CYCLOTOMIC IDEALS IN GROUP RINGS

BY S. BACHMUTH AND H. Y. MOCHIZUKI

Communicated by Walter Feit, May 5, 1966

Let G be a group of exponent n and $Z(G)$ the integral group ring of G. We call the ideal of $Z(G)$, which is generated by all elements of the form $1+u+u^2+ \cdots +u^{n-1}$ where u runs through all elements of the group G (including 1), the cyclotomic (or Burnside) ideal $\mathfrak{J}(n)$. The motivation for the study of these ideals is the close connection of $\mathfrak{J}(n)$ with the Burnside group $F/R'F^n$ where $G=F/R$ is represented as a factor group of a free group, R' is the commutator subgroup of R, and F^n is the group generated by nth powers of elements of F. Specifically, one considers $RF^n/R'F^n$ as a module over $Z_n(F/RF^n)$ coupled with the observation that for u in $RF^n/R'F^n$ and x in F/RF^n,

$$u^{1+x+\cdots+x^{n-1}} = u(xux^{-1})(x^2ux^{-2}) \cdots (x^{n-1}ux^{-n+1}) = (ux)^n.$$

We hope this announcement helps to initiate a study of the quotient rings $Z(G)/\mathfrak{J}(n)$. We restrict ourselves to the commutative case, where G is a direct product of two cyclic groups of order n. We denote by Σ the augmentation ideal of $Z(G)$; i.e., the ideal generated by $(1-g)$ for all g in G. Throughout, the letter F will always mean a free group of rank 2.

THEOREM 1.
(i) $\Sigma^{p-1}=\mathfrak{J}(p)$, $(\Sigma^{p-2}\not\subset\mathfrak{J}(p))$, *for any prime* p.
(i)i $\Sigma^4\subseteq\mathfrak{J}(4)$, $(\Sigma^3\not\subset\mathfrak{J}(4))$.
(iii) $\Sigma^{12}\subseteq\mathfrak{J}(9)$, $(\Sigma^{11}\not\subset\mathfrak{J}(9))$.

The application to the Burnside groups is the following:

COROLLARY 1.
(i) $F/F''F^p$ *has nilpotency class* $p-1$.
(ii) $F/F''F^4$ *has nilpotency class* 4 *or* 5.
(iii) $F/F''F^9$ *has nilpotency class* 12 *or* 13.

Part (i) of Corollary 1 is due to Meier-Wunderli [1], and Part (i) of Theorem 1 follows easily from his results. In general, our aim is to establish bounds on the nilpotency class of $F/F''F^n$ in the case where $n=p^e$ is a power of a prime, and in the case where n is a composite, to determine where the lower central series becomes stationary. Since we can show the latter is determined once one knows the nilpotency class of $F/F''F^{p_i^e}$ for each prime power p_i^e dividing n, we will restrict our attention to the case where $n=p^e$ for some prime p.

Theorem 1 provides, as far as our methods are concerned, the best possible information where the nilpotency class is determined within 1. (For the lower bounds we make use of the representation of R/R' as a $Z(F/R)$ module as given by Magnus [2].) To get this much information in the general situation seems difficult at the moment, but to get an error of p appears to be relatively easy. For example, in the case $p^3=8$ and $p^3=27$, it is easy to establish $\Sigma^{13}\subseteq\mathfrak{I}(8)$ $(\Sigma^{11}\not\subseteq\mathfrak{I}(8))$ and $\Sigma^{56}\subseteq\mathfrak{I}(27)$ $(\Sigma^{53}\not\subseteq\mathfrak{I}(27))$ which translates as follows: $F/F''F^8$ has nilpotency class at most 14 and at least 12, while $F/F''F^{27}$ has nilpotency class at most 57 and at least 54. However, the previous work makes one strongly suspect that $\Sigma^{54}\subseteq\mathfrak{I}(27)$ and $\Sigma^{12}\subseteq\mathfrak{I}(8)$. In the case of arbitrary p^2, we have shown the following:

THEOREM 2. *$\Sigma^{2p^2-p-1}\subseteq\mathfrak{I}(p^2)$ for arbitrary prime p.*

COROLLARY 2. *$F/F''F^{p^2}$ has nilpotency class at most $2p^2-p$ and at least $2p^2-2p$.*

For $p^2=4$, 9, Theorem 1 gives sharper information, and in general, we believe the following to be the case.

CONJECTURE 1. $\Sigma^{e(p^e-p^{e-1})}\subseteq\mathfrak{I}(p^e)$ for any prime p and integer $e\geqq 1$.

$$(\Sigma^{e(p^e-p^{e-1})-1}\not\subseteq\mathfrak{I}(p^e)).$$

A computation which yields results similar to Theorem 2 could be made for arbitrary p^e, but we have not done so yet. We can at this time state the following:

THEOREM 3. *For x in G, $(1-x)^{e(p^e-p^{e-1})}$ is in $\mathfrak{I}(p^e)$.*

COROLLARY 3. *$F/F''F^{p^e}$ has Engel length $e(p^e-p^{e-1})-1$ or $e(p^e-p^{e-1})$.*

COROLLARY 4. *$F/F^{(k)}F^{p^e}$ has Engel length $\leqq(k-1)e(p^e-p^{e-1})$.*

Here $F^{(k)}$ denotes the kth term of the derived series of F. The impetus for much of our work and for which Conjecture 1 would be an immediate corollary is

CONJECTURE 2. Let ω be a primitive p^e root of unity and $Z_{p^e}[\omega]\cong Z_{p^e}[x]/\theta_{p^e}(x)$, where $\theta_{p^e}(x)$ is the irreducible polynomial which ω satisfies. Let k be the kernel of a homomorphism of $Z(G)$ into $Z_{p^e}[\omega]$. Let K be the intersection of all such kernels. Then $K=\mathfrak{I}(p^e)$.

We have been able to prove Conjecture 2 only in simplest situations (e.g., $p=2$, 3, 5) and even in the simplest cases, where Z_{p^e} contains nilpotent elements (e.g., $p^e=4$), a great deal of complexity is introduced.

Added in proof (September 1966). Conjecture 1 has been proved by Professor H. Heilbronn. Thus it follows that $F/F''F^{p^e}$ has nilpotency class $e(p^e-p^{e-1})$) or $e(p^e-p^{e-1})+1$.

References

1. H. Meier-Wunderli, *Metabelsche Gruppen*, Comment. Math. Helv. **25** (1951), 1–10.

2. W. Magnus, *On a theorem of Marshall Hall*, Ann. of Math. (2) **40** (1939), 764–768.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA