** Let $Y_1, Y_2, \cdots$ be random variables, such that $|Y_i| \leqq 1$ and Y_i is $\mathfrak{F}_i$-measurable and $E(Y_i|\mathfrak{F}_{i-1}) = 0$. Let $V_i = E(Y_i^2|\mathfrak{F}_{i-1})$.

(6) For nonnegative a and b,

$$P\left\{\max_{j \leqq \tau} \sum_{i=1}^{j} Y_i \geqq a \text{ and } \sum_{i=1}^{\tau} V_i \leqq b\right\}$$
$$\leqq \left(\frac{b}{a+b}\right)^{a+b} e^a \leqq \exp\left[-\frac{a^2}{2(a+b)}\right].$$

To prove this, let $\lambda > 0$, and $e(\lambda) = e^\lambda - 1 - \lambda$. Confirm that $R_\lambda(v, y) = \exp[\lambda y - e(\lambda)v]$ is excessive: $R_\lambda(v, y) \geqq E\{R_\lambda(v + V, y + Y)\}$ for $v \geqq 0$, $-\infty < y < \infty$, and variables Y with $|Y| \leqq 1$, $E(Y) = 0$, and $E(Y^2) = V$.

This inequality is stronger than (2). It has, as a corollary, two other results of Lévy:

(7) $$\sum_{i=1}^{\infty} Y_i \text{ converges a.e. on } \left\{\sum_{i=1}^{\infty} V_i < \infty\right\}.$$

(8) $$\limsup_{n \to \infty} \sum_{i=1}^{n} Y_i \Big/ \left[2\left(\sum_{i=1}^{n} V_i\right) \log\log\left(\sum_{i=1}^{n} V_i\right)\right]^{1/2} \leqq 1$$
$$\text{a.e. on } \left\{\sum_{i=1}^{\infty} V_i = \infty\right\}.$$

Relations (7–8) are sharper than (3–4).

For $a > 0$, let τ_a be the least n if any with $Y_1 + \cdots + Y_n \geqq a$, and $\tau_a = \infty$ if none. Let $W_a = \sum_{i=1}^{\tau_a} V_i$. So W_a is the intrinsic time to reach a. Let $\varepsilon > 0$. There is a positive, finite number $c = c(\varepsilon)$ such that

(9) If $b/a > c$, and $a^2/b > c$, then

$$P\{W_a \leqq b\} \geqq \exp\left[-(1 + \varepsilon)\frac{a^2}{2b}\right].$$

To prove this, let $\lambda \geqq 0$ and let $f(\lambda) = e^{-\lambda} - 1 + \lambda$. Confirm that

$$R_\lambda(v, y) = \exp[\lambda y - f(\lambda)v]$$

is defective: $R_\lambda(v, y) \leqq E\{R_\lambda(v + V, y + Y)\}$ for $v \geqq 0$, $-\infty < y < \infty$, and variables Y with $|Y| \leqq 1$, $E(Y) = 0$, and $E(Y^2) = V$. In particular,

(10) $$E\{\exp[-f(\lambda)W_a]\} \geqq \exp[-\lambda(a + 1)].$$

So $P\{W_a > b\} < 5(a + 1)/b^{1/2}$ for $a > 0$ and $b > 1$.

This proves the companion results to (7) and (8), which are also due to Lévy:

$$\sum_1^\infty Y_i \text{ diverges a.e. on } \left\{\sum_1^\infty V_i = \infty\right\} \tag{11}$$

$$\limsup_{n\to\infty} \sum_1^n Y_i \Big/ \left[2\left(\sum_1^n V_i\right)\log\log\left(\sum_1^n V_i\right)\right]^{1/2} \geqq 1 \quad \text{a.e. on } \left\{\sum_{i=1}^\infty V_i = \infty\right\}. \tag{12}$$

The same arguments give upper and lower bounds for $E[\exp \lambda S_\tau]$, when $P\{b \leqq \sum_1^\tau V_i \leqq b'\}$ is 1 and b'/b is near 1: the conclusion being Lévy's central limit theorem, that $S_\tau/b^{1/2}$ is essentially normal with mean 0 and variance 1.

If $P\{\sum_1^\infty V_i = \infty\} = 1$, then

$$E\{\exp[-e(\lambda)W_a]\} \leqq \exp[-\lambda a]. \tag{13}$$

Consequently, for any $\varepsilon > 0$, there is a positive, finite number $c = c(\varepsilon)$ such that

(14) If $a > c$ and $b/a^2 > c$, then $(b^{1/2}/a)P\{W_a > b\}$ is between $(1 \pm \varepsilon)(2/\pi)^{1/2}$.

4. **Relaxing the boundedness conditions.** Result (3a) holds for any sequence X_i of nonnegative variables, such that X_i is $\mathfrak{F}_i$-measurable. Result (7) holds for any sequence of variables Y_i such that Y_i is $\mathfrak{F}_i$-measurable and $E(Y_i|\mathfrak{F}_{i-1}) = 0$.

To extend (3b) and (4), let

$$L(t) = \sup_\omega \sup_{n \leqq \sigma_t(\omega)} |X_n(\omega)|,$$

where σ_t is the sup of the nonnegative integers n if any with $M_1 + \cdots + M_n \leqq t$. Suppose X_i is nonnegative and $\mathfrak{F}_i$-measurable. Then (3b) holds if $L(t) = O(t)$ as $t \to \infty$. And (4) holds if $L(t) = o(t/\log\log t)$ as $t \to \infty$.

To extend (11) and (8/12), let

$$L(t) = \sup_\omega \sup_{n \leqq \sigma_t(\omega)} |Y_n(\omega)|,$$

where σ_t is the least n if any with $V_1 + \cdots + V_n \geqq t$, and $\sigma_t = \infty$ if none. Suppose Y_i is $\mathfrak{F}_i$-measurable, and $E(Y_i|\mathfrak{F}_{i-1}) = 0$ a.e. Then (11) holds if $L(t) = o(t^{1/2})$ as $t \to \infty$. And (8/12) holds if $L(t) = o(t^{1/2}/(\log\log t)^{1/2})$, the classical condition for the independent case.

Unbounded variables can be studied by the usual method of truncation.

5. **On the Poisson approximation.** If N and N^* are nonnegative, integer-valued random variables, let

$$d(N, N^*) = \frac{1}{2} \sum_{n=0}^{\infty} |P(N = n) - P(N^* = n)|.$$

Let $X_1, X_2, \cdots$ be random variables, taking only the values 0 and 1. Let X_i be $\mathfrak{F}_i$-measurable, and let $p_i = P\{X_i = 1 | \mathfrak{F}_{i-1}\}$. Let $0 \leqq a \leqq b$. Let $0 \leqq \delta \leqq 1$, and $0 \leqq \varepsilon \leqq 1/100$. Suppose

$$P\left\{a \leqq \sum_{i=1}^{\tau} p_i \leqq b \text{ and } \sum_{i=1}^{\tau} p_i^2 \leqq \varepsilon\right\} \geqq 1 - \delta.$$

Let $N = \sum_{i=1}^{\tau} X_i$, and let N^* be Poisson with parameter a. Then

$$d(N, N^*) \leqq 9\varepsilon/8 + 2\delta + b - a. \tag{15}$$

For the proof, you can embed the partial sums $X_1, X_1 + X_2, \cdots$ in a Poisson process, so $X_1 + \cdots + X_\tau$ is essentially the process at time a.

You might compare (15) with (5).

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