APPROXIMATION IN UNIFORM NORM BY SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

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Introduction. Let G be an open subset of the Euclidean n-space E^n, G_1 an open subset with compact closure in G. If $n=2$ and G is the whole of E^2, an important circle of theorems in the theory of analytic functions associated with the names of Walsh, Hartogs-Rosenthal, Lavrentiev, Keldych, and Mergelyan deals with the possibility of approximating analytic functions on G_1 continuous on its closure, uniformly on G_1 by polynomials in the complex variable z. Mergelyan's theorem [1], the most general of these results, asserts that if $\bar{G}_1$ does not disconnect E^2, then every such analytic function is uniformly approximable by polynomials on $\bar{G}_1$. More generally, if we replace $\bar{G}_1$ by any compact subset K of E^2, Mergelyan's result asserts that if K does not disconnect E^2, then every continuous function on K which is analytic at every interior point of K is uniformly approximable on K by polynomials in z. In view of the classical theorem of Runge on uniform approximation of analytic functions on compact subsets of G_1 by polynomials, Mergelyan's theorem is equivalent to the assertion that each function $f(z)$ which is continuous on K and analytic in the interior of K may be approximated uniformly on K by functions analytic on a prescribed open set G containing K in its interior.

From the point of view of differential equations, the class of analytic functions is merely the class of solutions of the homogeneous first-order linear elliptic differential equation with constant complex coefficients:

$$\frac{\partial u}{\partial \bar{z}} = 0,$$

where $\partial/\partial\bar{z}$ is the classical Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$$

in the plane. The existence of theorems of the Walsh-Lavrentiev-Mergelyan type for the Cauchy-Riemann operator raises the question of possible generalizations of such results for solutions of general

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linear elliptic differential equations. A significant step in this direction is the generalized Runge theorem due to Lax [9] and Malgrange [10] which asserts that if A is a linear elliptic operator with analytic coefficients on G and if G_1 does not disconnect G, then every solution of $Au=0$ in G_1 can be uniformly approximated on any compact subset of G_1 to any degree of closeness (together with its derivatives to any finite order) by solutions of $Au=0$ on G. Related results on L^2-approximation on closed lower dimensional manifolds in the interior have been given by H. Beckert [1] for solutions of second order elliptic equations.

It is the purpose of the present note to present the statement of some results obtained by the writer on uniform approximation for solutions of general linear elliptic equations. The results are given under very mild regularity conditions on A and are based upon the interior L^p-estimates for solutions of elliptic equations. The detailed proofs appear in the writer's forthcoming paper [4].

1. Let G be an open subset of E^n. An open subset G_1 with compact closure in G is said to be mildly regular if each point x_0 of the boundary of G_1 has a neighborhood N such that there exist arbitrarily small translations of $\mathrm{Cl}[N\cap G_1]$ into the interior of G_1 [7].

Let A be a linear differential operator of order r $(r\geqq 1)$ with complex coefficients defined on G. A may be written as usual in the form

$$A = \sum_{|\alpha|\leq r} a_\alpha(x) D^\alpha$$

where the summation is taken over n-tuples $\alpha=(\alpha_1, \cdots, \alpha_n)$ of nonnegative integers,

$$D^\alpha = \prod_{j=1}^n (i^{-1}\partial/\partial x_j)^{\alpha_j}, \qquad |\alpha| = \sum_j \alpha_j.$$

If $\xi=(\xi_1, \cdots, \xi_n)$ is a real n-vector, we set

$$\xi^\alpha = \prod_{j=1}^n \xi_j^{\alpha_j}.$$

Then the homogeneous characteristic form of A is defined by

$$a(x, \xi) = \sum_{|\alpha|=r} a_\alpha(x)\xi^\alpha,$$

and A is said to be elliptic in G if $a(x, \xi)\neq 0$ for any ξ different from zero, and any x in G.

The formal adjoint operator A' of A is the operator with (possibly) distribution coefficients given by

$$A'u = \sum_{|\alpha| \leq r} D^{\alpha}(\bar{a}_{\alpha} u).$$

We shall always assume below that the coefficients of A' are functions (i.e., that certain linear combinations of the distribution derivatives of the coefficients of A are functions).

The differential operator A' is said to satisfy the condition $(\mathrm{U})_s$ for uniqueness in the Cauchy problem in the small provided that if G' is any connected open subset of G, u an element of $C^r(G')$ such that $Au=0$ in G', and u vanishing on a nonvacuous open subset of G', then u must be identically zero in G'. Elliptic operators with analytic coefficients satisfy $(\mathrm{U})_s$. So do elliptic operators with real Hölder continuous coefficients satisfying an unique multiplicity condition for their complex characteristic roots, as Calderón has shown [**5**]. Results for some operators with double characteristics have been obtained by R. Pederson and others. Counter-examples with triple multiplicity have been constructed by P. Cohen and A. Plis.

THEOREM 1 (Theorem (3.22) of [**4**]). *Let A be an elliptic operator of order r on an open subset G of E^n with A and A' having coefficients in $C^1(G)$. Let K be a compact subset of G such that K has n-dimensional measure zero and $G-K$ has no components with compact closure in G. Suppose that A' satisfies the condition $(\mathrm{U})_s$ for uniqueness in the Cauchy problem in the small on G. Let S be the family of solutions of $Au=0$ in G with u in $C^r(G)$. Then the restrictions of the functions of S to the subset K form a dense family in $C^0(K)$, i.e., every continuous function on K can be uniformly approximated on K by solutions u of the equation $Au=0$ in G.*

THEOREM 2 (Theorem (3.23) of [**4**]). *Let A be an elliptic operator of order r on an open subset G of E^n with A and A' having coefficients in $C^{j+1}(G)$ for some j with $1 \leqq j \leqq r-1$. Let K be a compact set of zero n-measure contained in K such that $G-K$ has no components with compact closure in G. Suppose that A' satisfies the condition $(\mathrm{U})_s$ in G. Let S be the family of solutions u of $Au=0$ in G, $u \in C^r(G)$. Then for every v in $C^j(G)$ and each $\epsilon > 0$, there exists u_ϵ in S such that*

$$\left| D^{\alpha} v(x) - D^{\alpha} u_{\epsilon}(x) \right| < \epsilon$$

for $|\alpha| \leqq j$ and all x in K.

2. In the preceding results, we considered approximations on thin sets. We now proceed to the statement of a result along the lines of Walsh's theorem.

THEOREM 3 (Theorem (3.21) of [4]). *Let A be an elliptic operator of order r on an open subset G of E^n with A and A' having coefficients in $C^1(G)$. Suppose that the coefficients of the top-order terms of A are constants and that A' satisfies the condition* $(\mathrm{U})_s$ *for uniqueness in the Cauchy problem in the small on G. Let G_1 be a mildly regular open subset with compact closure in G and such that $G-\overline{G}_1$ has no components with compact closure in G. Let S be the family of solutions u in $C^r(G)$ of $Au=0$ in G, and let*

$$S_0 = \{u_1 : u_1 \in C^r(G_1) \cap C^0(\overline{G}_1),\ Au_1 = 0 \text{ in } G_1\}.$$

Then the restrictions of the functions of S to G_1 are dense in the uniform norm on G_1 in S_0, i.e., every solution u_1 of $Au_1=0$ in G_1 which is continuous on $\overline{G}_1$ can be uniformly approximated on G_1 by solutions u of $Au=0$ in G.

The hypothesis in Theorem 3 that the top-order coefficients should be constants is made necessary by the fact that we use the reproducing properties of elliptic operators in L^p (cf. [2; 3; 4; 6; 8; 12]), i.e., roughly the fact that if u and Au lie in $L^p_{\mathrm{loc}}(G)$, then $D^\alpha u$ must lie in $L^p_{\mathrm{loc}}(G)$ for $|\alpha| \leqq r$. In a very familiar sense, the uniform norm is a very unnatural one from the point of view of the study of the regularity properties of solutions of elliptic differential equations.

To illustrate this last point, let us consider two approximation theorems in different norms (for operators with variable top-order terms), both imposed however over the whole of the subdomain G_1 and not over compact subdomains.

THEOREM 4 (Theorem (3.16) of [4]). *Let A be an elliptic operator of order r on G with A and A' having coefficients in $C^1(G)$. Suppose that A' satisfies the condition* $(\mathrm{U})_s$. *Let G_1 be a mildly regular open subset with compact closure in G such that $G-\overline{G}_1$ has no components with compact closure in G. Let $1<p<\infty$, and let S be the family of solutions u in $C^r(G)$ of $Au=0$ in G, $S_p^{(1)}$ the family of u_1 in $C^r(G_1)\cap L^p(G_1)$ for which $Au_1=0$ in G_1.*

Then the restrictions of S to G_1 are a dense family in $S_p^{(1)}$ in the norm of $L^p(G_1)$.

Let $W^{j,p}(G_1)$ be the space of u in $L^p(G_1)$ such that $D^\alpha u \in L^p(G_1)$ for $|\alpha| \leqq j$. G_1 is said to be $W^{j,p}$ extension regular if $W^{j,p}(G_1) = W^{j,p}(E^n)\,|\,G_1$. Extension regularity follows, for example, from the condition that the boundary of G_1 is C^1 and G_1 lies locally on one side of the boundary.

THEOREM 5 (Theorem (3.19) of [4]). *Let A and A' have coefficients in $C^j(G)$, A elliptic, A' satisfying* $(U)_s$, *G_1 mildly regular and $W^{j,p}$ extension regular with compact closure in G. Then the restrictions to G_1 of the solutions u of $Au=0$ in G lie in the sub-family $S_{j,p}^{(1)}$ of $W^{j,p}(G_1)$ of solutions of $Au_1=0$ in G_1 and are dense in that sub-family in the $W^{j,p}(G_1)$-norm, i.e., essentially in the norm*

$$\sum_{|\alpha|\leq j} \|D^\alpha u\|_{L^p(G_1)}.$$

We may also obtain results as in [1] in which we impose boundary conditions of various types on the approximating functions u on a proper part of the boundary of G.

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