ON THE REPRESENTATION PROBLEM FOR STATIONARY STOCHASTIC PROCESSES WITH TRIVIAL TAIL FIELD[1]

BY D. L. HANSON

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Let $\{X_n\}$ be a real valued strictly stationary stochastic process on the probability space (Ω, Σ, P) and let $\{\xi_n\}$ be an independent sequence of random variables uniformly distributed on $[0, 1]$ where $n=0, \pm 1, \cdots$. When does there exist a function f on the sequence $\{\xi_n\}$ such that the sequences $\{X_n\}$ and $\{f(T^n\xi)\}$ have the same probability structure where $\xi=(\cdots, \xi_{-1}, \xi_0, \xi_1, \cdots)$ and $T\xi=(\cdots, \xi_0, \xi_1, \xi_2, \cdots)$ (i.e. such that the joint distribution of $X_{i_1}, \cdots, X_{i_k}$ is the same as the joint distribution of $f(T^{i_1}\xi), \cdots, f(T^{i_k}\xi)$ for all k and all sequences $i_1, \cdots, i_k$)?

Let Σ_n be the smallest σ-field of subsets of Ω with respect to which X_k is measurable for all $k\leqq n$ and let $\Sigma_{-\infty}=\bigcap\Sigma_n$. $\Sigma_{-\infty}$ is called the tail field of the process $\{X_n\}$ and is said to be trivial if $A\in\Sigma_{-\infty}$ implies $P(A)=0$ or 1. It has been shown (see [1] and [2]) that if $\{X_n\}$ is a stationary Markov chain with a denumerable state space and whose tail field is trivial then a representation of the above type holds and in fact $f(T^n\xi)=f(\cdots, \xi_{n-1}, \xi_n)$.[2]

By use of a fairly simple transformation an arbitrary stationary process $\{X_n\}$ with trivial tail field can be converted to a stationary Markov process $\{Y_n\}$ with trivial tail field and from which the $\{X_n\}$ process can be recovered. Thus the seeming preoccupation with Markov processes.

The following theorem generalizes Rosenblatt's results to a class of Markov process with nondenumerable state space. $\overline{P}$ is the stationary measure induced by the process on the state space and $P_X(A')$ is the stationary conditional probability that $X_n\in A'$ given $X_{n-1}=X$.

THEOREM. *Let* $\{X_n\}$, $n=0, \pm 1, \cdots$ *be a real stationary Markov process such that*

(i) $\Sigma_{-\infty}$ *is trivial.*

(ii) *There exist Borel subsets A and B of the state space and a nonnegative measure ϕ on the state space such that* $\overline{P}(B)>0$, $\phi(A)>0$, *and for all* $X\in B$ *and* $A'\subset A$ *we have* $P_X(A')\geqq\phi(A')$.

[1] This work was performed under the auspices of the United States Atomic Energy Commission.

[2] A stationary Markov chain with denumerable state space has a trivial tail field if and only if it is ergodic and aperiodic.

Then if $\{\xi_n\}$ is an independent sequence of random variables uniformly distributed $[0, 1]$ there exists a function $g = g(\cdots, \xi_{-1}, \xi_0)$ such that the sequences $\{X_n\}$ and $\{g(\cdots, \xi_{n-1}, \xi_n)\}$ have the same probability structure.

COROLLARY 1. *In the above theorem it is sufficient to replace condition* (ii) *with*

(iia) *The state space of* $\{X_n\}$ *has an atom under the stationary probability* $\overline{P}$.

COROLLARY 2. *If* $\{X_n\}$ *is a stationary ergodic aperiodic Markov chain with a denumerable state space then conditions* (i) *and* (ii) *hold and the above theorem is true.*

Detailed proofs will appear elsewhere.

REFERENCES

1. M. Rosenblatt, *Stationary processes as shifts of functions of independent random variables*, J. Math. Mech. 8 (1959), 665–681.

2. ———, *Stationary Markov chains and independent random variables*, J. Math. Mech. 9 (1960), 945–949.

SANDIA CORPORATION