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CONSTRUCTING ISOTOPIES IN NONCOMPACT 3-MANIFOLDS[1]

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Introduction. Let M be a noncompact, orientable 3-manifold with a (possibly empty) boundary ∂M. Suppose g and h are homeomorphisms of M onto itself. When is g isotopic to h? This question was essentially answered in the compact case by Waldhausen in [**3**]; roughly the answer given was—when g is homotopic to h. We will show that essentially the same answer can be given for a large and interesting class of noncompact manifolds; these manifolds include Whitehead-type contractible open subsets of R^3. Full proofs of the theorems stated below will be given elsewhere.

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Preliminaries. The ambient manifolds considered here are orientable, triangulable and 3-dimensional. By a *surface in M*, we mean a 2-dimensional, triangulable manifold which is properly imbedded in M. (Everything is considered from the piecewise linear point of view.) M is an *irreducible manifold* if every 2-sphere in M bounds a ball in M. For noncompact manifolds this implies that M is aspherical. A surface F in M or ∂M different from a 2-sphere is *incompressible* in M if $\pi_1(F) \to \pi_1(M)$ is a monomorphism. M is *boundary-irreducible* if each component of ∂M is an incompressible surface. Finally we need the notion of a *hierarchy* for a manifold. The triple $(F_j, U(F_j), M_j), j = 1, 2, \ldots$, is a hierarchy for $M = M_1$ if each F_j is a compact incompressible orientable surface in M_j, $M_{j+1} = \mathrm{cl}(M_j - U(F_j))$, where $U(F_j)$ is a regular neighborhood of F_j in M_j [**4**], and $M - \bigcup_j \mathring{U}(F_j)$ is a collection of balls. If M is compact we require the sequence F_j to be finite. For M compact these surfaces have been constructed by Haken when M is irreducible and has an incompressible surface. Waldhausen uses the hierarchy to prove the isotopy theorem in the compact case.

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Noncompact manifolds admitting a hierarchy are called *end-irreducible*; they have been introduced by E. M. Brown, for manifolds of any dimension using his notions of proper fundamental groups. It can be seen from [**1**] that the name end-irreducible is appropriate in that it generalizes the idea of boundary-irreducible to ends of a manifold. Moreover any irreducible manifold which is obtained from a compact 3-manifold by removing some incompressible boundary components is an end-irreducible manifold.

Results for irreducible, end-irreducible manifolds. If the manifold M has vacuous boundary, our result asserts that any orientation-preserving homeomorphism of an irreducible and end-irreducible manifold which is *homotopic* to the identity homeomorphism is *isotopic* to the identity. Notice that since M is aspherical we are saying that any orientation-preserving homeomorphism which induces the "identity" map on $\pi_1(M)$ is isotopic to the identity homeomorphism. More precisely we prove the following two isotopy theorems.

THEOREM 1. *Let M be an irreducible, end-irreducible manifold and $H:(M \times I, \partial M \times I) \to (M, \partial M)$ be a homotopy of an orientation-preserving homeomorphism h to the identity. Then h is isotopic to the identity. [If $H|\partial M \times I$ is already the constant homotopy, the isotopy of h to the identity may be chosen fixed on ∂M.]*

THEOREM 2. *Let M be an irreducible, end-irreducible, and boundary-irreducible manifold. Assume $\partial M \neq \emptyset$. Suppose $H:M \times I \to M$ is a homotopy of an orientation-preserving homeomorphism h to the identity and suppose that H is a proper map when restricted to each component of $\partial M \times I$. Then h is isotopic to the identity.*

(By a *proper map* we mean that the inverse image of compact sets are compact.) In the above theorems we only use the fact that h is orientation-preserving for manifolds that

(a) are bundles with fiber R over closed surfaces, (in Theorem 1),

(b) are bundles with fiber I over open surfaces, (in Theorem 2) or

(c) when some component of ∂M is a plane or an open annulus, (in Theorem 1).

To see that in Theorem 2 it is necessary to assume H is a proper map when restricted to components of $\partial M \times I$ we look at the following example. Let M be a solid torus with two disjoint longitudinal curves removed from ∂M. Then ∂M consists of two open annuli. Let h be the homeomorphism of M which rotates the torus so that the boundary components are interchanged. Then h is homotopic to the identity but not isotopic to the identity.

Other examples of end-irreducible manifolds will be given in the next section.

An outline of the proof of Theorem 1 for a manifold M with vacuous boundary. In [1], it is shown that M has an exhausting sequence $\{C_n\}$ of submanifolds with the following properties:

(1) C_n is a compact, connected manifold,

(2) $C_n \subset \mathring{C}_{n+1}$ and $\bigcup_n C_n = M$,

(3) components of ∂C_n are incompressible.

Now one can show that $\{C_n\}$ may be chosen so that

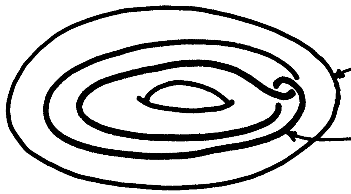
(4) $H(C_n \times I) \subset \mathring{C}_{n+1}$.

We show by an inductive procedure that if $h|C_n$ is the identity and $H|C_n \times I$ is the constant homotopy, then there is an isotopy of h fixed on C_n so that the new homeomorphism (which we will still call h) is the identity on C_{n+1}. Moreover we show that one can change the homotopy to be constant on C_{n+1}. Let $F_1, \ldots, F_k$ be the components of ∂C_{n+1}. We first show that since F_1 and $h(F_1)$ are incompressible surfaces which are homotopic via H in C_{n+2}, there is an isotopy of M fixed on C_n which carries $h(F_1)$ onto F_1. (We use here that if M is a product bundle, then h is assumed orientation-preserving.) Changing h by this isotopy—still call the homeomorphism h—we now have $h(F_1) = F_1$.

The fact that this isotopy can be chosen fixed on C_n rests heavily on the fact that $H|C_n \times I$ is already constant. Thus we now try to change $H|F_1 \times I$ by a homotopy to the constant homotopy. First we attempt to homotope $H|F_1 \times I$ rel $F_1 \times \partial I$ to a map into F_1. We show the only difficulties arise when M is a bundle over a closed, nonorientable manifold. Here we again use the assumption that h is orientation-preserving. Next we show that unless F_1 is a torus, $H|F_1 \times I$ is homotopic rel $F_1 \times \partial I$ to the constant homotopy. If F_1 is a torus it may be necessary to change $h|F_1$ by an isotopy before one can homotope $H|F_1 \times I$ to the desired state. We continue—by induction on the number of components in ∂C_{n+1}—to change h by an isotopy fixed on C_n and to change $H|\partial C_{n+1} \times I$ by a homotopy so that in the end, $h|\partial C_{n+1}$ is the identity and $H|\partial C_{n+1} \times I$ is the constant homotopy.

Since $\mathrm{cl}(C_{n+1} - C_n)$ is a manifold with boundary, it admits a hierarchy $G_1, \ldots, G_r$ (see [2]). Again we inductively isotope h and homotope H without disturbing our previous changes so that the end result is that h is the identity on all the G_i. Hence h is the identity on C_{n+1} except for a collection of 3-cells. Using Alexander's Theorem, we conclude h is isotopic to the identity on C_{n+1}; using the fact that M is aspherical we have no obstructions to homotoping H to the constant homotopy. This concludes the argument.

Results for irreducible, eventually end-irreducible manifolds. M is *eventually end-irreducible* if there is a compact subset C of M such that $M - C$ is end-irreducible, or equivalently that M eventually has a hierarchy, i.e., $M - C$ has a hierarchy. For the isotopy results for such manifolds the following example shows that it is necessary that we assume the homeomorphism h is proper homotopic to the identity. Let T_0 be a solid torus linked in the solid torus T_1. (See Figure below.)



Let $h: R^3 \to R^3$ be a homeomorphism of R^3 such that $h(T_0) = T_1$. Then $W = \bigcup h^i(T_0)$ is the contractible open subspace of R^3 described by Whitehead in [**5**]. By results in [**6**], one can show that W is an eventually end-irreducible manifold. The homeomorphism h maps W onto itself. Moreover since W is contractible h is homotopic to the identity. Again the results in [**6**] show that h is not proper homotopic to the identity and hence it certainly is not isotopic to the identity.

THEOREM 3. *Let M be an irreducible, eventually end-irreducible manifold and $H:(M \times I, \partial M \times I) \to (M, \partial M)$ a proper homotopy of a homeomorphism h to the identity. Then h is isotopic to the identity. If $H | \partial M \times I$ is the constant homotopy, then the isotopy of h to the identity may be chosen fixed on ∂M.*

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