CHARACTERISTIC ROOTS AND FIELD OF VALUES OF A MATRIX

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1. **Introduction.** Let $A=(a_{ij})$ be a square matrix of order n whose elements are in the field of complex numbers. The complex number λ is a *characteristic root* of the matrix A if the determinant of the matrix $\lambda I-A$ is zero. It follows that λ is a characteristic root of A if, and only if, there exists a vector x such that $xx^*=1$ and

$$Ax^* = \lambda x^*, \tag{1}$$

where * is used to denote transposed conjugate. By taking transposed conjugates on both sides in (1) we obtain

$$xA^* = \bar{\lambda} x. \tag{2}$$

From (1) it follows that $\lambda = xAx^*$. The set of all complex numbers zAz^* where $zz^*=1$ is called the *field of values* [**25**][1] of the matrix A. It follows that *the characteristic roots of A belong to the field of values of A*.

Beginning with Bendixson [**3**] in 1900, many writers have obtained limits for the characteristic roots of a matrix. In many cases these were actually limits for the field of values of the matrix [**14**]. In an address delivered before the Mathematical Association of America in 1938, Browne [**10**] gave a summary of these results up to that time. It is the purpose here to discuss some of the results obtained since the time of Browne's paper.

2. **Some well known results.** If x and y are two vectors such that $xx^*=yy^*=1$ and X and Y are unitary matrices with leading vectors x and y respectively, then $xX^*=yY^*$ and hence $y=xX^*Y$, or $y=xU$ where U is a unitary matrix. Also, if $xx^*=1$ and $y=xU$ where U is a unitary matrix, then $yy^*=xUU^*x^*=xx^*=1$. It follows, therefore, that A and UAU^* have the *same field of values and the same characteristic roots* for every unitary matrix U. It may be readily shown [**21**] that *the field of values of A is identical with the set of all diagonal elements of the matrices UAU^* where $UU^*=I$*. If A is *Hermitian* there exists a unitary matrix U such that $UAU^*=\text{diag.}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

where $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ are the (real) characteristic roots of A. It follows immediately that the *field of values of an Hermitian matrix is the shortest segment of the real axis containing all the characteristic roots of A.*

From (1) and (2) the following relations are obtained

$$
(3)\qquad \begin{gathered}
\lambda = xAx^*, \qquad \bar{\lambda} = xA^*x^*, \qquad \bar{\lambda}\lambda = xA^*Ax^*, \\
\frac{\lambda+\bar{\lambda}}{2} = x\left(\frac{A+A^*}{2}\right)x^*, \qquad \frac{\lambda-\bar{\lambda}}{2i} = x\left(\frac{A-A^*}{2i}\right)x^*.
\end{gathered}
$$

From the first three of these relations we obtain two well known theorems. If $A=A^*$, then $\lambda=\bar{\lambda}$ and hence *the characteristic roots of an Hermitian matrix are all real.* If $A^*A=I$, then $\bar{\lambda}\lambda=1$, and hence *the characteristic roots of a unitary matrix are all in absolute value one.* Again from the third of these relations we obtain at once Browne's theorem [8] *that $|\lambda|$ is bounded by the least and greatest of the characteristic roots of the positive semi-definite Hermitian matrix B defined by $B^2=A^*A$.* It may also be shown [20] that for a given B there exists a matrix A which has as a characteristic root any number in this region. From the last two relations in (3) it follows that the characteristic roots of A lie in a rectangular region of the complex plane determined by the characteristic roots of $(A+A^*)/2$ and $(A-A^*)/2$ [16]. In fact the entire field of values of A lies in this rectangle.

3. **Limits for the characteristic roots of a matrix.** In 1930 Browne [9] obtained a limit for the characteristic roots of the matrix A in terms of the sums of the absolute values of the elements in the rows and the columns of A. Write

$$
(4)\qquad \begin{gathered}
R_i = \sum_j |a_{ij}|, \qquad T_j = \sum_i |a_{ij}|, \qquad 2S_i = R_i + T_i, \\
R_i' = R_i - |a_{ii}|, \qquad T_j' = T_j - |a_{jj}|
\end{gathered}
$$

and let R, T, and S denote the maximum R_i, T_i, and S_i, respectively. Browne showed that $|\lambda| \leqq (R+T)/2$. This result was improved by this author in 1937 [18] when it was shown that $|\lambda| \leqq S \leqq (R+T)/2$. Another improvement of Browne's result was given by Farnell in 1944 when he showed that $|\lambda| \leqq (RT)^{1/2} \leqq (R+T)/2$. In 1945 Barankin [1] further improved Farnell's result by showing $|\lambda| \leqq \max\, (R_iT_i)^{1/2}$.

The first two of these are actually limits for the field of values of A. Suppose that $\mu = xAx^*$ is any number in the field of values of A. Write $x=(x_1, x_2, \cdots, x_n)$, then $\mu = \sum_{i,j} a_{ij}x_i\bar{x}_j$. If $|x_i| = \xi_i$ it follows that

$$|\mu| \leqq \sum_{i,j} |a_{ij}| \xi_i \xi_j \leqq \frac{1}{2} \sum_{i,j} |a_{ij}| (\xi_i^2 + \xi_j^2)$$
$$= \frac{1}{2} \sum_i R_i \xi_i^2 + \frac{1}{2} \sum_j T_j \xi_j^2 = \sum_i S_i \xi_i^2 \leqq S \sum_i \xi_i^2 = S.$$

Relation (1) may be written as a system of linear equations in the form

$$\lambda \bar{x}_i = \sum_j a_{ij} \bar{x}_j \qquad (i = 1, 2, \cdots, n). \tag{5}$$

If ξ_k is the greatest of the ξ_i, then since $\lambda \bar{x}_k = \sum_j a_{kj} \bar{x}_j$

$$|\lambda \bar{x}_k| = |\lambda| \xi_k \leqq \sum_j |a_{kj}| \xi_j \leqq R_k \xi_k$$

and hence $|\lambda| \leqq R_k$. In a similar manner $|\lambda| \leqq T_m$ for some $j = m$. This establishes the following theorem.

THEOREM 1. *If λ is a characteristic root of the matrix A and R is the greatest sum obtained for the absolute values of the elements of a row and T is the greatest sum obtained for the absolute values of the elements of a column, then* $|\lambda| \leqq \min (R, T)$.

This theorem was proved by the author in 1943. It was subsequently given by Barankin [1] and later again by Brauer [4]. In an added note Brauer pointed out that the theorem was proved by Perron in the second edition of his *Algebra* in 1933 and also by Specht in 1938. In this same paper Brauer obtained a better limit given by the following theorem.

THEOREM 2. *Let M_r be the maximum of the sums of the absolute values of the elements in each row of the matrix A^{2^r}. Then each characteristic root λ of A satisfies* $|\lambda| \leqq (M_r)^{1/2^r}$.

From (5) it follows that

$$(\lambda - a_{kk}) \bar{x}_k = \sum_{j \neq k} a_{kj} \bar{x}_j$$

so that

$$|\lambda - a_{kk}| \xi_k \leqq \sum_{j \neq k} |a_{kj}| \xi_j \leqq \xi_k \sum_{j \neq k} |a_{kj}| = R_k' \xi_k$$

and hence

$$|\lambda - a_{kk}| \leqq \sum_{j \neq k} |a_{kj}| = R_k - |a_{kk}| = R_k'.$$

In a similar fashion

$$|\lambda - a_{mm}| \leqq T_m - |a_{mm}| = T'_m.$$

This establishes the following theorem.

THEOREM 3 (BRAUER [4]). *Let $A = (a_{ij})$ be an arbitrary matrix and R'_k the sum of the absolute values of the non-diagonal elements of the kth row and T'_m the sum of the absolute values of the non-diagonal elements of the mth column of A. Each characteristic root of A lies in at least one of the circles $|z - a_{kk}| \leqq R'_k$ and in at least one of the circles $|z - a_{mm}| \leqq T'_m$.*

The equations (5) may also be written as

$$(\lambda - a_{ii})\bar{x}_i = \sum_{j \neq i} a_{ij}\bar{x}_j \qquad (i = 1, 2, \cdots, n). \tag{6}$$

If $\xi_k \geqq \xi_l \geqq \xi_j$, $j \neq k$, l, then from (6)

$$|\lambda - a_{kk}|\, \xi_k \leqq R'_k \xi_l \quad \text{and} \quad |\lambda - a_{ll}|\, \xi_l \leqq R'_l \xi_k.$$

It is assumed now that the characteristic root λ is not the diagonal element a_{kk}, so that $\xi_l \neq 0$. Then since

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}|\, \xi_k \xi_l \leqq R'_k R'_l \xi_k \xi_l,$$

it follows that

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}| \leqq R'_k R'_l.$$

The inequality is obviously true of $\lambda = a_{kk}$. In a similar manner

$$|\lambda - a_{mm}| \cdot |\lambda - a_{qq}| \leqq T'_m T'_q$$

for some m and q. This establishes the following theorem.

THEOREM 4 (BRAUER II [5]). *Each characteristic root λ of the matrix A lies in at least one of the $n(n-1)/2$ ovals of Cassini*

$$|z - a_{kk}| \cdot |z - a_{ll}| \leqq R'_k R'_l$$

and in at least one of the ovals

$$|z - a_{kk}| \cdot |z - a_{ll}| \leqq T'_k T'_l.$$

In his third paper Brauer [6] shows how the above theorems may be applied to specially selected polynomials in A to obtain better limits for the characteristic roots. He proved the following theorem.

THEOREM 5 (BRAUER III [6]). *Let $A=(a_{ij})$ be a square matrix of order n and $f_1(y), f_2(y), \cdots, f_n(y)$ be arbitrary polynomials. Denote the elements of the matrix $f_r(A)$ by $a_{ij}^{(f_r)}$ and set*

$$\sum_{j=1, j\neq i}^{n} \left| a_{ij}^{(f_r)} \right| = P_i^{(f_r)} \qquad (i, r = 1, 2, \cdots, n)$$

Each characteristic root λ of A satisfies at least one of the n inequalities

$$\left| f_s(\lambda) - a_{ss}^{(f_s)} \right| \leqq P_s^{(f_s)} \qquad (s = 1, 2, \cdots, n).$$

and at least one of the $n(n-1)/2$ inequalities

$$\left| f_s(\lambda) - a_{ss}^{(f_s)} \right| \cdot \left| f_t(\lambda) - a_{tt}^{(f_t)} \right| \leqq P_s^{(f_s)} P_t^{(f_t)}$$
$$(s, t = 1, 2, \cdots, n; s \neq t).$$

4. **Limits for the field of values of a matrix.** As pointed out above, several of the earlier theorems actually gave upper bounds for the field of values of A. Farnell [14] gives two limits for the field of values of A. *Let $d_{rs}=(|a_{rs}|+|a_{sr}|)/2$, and let $S_r=\sum_s d_{rs}^2$. Then if λ lies in the field of values of A, $|\lambda| \leqq \max (S_r)$ and $|\lambda| \leqq (\sum_{r,s} d_{rs}^2)^{1/2}$.*

In most instances the above bounds are obtained in terms of the distance from zero. However zero may lie entirely outside the field of values of A. Since the field of values of A is convex and contains all the elements of the diagonal of A, it also contains the centroid of these elements. That is, $c=(1/n)\sum_i a_{ii}$ belongs to the field of values of A. Since the trace, $T(A)=\sum_i a_{ii}$, is invariant under unitary transformations, the same c is obtained for every UAU^* where U is unitary. In fact it is possible to select U so that every diagonal element of UAU^* will be c [22].

If $B=A-cI$, the field of values of B is obtained by shifting the field of values of A the distance $|c|$ in the direction of $-c$. Applying the above theorems to B now gives bounds for the characteristic roots of A and the field values of A in terms of the distance from c. This was used by the author [22] to obtain a theorem on the *spread* of the characteristic roots of an Hermitian matrix.

THEOREM 6. *If $A=(a_{ij})$ is an Hermiitan matrix with characteristic roots $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, then*

$$(\lambda_n - \lambda_1)/2 \geqq \max |a_{ij}|, \qquad i \neq j.$$

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