ON REPRESENTATIONS ASSOCIATED WITH SYMMETRIC SPACES

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1. **Introduction.** The results of [2] on the structure of the symmetric algebra or universal enveloping algebra over a complex reductive Lie algebra, as a module for the adjoint group, are generalized to the symmetric space case. In particular one obtains a separation of variables theorem (freeness over the ring of invariants). Also multiplicities of the various representations are given as well as the degrees of the homogeneous subspaces on which they occur. We use the notation of [3].

2. **Sections and the principal normal TDS.**

2.1. Now if $\mathfrak{u}$ is a principal normal TDS in $\mathfrak{g}$ then one can show that any nonzero element in $\mathfrak{u}\cap\mathfrak{s}$ is necessarily regular. In fact up to a scalar it is K-conjugate to the unique element $w\in\mathfrak{g}$ defined by the relations

(1) $w\in\mathfrak{a}\cap[\mathfrak{g},\mathfrak{g}]$ and

(2) $\langle w,\alpha_i\rangle=1$ where $\{\alpha_1,\cdots,\alpha_d\}=\Sigma$ is the set of simple roots. In particular w can be embedded in a principal normal TDS $\mathfrak{u}$.

Let such a $\mathfrak{u}$ be fixed. Then $\mathfrak{u}$ has as a basis a principal normal S-triple (x,e,f) where $w=(e+f)/2$.

Now let $\tilde{\mathfrak{g}}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{a}$ and $\mathfrak{u}$.

THEOREM 1. *$\tilde{\mathfrak{g}}$ is a reductive Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{a}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$. (Also $\tilde{\mathfrak{g}}$ is semisimple in case $\mathfrak{g}$ is semisimple). Moreover the roots $\Delta\subseteq\mathfrak{a}'$ of $\tilde{\mathfrak{g}}$ is exactly the subset $\Lambda^1\subseteq\Lambda$ of all restricted roots $\phi\in\Lambda$ such that $\phi/2$ is not a root. Furthermore the Weyl group of* $(\tilde{\mathfrak{g}},\mathfrak{a})$ *is just the Weyl group of $\mathfrak{g}$ associated with $\mathfrak{a}$* ("baby Weyl group").

Finally $\mathfrak{u}$ is a principal TDS of $\tilde{\mathfrak{g}}$ in the sense of [1].

REMARK 1. If $\mathfrak{u}$ is chosen so that $\mathfrak{u}$ is the complexification of $\mathfrak{u}\cap\mathfrak{g}_R$ (and it can be so chosen) then $\tilde{\mathfrak{g}}$ is the complexification of $\tilde{\mathfrak{g}}_R=\tilde{\mathfrak{g}}\cap\mathfrak{g}_R$ and $\tilde{\mathfrak{g}}_R$ is the normal real form of $\tilde{\mathfrak{g}}$. That is $\tilde{\mathfrak{g}}$ is defined and split over $\boldsymbol{R}$. (It is a maximal such subalgebra of $\mathfrak{g}$.)

2.2 Now in [2] a cross-section was found for the set of all regular elements (called quasi-regular in [2] but henceforth called regular since the case at hand here generalizes the case in [2]). In fact by Theorem 8 in [2] $f+\tilde{\mathfrak{g}}^e$ is a cross-section of the set of all regular elements in $\tilde{\mathfrak{g}}$. On the other hand if we consider the complex Cartan decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} + \tilde{\mathfrak{p}}$ where $\tilde{\mathfrak{k}} = \mathfrak{k} \cap \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}} = \mathfrak{p} \cap \tilde{\mathfrak{g}}$ then since $\mathfrak{a}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$ it follows that $\tilde{\mathfrak{g}}^e = \tilde{\mathfrak{p}}^e$ and that furthermore if $x \in \tilde{\mathfrak{p}}$ then x is regular in $\tilde{\mathfrak{p}}$ if and only if it is regular in $\tilde{\mathfrak{g}}$. But one can also show that $\tilde{\mathfrak{p}}^e = \mathfrak{p}^e$ (so that $f + \tilde{\mathfrak{g}}^e = f + \tilde{\mathfrak{p}}^e$) since x is regular in $\tilde{\mathfrak{p}}$ if and only if it is regular in $\mathfrak{p}$. Applying the results of [2], (see Theorem 8) we can prove

THEOREM 2. *Let (x, e, f) be any principal normal S-triple. Then $f + \mathfrak{p}^e$ is a cross-section for the set of all regular elements in $\mathfrak{p}$. That is any element in $f + \mathfrak{p}^e$ is regular and every regular element in $\mathfrak{p}$ is K_θ-conjugate to one and only one element in $f + \mathfrak{p}^e$. Furthermore the affine space $f + \mathfrak{p}^e$ has dimension r ($= \dim \mathfrak{a}$) and if $S'(f + \mathfrak{p}^e)$ is the affine algebra of this manifold then the map $J' \to S'(f + \mathfrak{p}^e)$ (where $J' = (S')^{K_\theta}$) given by restriction is an algebra isomorphism. In particular where $J' = \mathbf{C}[u_1, \cdots, u_r]$ one has that the differentials du_i, $i = 1, 2, \cdots, r$ are linear independent for every regular element $x \in \mathfrak{p}$.*

2.3. As a consequence of the last statement of Theorem 2 we can apply the theory of §1 in [2] to determine the K_θ-module structure of S'. By Proposition 3 in [3] the ideal $J'_+ S'$ defines the variety $\mathfrak{N} \subseteq \mathfrak{p}$. But more than that one has

THEOREM 3. *The ideal $J'_+ S'$ in S' is radical and hence is the ideal associated with the variety $\mathfrak{N}$ of all nilpotent elements in $\mathfrak{p}$ (i.e. $J'_+ S' = \{f \in S' \mid f(x) = 0$ for all $x \in \mathfrak{N}\}$).*

Now let S be the symmetric algebra over $\mathfrak{p}$ regarded as a K_θ-module in the obvious way. Let $J = S^{K_\theta}$ and let J_+ be the ideal in J of all elements without constant term. To each element $v \in S$ one naturally associates a differential operator ∂_v on $\mathfrak{p}$ with constant coefficients. A polynomial $f \in S'$ on $\mathfrak{p}$ is called harmonic in case $\partial_v f = 0$ for all $v \in J_+$. Let $H' \subseteq S'$ be the space of all harmonic polynomials on $\mathfrak{p}$. Clearly H' is a K_θ-submodule. Theorem 3 then yields

THEOREM 4. *The map $J' \otimes H' \to S'$ defined by $u \otimes f \to uf$ is a K_θ-isomorphism.*

2.4. Now if $x \in \mathfrak{p}$ then the orbit $O_x = K_\theta \cdot x$ is an algebraic variety and if $R(O_x)$ denotes the ring of all everywhere defined rational functions, then $R(O_x)$ is a K_θ-module where if $g \in R(O_x)$, $a \in K_\theta$, and $y \in O_x$, one has $(a \cdot g)(y) = g(a^{-1} \cdot y)$. The correspondence $f \to f \mid O_x$ clearly defines a K_θ-map $S' \to R(O_x)$ and hence by restriction to H' a K_θ-map $H' \to R(O_x)$.

Since J' reduces to the constants on O_x it follows that the image of H' is the same as the image of S'. But if O_x is closed then the latter is onto. Hence by Theorems 4 in [3] and 4 here one has

PROPOSITION 1. *If* $x \in \mathcal{S}$ *one has a surjection*

$$H' \to R(O_x) \to 0.$$

On the other hand, we can prove the following theorem which plays an important role in applications to representation theory.

THEOREM 5. *If* $x \in \mathcal{R}$ *one has an injection*

$$0 \to H' \to R(O_x).$$

Now let M_θ be the centralizer of $\mathfrak{a}$ in K_θ.

It is then immediate that if $x \in \mathfrak{a}$ is regular (i.e. $\phi(x) \neq 0$ for all roots $\phi \in \Lambda$) then the isotropy subgroup of K_θ at x is M_θ so that $O_x \cong K_\theta / M_\theta$. But now let Γ be the set of all equivalence classes of irreducible holomorphic finite dimensional K_θ-modules V such that $V_\gamma^{M_\theta} \neq 0$. For each $\gamma \in \Gamma$ fix a module V_γ in the class γ and let $l(\gamma) = \dim V_\gamma^{M_\theta}$. One then knows that $R(O_x) \cong R(K_\theta/M_\theta)$ is completely reducible as a K_θ-module and only elements in Γ occur and that in fact $\gamma \in \Gamma$ occurs with multiplicity $l(\gamma)$. But now $x \in \mathcal{S} \cap \mathcal{R}$ and hence by Proposition 1 and Theorem 5 H' is isomorphic to $R(O_x)$ as a K_θ-module yielding the K_θ-module structure of H' and hence of S' by Theorem 4. But S is equivalent to S' as a K_θ-module and consequently one can prove

THEOREM 6. *Let S be the symmetric algebra over $\mathfrak{p}$ and let $J = S^{K_\theta}$. Next let $H \subseteq S$ be the subspace spanned all power x^k where $x \in \mathfrak{p}$ is nilpotent. Then where $\otimes$ is realized by multiplication one has $S = J \otimes H$. Furthermore H is a K_θ-module and for any $\gamma \in \Gamma$ let H_γ be the space of all elements $v \in H$ which transform under K_θ according to γ. Then*

$$H = \oplus_{\gamma \in \Gamma} H_\gamma$$

and in fact γ occurs with multiplicity $l(\gamma)$ $(=\dim V_\gamma^{M_\theta})$ in H_γ (so that in particular H_γ is finite dimensional).

2.5. Now S is more than just a K_θ-module. It is a graded K_θ-module. Since one knows the graded structure of J to determine the latter it is enough to determine the graded K_θ-structure of H. Let $\gamma \in \Gamma$. Although H_γ is, in general, not uniquely a direct sum on irreducible K_θ-modules there clearly exists a unique monotonic sequence of increasing nonnegative integers $d_i(\gamma)$, $i = 1, 2, \cdots, l(\gamma)$, such that there exists an irreducible K_θ-module $H_{\gamma,i} \subseteq H_\gamma$, homogeneous of degree $d_i(\gamma)$ and such that $H_\gamma = \sum_{i=1}^{l(\gamma)} H_{\gamma,i}$.

Generalizing Theorem 17 of [2] we will see that the integers $d_i(\gamma)$ can be obtained from the abstract K_θ-module V_γ in the manner now to be described.

Now $H' \subseteq S'$ is a space of polynomials on p. Let $F_\gamma = \mathrm{Hom}_{K_\theta}((V_\gamma)', H')$ where $(V_\gamma)'$ is the dual to V_γ. But since H' and H are nonsingularly paired it follows that F_γ is an $l(\gamma)$-dimensional vector space.

Now, for each $x \in p$, we define a linear map

$$\beta_x : F_\gamma \to V_\gamma$$

by the relation

$$\langle \beta_x(\sigma), g \rangle = \sigma(g)(x)$$

holding for all $g \in (V_\gamma)'$ and $\sigma \in F_\gamma$. Hence to each $x \in \mathfrak{p}$ we can associate a subspace $V_\gamma(x) \subseteq V_\gamma$ by putting $V_\gamma(x) = \beta_x(F_\gamma)$.

As a consequence of Theorem 5 one has

THEOREM 7. *For any $x \in \mathcal{R}$ the map β_x is a linear isomorphism so that $V_\gamma(x)$ is an $l(\gamma)$-dimensional subspace of V_γ. Moreover if $x \in \mathcal{R} \cap \mathcal{S}$ then $V_\gamma(x) = V_\gamma^{K_\theta^x}$ where K_θ^x is the centralizer of x in K_θ and $V_\gamma^{K_\theta^x}$ is the space of K_θ^x invariants in V_γ.*

REMARK 3. In [2] one had that $V_\gamma(x) = V_\gamma^{K_\theta^x}$ for all $x \in \mathcal{R}$. This was true because in [2] the orbit of any $x \in \mathcal{R}$ was a normal variety. In the case at hand this is not true and one can only conclude that $V_\gamma(x) \subseteq V_\gamma^{K_\theta^x}$, the latter having possibly a larger dimension than $l(\gamma)$. One notes, however, that one can obtain $V_\gamma(x)$ for all $x \in \mathcal{R}$ from $V_\gamma(x)$ where $x \in \mathcal{R} \cap \mathcal{S}$ (in which case $V_\gamma(x) = V_\gamma^{K_\theta^x}$) since $\mathcal{R} \cap \mathcal{S}$ is dense in $\mathcal{R}$ and the map $x \to V_\gamma(x)$ is continuous from $\mathcal{R}$ into the Grassmannian of all $l(\gamma)$ dimensional subspaces of V_γ.

For the purpose of obtaining the integers $d_i(\gamma)$ we need only consider $V_\gamma(e)$ where e is a principal nilpotent element (i.e., $e \in \mathcal{R} \cap \mathfrak{N}$). One knows (Proposition 3) that for any nilpotent element $f \in p$ there exists $z \in \mathfrak{k}$ such that $[z, f] = f$. Now of course V_γ is a $\mathfrak{k}$-module so that z operates on V_γ.

THEOREM 8. *Let $\gamma \in \Gamma$ be arbitrary and let $e \in \mathfrak{p}$ be a principal nilpotent element so that $V_\gamma(e) \subseteq V_\gamma$ is an $l(\gamma)$-dimensional subspace. Let $z \in \mathfrak{k}$ be any element such that $[z, e] = e$. (Such an element exists by Proposition 3.) Then $V_\gamma(e)$ is stable under the action of z. Moreover z is diagonalizable on $V_\gamma(e)$ and the eigenvalues are exactly the nonnegative integers $d_i(\gamma)$, $i = 1, 2, \cdots, l(\gamma)$, defined above describing the graded structure of H_γ.*

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Remark (i) at the end of Theorem 1 is incorrect. E is a normal family, but it need not be compact. The conclusion of part (i) of Theorem 2 should read

$$\liminf_{p\to\infty}(n_1 n_2 \cdots n_p)^{1/n_p} \leqq 4R.$$