DERIVATIVES OF INFINITE ORDER

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Let $f(x)$ have derivatives of all orders in (a, b). If, as $n\to\infty$, $f^{(n)}(x)\to g(x)$ uniformly, or even boundedly, dominatedly or in the mean, then $g(x)$ is necessarily of the form ke^x, where k is a constant; in fact, if $c\in(a, b)$,

$$f^{(n-1)}(x) - f^{(n-1)}(c) \to \int_c^x g(t)dt$$

and so

$$g(x) - g(c) = \int_c^x g(t)dt.$$

It then follows first that $g(x)$ is continuous, then that $g(x)$ is differentiable in (a, b), finally that $g'(x) = g(x)$ and so $g(x) = ae^x$.

If $f^{(n)}(x)$ approaches a limit only for one value of x, however, it does not necessarily do so for other values of x. On the other hand, G. Vitali [10][1] and V. Ganapathy Iyer [6] showed that if $f(x)$ is analytic in (a, b) and $f^{(n)}(x)$ approaches a limit for one $x_0\in(a, b)$, then $f^{(n)}(x)$ converges uniformly in each closed subinterval of (a, b). Ganapathy Iyer asked two questions in this connection:

(I) If $f^{(n)}(x)\to g(x)$ for each x in (a, b), where $g(x)$ is finite, does $g(x) = ke^x$?

(II) If $f(x)$ belongs to a quasianalytic class in (a, b) and $\lim_{n\to\infty} f^{(n)}(x_0)$ exists for a single x_0, does $\lim_{n\to\infty} f^{(n)}(x)$ exist for every x in (a, b)?

We shall show that the answer to both questions is yes. We also indicate some possible generalizations.

We first answer (I).

THEOREM 1. *If $f^{(n)}(x)\to g(x)$ for each x in (a, b), where $g(x)$ is finite, then $f(x)$ is analytic in (a, b).*

It follows from Ganapathy Iyer's result that then $g(x) = ke^x$.

PROOF. At each point x of (a, b) form the Taylor series of $f(x)$. The radius of convergence of this series, as a function of x, has a positive

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[1] Numbers in brackets refer to the references cited at the end of the paper.

lower bound; in fact, it is infinite for each x. By a known theorem [2, 5, 11] (stated with an incomplete proof by Pringsheim [9, p. 180]), $f(x)$ is analytic in (a, b).

Next we answer (II).

THEOREM 2. *If $f(x)$ belongs to a Denjoy-Carleman quasianalytic class in the (open) interval (a, b), and if $f^{(n)}(x_0) \to L$ for one x_0 in (a, b), then $f(x)$ is analytic in (a, b).*

Again, by the result of Vitali and Ganapathy Iyer it follows that $f^{(n)}(x) \to Le^{x-x_0}$ in (a, b).

PROOF. We say that $f(x) \in C\{M_n\}$ if $|f^{(n)}(x)| \leqq k^n M_n$, $x \in I$, for each closed subinterval I of (a, b), where k depends on $f(x)$ and on I. The class $C\{M_n\}$ is quasianalytic if $\sum M_n^{-1/n} < \infty$; in this case any two functions of the class which coincide, together with all their derivatives, at $x_0 \in (a, b)$, are identical. It is known [3, 8] that $C\{M_n\}$ is identical with the class $C\{M_n^o\}$ obtained by a certain regularizing process; the only property of M_n^o which we need here is that M_{n+1}^o / M_n^o is nondecreasing. It follows that every class $C\{M_n\}$, except the trival class $C\{0\}$, contains $C\{1\}$. This seems to have been first pointed out explicitly by T. Bang [1, p. 16]; we reproduce the simple proof.

We have to show that $k_1^n \leqq k_2^n M_n$, or that $M_n^o \geqq k_3^n$ for some k_3. Now we have $M_n^o / M_{n-1}^o \geqq M_1^o / M_0^o = \alpha$, say. Hence $M_n^o \geqq M_{n-1}^o \alpha \geqq M_{n-2}^o \alpha^2 \geqq \cdots \geqq \alpha^n M_0^o$.

Now suppose that $f^{(n)}(x_0) \to L$ and let

$$g(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0)(x - x_0)^k / k!.$$

For some number Q, $|f^{(n)}(x_0)| \leqq Q$. Hence

$$| g^{(n)}(x) | = \left| \sum_{k=0}^{\infty} f^{(n+k)}(x_0)(x - x_0)^k / k! \right| \leqq Qe^{x-x_0}$$

and so $g(x) \in C\{1\}$; hence $g(x) \in C\{M_n\}$. But $g^{(n)}(x_0) = f^{(n)}(x_0)$ for every n and so $f(x) \equiv g(x)$, an analytic function.

A natural generalization of the problem is to interpret the relation $f^{(n)}(x) \to g(x)$ in a generalized sense. For example, if $f^{(n)}(x) \to g(x)\,(C, 1)$, dominatedly, the proof given in §1 shows that $g(x) = ke^x$; this proof, in fact, applies to any generalized limit such that $s_{n-1}(x)$ converges to the same limit as $s_n(x)$ (see [4, p. 418], [7] for discussions of such generalized limits, which include, in particular, (C, k), $k > -1$).

We can also replace $f^{(n)}(x) \to g(x)$ by $f^{(n)}(x)/\lambda_n \to g(x)$, $\{\lambda_n\}$ a given sequence of constants. We give two simple theorems in this direction.

THEOREM 3. *Let*

$$\lim_{n\to\infty} f^{(n)}(x)/\lambda_n = g(x), \qquad a \leqq x \leqq b. \tag{1}$$

(i) *If* $\liminf |\lambda_{n-1}/\lambda_n| = 0$ *and* (1) *holds uniformly*, $g(x) \equiv 0$ *in* $a \leqq x \leqq b$.
(ii) *If* $\liminf |\lambda_{n-1}/\lambda_n| > 0$ *and* (1) *holds dominatedly*, $g(x) = ke^{bx}$.

The example $f(x) = 1/x$, $\lambda_n = (-1)^n n!$, $a = 1$, $b = 2$ shows that uniformity is essential in (i). It would be interesting to know whether (without uniformity) there can be an exceptional point in the interior of (a, b); if $f(x)$ is analytic, there cannot, as the next theorem shows.

THEOREM 4. *If*

$$\limsup_{n\to\infty} n^{-1} |\lambda_n|^{1/n} < \infty \tag{2}$$

and (1) *is true for each* x *in* $a < x < b$, *then* $g(x) = ke^{bx}$ *in* $a < x < b$.

If $f(x)$ is analytic, $\limsup n^{-1} |f^{(n)}(x)|^{1/n} < \infty$ for each x and hence either (2) is true or (1) implies $g(x) \equiv 0$.

PROOF OF THEOREM 3. We observe that if $a < c < b$

$$\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} \left\{ \frac{f^{(n-1)}(x)}{\lambda_{n-1}} - \frac{f^{(n-1)}(c)}{\lambda_{n-1}} \right\} = \int_c^x g(t)dt. \tag{3}$$

If (i) of Theorem 3 is true, the left side of (3) approaches zero as $n \to \infty$ through a suitable sequence; hence $g(x) = 0$ almost everywhere; but $g(x)$ is continuous because (1) holds uniformly, and so $g(x) \equiv 0$.

If (ii) is true and $\lim |\lambda_{n-1}/\lambda_n| = \infty$, $\lambda_n \to 0$ and so $f^{(n)}(x) \to 0$; otherwise, for some sequence of n's, $\lambda_{n-1}/\lambda_n \to L$, where $L \neq 0$, $L \neq \infty$. Then (3) gives

$$L\{g(x) - g(c)\} = \int_0^x g(t)dt$$

and hence $g(x) = ke^{x/L}$.

PROOF OF THEOREM 4. We have from (1), for each x and for $n > n_x$, $|f^{(n)}(x)| \leqq (1 + g(x))|\lambda_n|$, and so

$$\limsup_{n\to\infty} | f^{(n)}(x)/n! |^{1/n} \leqq \limsup_{n\to\infty} | \lambda_n |^{1/n}/(n/e) < \infty.$$

The reasoning given for Theorem 1 now applies.

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