APPROXIMATE ISOMETRIES

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In a recent paper [1][1] Hyers and Ulam formulated the problem of approximate isometries. Thus if E_1 and E_2 are metric spaces, a transformation T on E_1 to E_2 is an ϵ isometry if $|d_1(T(x), T(x')) - d(x, x')| < \epsilon$, for all x, x' in E_1. These authors analyzed the ϵ isometries defined on a complete abstract Euclidean space E and showed that if T maps E onto itself and $T(\theta) = \theta$, then there exists an isometry [2, p. 165], U, of E onto E such that $\|T(x) - U(x)\| < 10\epsilon$. The analysis depends on the properties of the scalar product. In the present work we show, first, that similar results hold when $E_1 = E_2 = L_r(0, 1)$, $1 < r < \infty$, though, except of course for $r = 2$, a scalar product no longer exists. It is shown further that it is sufficient that E_2 belong to a restricted class of uniformly convex Banach spaces and that E_1 be a Banach space.

THEOREM 1. *Let $T(x)$ be an ϵ isometry of $L_r(0, 1)$, $1 < r < \infty$, into itself with $T(\theta) = \theta$. Then $U(x) = L_{n\to\infty} T(2^n x)/2^n$ exists for each x and $U(x)$ is an isometric, linear transformation.*

Our fundamental assumption is that

$$(1.01) \qquad \big|\, \| T(x) - T(x') \| - \| x - x' \| \,\big| < \epsilon, \qquad T(\theta) = \theta.$$

The following inequality is due to Clarkson [3, 4],

$$(1.02) \qquad \|\alpha + \beta\|^p + \|\alpha - \beta\|^p \leqq 2(\|\alpha\|^q + \|\beta\|^q)^{p-1},$$

where here and later we understand that

$$p = \sup (r/(r-1)) \geqq 2 \geqq q = \inf (r, r/(r-1)).$$

Let

$$2\alpha = T(x),\ 2\beta = T(x) - T(2x).$$

Then

$$\begin{aligned} &\|T(x) - T(2x)/2\|^p \\ (1.03) \qquad &\leqq 2^{1-q(p-1)}(\|T(x)\|^q + \|T(x) - T(2x)\|^q)^{p-1} - \|T(2x)/2\|^p \\ &\leqq (\|x\| + \epsilon)^p - (\|x\| - \epsilon/2)^p. \end{aligned}$$

If $\|x\| \leqq \epsilon$ then the right-hand side of equation (1.03) is surely in-

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[1] Numbers in brackets refer to the Bibliography.

ferior to $(2\epsilon)^p$. For $0 \leqq z \leqq 1$, $r \geqq 1$, the following inequalities are easy to establish,

$$(1 + z)^r \leqq 1 + (2^r - 1)z, \tag{1.04}$$

$$(1 - z)^r \geqq 1 - rz. \tag{1.05}$$

Hence for $||x|| > \epsilon$ the right-hand side of equation (1.03) is dominated by

$$\left(\frac{2^{p+1} + p - 2}{2}\right) \epsilon ||x||^{p-1}.$$

Accordingly in both cases

$$||T(x) - T(2x)/2|| \leqq k||x||^{1/q} + 2\epsilon$$

where

$$k = \left(\frac{\epsilon(2^{p+1} + p - 2)}{2}\right)^{1/p}.$$

Write

$$||T(2^n x)/2^n - T(x)|| \leqq k_n||x||^{1/q} + l_n\epsilon$$

and

$$\begin{aligned} &||T(2^{n+1}x)/2^{n+1} - T(x)|| \\ &\leqq ||T(2^{n+1}x)/2^{n+1} - T(2x)/2|| + ||T(2x)/2 - T(x)|| \\ &\leqq (2^{-1/p}k_n + k)||x||^{1/q} + (l_n/2 + 2)\epsilon. \end{aligned} \tag{1.06}$$

On setting the right-hand side of Equation (1.06) equal to $k_{n+1}||x||^{1/q} + l_{n+1}\epsilon$, we have the difference equations

$$k_{n+1} = 2^{-1/p}k_n + k, \; l_{n+1} = l_n/2 + 2.$$

The solutions of these equations are

$$k_n = k \sum_{j=0}^{n} 2^{-j/p} \leqq k/1 - 2^{-1/p} = A,$$

$$l_n = 2 \sum_{j=0}^{n} 2^{-j} \leqq 4.$$

Hence

$$||T(2^{n+m}x)/2^{n+m} - T(2^n x)/2^n|| \leqq A2^{-n/p}||x||^{1/q} + 4(2^{-n}\epsilon). \tag{1.07}$$

Since $L_r(0, 1)$ is complete we can define $U(x)$ by

$$U(x) = L_{n\to\infty}T(2^n x)/2^n, \qquad U(\theta) = \theta.$$

Moreover in view of equations (1.01) and (1.07) we can establish directly that $U(x)$ is an isometry (and is linear [2, p. 166]). We shall make frequent use of

$$\|U(x) - T(x)\| \leqq A\|x\|^{1/q} + 4\epsilon. \tag{1.08}$$

THEOREM 2. *If T is an ϵ isometry of $L_r(0,1)$ on $L_r(0,1)$ then U is an isometry of $L_r(0, 1)$ on $L_r(0, 1)$.*

The proof given by Hyers and Ulam for their Theorem 3 is obviously valid here. A more general situation is covered by our Theorem 5.

THEOREM 3. *If $T(x)$ is an ϵ isometry of $L_r(0, 1)$ on $L_r(0, 1)$ then $\|T(x) - U(x)\| \leqq 12\epsilon$.*

We shall tacitly follow the convention that $z\in L_r(0, 1)$ has as its representative the function $z_r(s)$. Choose $x\neq\theta$ arbitrarily and write

$$U(x) = y_1, \quad T(x) = y_2.$$

We first assume that y_1 and y_2 are not collinear with θ. Then $\|y_2 - vy_1\|$ has a unique, positive minimum for some value of v, say u. For instance, this is a consequence of the Alaoglu-Birkhoff lemma [5, p. 299] *that in a uniformly convex Banach space a closed convex set, here* $\{y_2 - vy_1 \mid |v| < \infty\}$, *contains an element of least norm* (reflexivity would be sufficient for this) *and this element is unique.* Let

$$y_0 = (y_2 - uy_1)/\|y_2 - uy_1\|. \tag{3.01}$$

Then

$$y_2 = uy_1 + \|y_2 - uy_1\|y_0 = uy_1 + wy_0. \tag{3.02}$$

It is significant for our developments that $w \geqq 0$. In view of Theorem (2) a unique element, x_0, is defined by $U^{-1}y_0$.

Since $\inf \|y_0 - hy_1\| = 1$ there is a linear functional [2, p. 57] of unit norm, f_0, such that $f_0(y_0) = \|y_0\| = 1$ and $f_0(y_1) = 0$. Let $E_0 = \{y \mid f_0(y) = 0\}$. Since L_r^* is strictly convex it is easy to show that f_0 is unique and an explicit representation is

$$f_0(y) = \int_0^1 |\, y_0(s)\,|^{r-1} \operatorname{sign} y_0(s) y(s) ds. \tag{3.03}$$

An obvious argument shows every element in E is uniquely expressible as a sum of a multiple of y_0 and an element in E_0. Moreover, if

$y \in E_0$, then

$$\|y_0 + y\| \geqq |f_0(y_0 + y)| = 1. \tag{3.04}$$

We have from Equation (1.08)

$$\begin{aligned} \|T(x) - U(x)\| &= \|(u-1)y_1 + wy_0\| \\ &\leqq A\|x\|^{1/q} + 4\epsilon \\ &= A\|y_1\|^{1/q} + 4\epsilon. \end{aligned} \tag{3.05}$$

Moreover

$$\big| \|vy_1 + wy_0\| - \|y_1\| \big| = \big| \|(Tx)\| - \|x\| \big| < \epsilon. \tag{3.06}$$

We write

$$T(2^n x_0) = 2^n x_0 + l_n y_0 + Y_n \tag{3.07}$$

where $Y_n \in E_0$. Accordingly

$$\epsilon > \big| \|2^n y_0 + l_n y_0 + Y_n - uy_1 - wy_0\| - \|2^n y_0 - y_1\| \big|. \tag{3.08}$$

Also

$$\big| \|2^n y_0 + l_n y_0 + Y_n\| - 2^n \big| < \epsilon, \tag{3.09}$$

$$\|l_n y_0 + Y_n\| \leqq A 2^{n/q} + 4\epsilon. \tag{3.10}$$

From equations (3.02), (3.09), and (3.10) it is manifest that

$$-(A2^{n/q} + 4\epsilon) \leqq l_n \leqq \epsilon, \quad \|Y_n\| \leqq 2(A2^{n/q} + 4\epsilon).$$

Actually it is sufficient for our purpose that $\|Y_n\|/2^n$, $l_n/2^n$ go to 0 as $n \to \infty$.

We remark that

$$\|2^n y_0 - y_1\| = 2^n + o(1). \tag{3.11}$$

Indeed equation (3.02) entails

$$\frac{d}{dt}\|y_0 - ty_1\| \Big|_{t=0} = -\int_0^1 (|y_0(s)|^{r-1} \operatorname{sign} y_0(s)) y_1(s)\, ds = 0.$$

Hence we have from equations (3.08), (3.09), and (3.11)

$$\begin{aligned} &\|(2^n + l_n)y_0 + Y_n\|^r - \|(2^n + l_n)y_0 + Y_n - uy_1 - wy_0\|^r \\ &\qquad \leqq (2r\epsilon)2^{n(r-1)} + o(2^{n(r-1)}). \end{aligned} \tag{3.12}$$

The crucial step in our demonstration is the justification of the assertion that the left-hand side of equation 3.12 is

$$2^{n(r-1)}(rw) + o(2^{n(r-1)}).$$

Write $t = 2^{-n}$, $l_t = l_n/2^n$, $Y_t = Y_n/2^n$. Then l_t and $\|Y_t\|$ go to 0 with t.

Write $V=tu$, $W=tw$ and α and β for real numbers between 0 and V and 0 and W respectively. Write also $L_t=l_t-\beta$ and

$$\psi(s,t) = y_0(s)(1+L_t)+Y_t(s)-\alpha y_1(s),$$

$$f_t(y) = \int_0^1 |\psi(s,t)|^{r-1} \operatorname{sign} \psi(s,t) y(s) ds.$$

Denote the rectangle $|u|\leqq A$, $w\leqq B$ by Q. For each choice of t the theorem of the mean guarantees that α and β exist such that the left-hand side of equation (3.12) has the value

$$(3.13)\quad -\left(V\frac{\partial}{\partial\alpha}+W\frac{\partial}{\partial\beta}\right)\|(1+L_t)y_0+Y_t-\alpha y_1\|^r = r(Vf_t(y_1)+Wf_t(y_0)).$$

For arbitrary positive δ and all sufficiently small t values

$$\sup(|l_t|+tB, \|Y_t\|, t\|Ay_1\|) < \delta.$$

Let

$$S = \{s \mid y_0(s) \leqq |L_ty_0(s)+Y_t(s)-\alpha y_0(s)|\}$$

and write R for the complement of S in $0\leqq s\leqq 1$. Then

$$|f_t(y_1)| \leqq \left|\int_R |\psi(s,t)|^{r-1}\operatorname{sign} y_0(s)y_1(s)ds\right| + \left|\int_S |\psi(s,t)|^{r-1}\operatorname{sign}\psi(s,t)y_1(s)ds\right|.$$

It may be verified that

$$\int_S |\psi(s,t)|^{r-1}|y_1(s)|ds \leqq 2^{r-1}\int_S |L_ty_0(s)+Y_t(s)-\alpha y_1(s)|^{r-1}|y_1(s)|ds$$

$$(3.14)\qquad \leqq 2^{r-1}(3\delta)^{r/(r-1)}\|y_1\|.$$

$$\left|\int_R |\psi(s,t)|^{r-1}\operatorname{sign} y_0(s)y_1(s)ds\right| \leqq \left|\int_0^1 |\psi(s,t)|^{r-1}\operatorname{sign} y_0(s)y_1(s)ds\right| + \left|\int_S |\psi(s,t)|^{r-1}\operatorname{sign} y_0(s)y_1(s)ds\right|.$$

The first integral on the right-hand side can be written

$$\int_0^1 (|\psi(s,t)|^{r-1}-|y_0(s)|^{r-1})\operatorname{sign} y_0(s)y_1(s)ds + \int_0^1 |y_0(s)|^{r-1}\operatorname{sign} y_0(s)y_1(s)ds.$$

Since $y_1\in E_0$ the last integral vanishes, and we may dominate by

$$\int_R (|\psi(s, t)|^{r-1} - |y_0(s)|^{r-1})|y_1(s)|ds + \int_S (|\psi(s, t)|^{r-1} - |y_0(s)|^{r-1})|y_1(s)|ds.$$

The first integral may be written

$$\int_R \left| \left| 1 + \frac{L_t y_0(s) + Y_t(s) - \alpha y_1(s)}{y_0(s)} \right|^{r-1} - 1 \right| |y_0(s)|^{r-1} |y_1(s)| ds.$$

For $|z| \leqq 1$ and positive k we have

$$(1 + |z|)^k \leqq 1 + k|z|,\ 1 + (2^k - 1)|z|,\ (1 - |z|)^k \geqq 1 - |z|,\ 1 - k|z|,$$

according as k is less than or greater than 1. Hence for some positive K the last integral written is bounded by

$$K \int_R |L_t y_0(s) + Y_t(s) - \alpha y_1(s)|\ |y_0(s)|^{r-2} |y_1(s)| ds$$
$$\geqq K \| L_t y_0 + Y_t - \alpha y_1 \|^{r/(r-1)} \| y_0 \|^{r(r-2)/(r-1)} \| y_1 \|$$
$$\leqq K(3\delta)^{r/(r-1)} \| y_1 \|.$$

Since all the integrals over S are covered by equation (3.14),

$$|f_t(y_1)| \leqq C\delta^{r/(r-1)}, \tag{3.15}$$

uniformly in $(u, w) \in Q$ for all sufficiently small t. Hence

$$utf_t(y_1) = o(t).$$

Now

$$\begin{aligned} f_t(y_0) &= \int_R + \int_S (|\psi(s, t)|^{r-1} - |y_0(s)|^{r-1}) \operatorname{sign} \psi(s, t) ds \\ &\quad + \int_0^1 |y_0(s)|^{r-1} \operatorname{sign} \psi(s, t) y_0(s) ds. \end{aligned} \tag{3.16}$$

Each of the first two integrals on the right-hand side is readily shown to be inferior in absolute value to $C_1\delta^{r/r-1}$. The last integral on the right-hand side may be written

$$\|y_0\|^r + \int_S |y_0(s)|^{r-1} (\operatorname{sign} \psi(s, t) - \operatorname{sign} y_0(s)) y_0(s) ds.$$

Evidently the integral over S is inferior in absolute value to

$$2 \int_S |\psi(s, t)|^{r-1} |y_0(s)| ds \leqq 2(3\delta)^{r/(r-1)}.$$

Therefore

$$f_t(y_0) = 1 + \rho, \qquad |\rho| \leqq C_2\delta^{r/(r-1)},$$

for all sufficiently small t values uniformly for $(u, w) \in Q$. Thus

$$wtf_t(y_0) = wt + o(t). \tag{3.17}$$

Hence the right-hand side of equation (3.13) is $rwt+o(t)$. Accordingly, equation (3.12) may be written

$$2^{n(r-1)}rw + o(2^{n(r-1)}) \leqq 2^{n(r-1)}2r\epsilon + o(2^{n(r-1)}).$$

Therefore

$$w \leqq 2\epsilon. \tag{3.18}$$

We have then

$$\|y_2 - y_1\| \leqq |u - 1| \, \|y_1\| + 2\epsilon.$$

From equations (3.06) and (3.18) we infer

$$\big| |u| - 1 \big| \|y_1\| \leqq 3\epsilon. \tag{3.19}$$

Hence if $u \geqq 0$ we have

$$\|y_2 - y_1\| \leqq 5\epsilon. \tag{3.20}$$

The case that y_1, y_2 and θ are collinear offers no exception. Here $y_2 = uy_1$ and, for $u \geqq 0$, equation (3.06) surely implies equation (3.20).

Suppose now that $u<0$. It may be shown from equations (3.05), (3.06), and (3.18) that the maximum value of $\|y_1\|$, denoted by B, consistent with $u<0$ is given by

$$2B = AB^{1/q} + 9\epsilon. \tag{3.21}$$

In view of equations (1.08) and (3.19) it follows that in all cases

$$\|y_2 - y_1\| \leqq \sup (5\epsilon, 2B - 5\epsilon). \tag{3.22}$$

Evidently B depends on p and ϵ alone and goes to 0 with ϵ. Since $1<q\leqq 2$ it can be verified that for $A+9\epsilon<2$, for instance,

$$B \leqq ((A + (A^2 + 72\epsilon)^{1/2})/4)^q.$$

Instead of continuing with the determination of explicit bounds for B from equation (3.21) it seems preferable to present an alternative argument which has the merit of yielding a convenient bound directly. The idea behind this argument is borrowed from [1] and consists of the observation that for all sufficiently large multiples of x, say lx, $T(lx)$ has a positive y_1 component. This is of course obvious, since we need merely satisfy $l\|x\| \geqq B$. Accordingly if $u<0$, there is an integer m such that in the general case

$$T(2^m x) = -u_2 2^m y_1 + w_2 z_0, \tag{3.23}$$
$$T(2^{m+1}x) = u_1 2^{m+1} y_1 + w_1 Z_0,$$

where u_1 and u_2 are *non-negative* and z_0 and Z_0 have unit norm and are determined in the same way as y_0. The argument leading to equation (3.18) shows that $0 \leqq w_j \leqq 2\epsilon$, $j=1, 2$. The possibility that either (or both) of $T(2^m x)$ and $T(2^{m+1}x)$ is collinear with y_1 and θ is formally included by taking the corresponding w as 0. We write z for $2^m y_1$. Thus $\|y_1\| \leqq \|z\|$. Then, in view of equations (1.01) and (3.23),

$$\left| \, \|(2u_1 + u_2)z - w_2 z_0 + w_1 Z_0\| - \|z\| \, \right| < \epsilon$$

and

$$|2u_1 + u_2 - 1| \, \|z\| \leqq 5\epsilon. \tag{3.24}$$

Similarly the analogues of equation (3.19) are

$$|2u_1 - 2| \, \|z\| \leqq 3\epsilon, \tag{3.25}$$

$$|u_2 - 1| \, \|z\| \leqq 3\epsilon. \tag{3.26}$$

There are several cases to consider, depending on whether (u_1, u_2) and $2u_1+u_2-1$ are larger or smaller than 1. The largest value of $\|z\|$ is admitted in the event sup $(u_1, u_2) \leqq 1 \leqq 2u_1+u_2 \leqq 3$. On combining the inequalities in (3.24), (3.25), and (3.26) we obtain in this case

$$\|z\| \leqq 11\epsilon/2.$$

Since $\|y_1\| \leqq \|z\|$ we infer for $u<0$ and then, by equation 3.20, for all u,

$$\|T(x) - U(x)\| \leqq 2\|y_1\| + \epsilon \leqq 12\epsilon. \tag{3.27}$$

The developments just concluded motivate the generalization presented below. Elements in the Banach spaces E_1 and E_2 are denoted by x and by y or z respectively. Our restrictions bear on E_2 alone. Henceforth we assume E_2 is a uniformly convex Banach space [3], that is to say $\|z_1 - z_2\| \geqq \gamma$ sup $(\|z_1\|, \|z_2\|)$, $\gamma > 0$, implies $\|z_1 + z_2\| \leqq 2\,(1-\delta(\gamma))$ sup $(\|z_1\|, \|z_2\|)$, where $\delta(\gamma)$ is strictly monotone with $\delta(0)=0$, $\delta(2)=1$. We define γ' for all positive δ as sup $\{\gamma \mid \delta(\gamma) \leqq \inf(1, \delta)\}$ and write $\gamma' = \psi(\delta)$. We drop the primes in the sequel. We require that E_2 satisfy the following restrictions, also,

$$\sum_{n=1}^{\infty} \psi(2^{-n}C) < \infty \tag{A}$$

for every positive C,

$$\text{(B)} \qquad L_{\lambda\to 0,\ \|z\|\to 0}(\|y_0 + z - \lambda y_1\| - \|y_0 + z\|)/\lambda = 0,$$

where $\|y_0\| = 1$ and z and y_1 lie in the linear space $\{y \mid f_0(y) = 0, f_0(y_0) = 1, \|f_0\| = 1\}$. (It is easy to verify that these conditions are satisfied by $L_r(0, 1)$, $1 < r < \infty$.)

THEOREM 4. *Let $T(x)$ be an ϵ isometry of E_1 into E_2 with $T(\theta_1) = \theta_2$. Then $U(x) = L_{n\to\infty}T(2^n x)/2^n$ exists for each x and $U(x)$ is an isometric (linear) transformation.*

Evidently sup $(\|T(2x) - T(x)\|, \|T(x)\|)$ can be written $\|x\| + \rho$ where $|\rho| < \epsilon$. Then

$$\|(T(2x) - T(x)) - T(x)\| \geqq \gamma(\|x\| + \rho)$$

implies

$$\|x\| - \epsilon/2 \leqq \|T(2x)/2\| \leqq (1 - \delta(\gamma))(\|x\| + \rho).$$

Hence

$$\delta(\gamma) \leqq 3\epsilon/2\|x\|.$$

Let $\delta_n = 3\epsilon/2^{m+1}\|x\|$. Then, since $|\rho| < \epsilon$,

$$\|T(2^{m+1}x) - 2T(2^m x)\| \leqq \gamma_m(2^m\|x\| + \epsilon).$$

Evidently γ_m is a monotone nonincreasing function on the positive real axis with $L_{\|x\|\to\infty}\ \gamma_m = 0$. In view of (A)

$$\text{(4.01)} \qquad \sum_{j=1}^{\infty} 2^{-j}\|T(2^j x) - 2T(2^{j-1}x)\| \leqq (\|x\| + \epsilon) \sum_{j=1}^{\infty} \gamma_j = k(\|x\|)(\|x\| + \epsilon).$$

It is important to observe that $k(\|x\|)$ is a monotone nonincreasing function of $\|x\|$ and

$$\text{(4.02)} \qquad L_{\|x\|\to\infty}k(\|x\|) = 0.$$

Since E_2 is complete, the demonstrated convergence of the left-hand side of equation (4.01) ensures the existence of $U(x)$ given by

$$U(x) = L_{n\to\infty}T(2^n x)/2^n.$$

It is easy to see that U is an isometry on E_1 to E_2 and that

$$\text{(4.03)} \qquad \|T(x) - U(x)\| \leqq k(\|x\|)(\|x\| + \epsilon).$$

THEOREM 5. *If T is onto E then U is onto E.*

The simple demonstration below covers more general E_2 spaces than those of our hypotheses. Let E_2' be the range of U. Then E_2' is closed in E_2 [2, p. 145]. Assume the assertion of the theorem invalid, that is to say E_2' is a proper subspace of E_2. Evidently for some positive s_0, $s \geqq s_0$ implies $k(s)(s+\epsilon) \leqq s+\epsilon-2$. It is well known [2, p. 83] that for some z_0 of unit norm, $\|z_0-z\| \geqq 1-(s_0+\epsilon)^{-1}$ for all z in E_2'. Let x satisfy $T(x)=(s_0+\epsilon)z_0$. Then $s_0 \leqq \|x\| \leqq s_0+2\epsilon$ and

$$(5.01)\quad \|T(x) - U(x)\| = (s_0 + \epsilon)\|z_0 - U(x)/(s_0 + \epsilon)\| \geqq s_0 + \epsilon - 1.$$

On the other hand, using equation (4.03),

$$(5.02)\qquad \|T(x) - U(x)\| \leqq s_0 + \epsilon - 2.$$

Since equations (5.01) and (5.02) are incompatible, our theorem is established. Hence incidentally E_1 and E_2 are equivalent [2, p. 180].

THEOREM 6 *If T defines an ϵ isometry of E_1 onto E_2 with $T(\theta_1)=\theta_2$ then $\|T(x)-U(x)\| \leqq 12\epsilon$.*

We continue the notation introduced in the course of the proof of Theorem 3. The results through equation (3.09) hold subject to the trivial modification of replacing $A\|y_1\|^{1/q}+4\epsilon$ by $k(\|y_1\|)(\|y_1\|+\epsilon)$. Moreover the analogues of equations (3.09) and (3.10) establish

$$-k(2^n)(2^n+\epsilon) \leqq l_n \leqq \epsilon, \quad \|Y_n\| \leqq 2k(2^n)(2^n+\epsilon).$$

In view of equation (4.02) we can assert $\|Y_n\|/(2^n-|l_n|)$ and $l_n/2^n$ go to 0 with n. Then by (B)

$$(6.01)\quad \begin{aligned} \|T(2^nx_0)\| - \|T(2^nx_0) - T(x)\| &\leqq \|2^ny_0\| + \epsilon - \|2^ny_0 - y_1\| + \epsilon \\ &\leqq 2\epsilon + o(1). \end{aligned}$$

The left-hand side of equation (6.01), for large n, is

$$(6.02)\quad \begin{aligned} &\left\{(2^n + l_n)\left(\left\|y_0 + \frac{Y_n}{2^n + l_n}\right\| - \left\|y_0 + \frac{Y_n - uy_1}{2^n + l_n - w}\right\|\right)\right\} \\ &\qquad + \left\{w\left\|y_0 + \frac{Y_n - uy_1}{2^n + l_n - w}\right\|\right\}. \end{aligned}$$

With $z=Y_n/(2^n+l_n-w)$, $\lambda=u/(2^n+l_n-w)$ it follows readily from (B) that the first brace of terms goes to 0 with n and hence $w+o(1) \leqq 2\epsilon+o(1)$ or

$$w \leqq 2\epsilon.$$

For some positive B, $k(B)(B+\epsilon)+2\epsilon \leqq B$ in view of equation (4.02).

Then a simple argument using equations (4.03) and (3.18) shows $u>0$ for $\|y_1\|>B$. The remainder of the proof follows the pattern of the proof of Theorem 3 in detail and is therefore omitted. It will be noted that the possibility that the functional f_0 (cf. Theorem 3) may not be unique does not disturb the proof.

REMARK. Since U is linear T is a 36ϵ linear transformation [6]. Moreover $U(x)$ is the unique distributive operation satisfying $\overline{L}_{\|x\|\to\infty}\|T(x)-V(x)/\|x\|<\infty$.

BIBLIOGRAPHY

1. D. H. Hyers and S. M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 288–292.

2. S. Banach, *Théorie des opérations linéaires*, 1932.

3. J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 396–414.

4. R. P. Boas, *Some uniformly convex spaces*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 304–311.

5. L. Alaoglu and G. Birkhoff, *General ergodic theorems*, Ann. of Math. vol. 41 (1940) pp. 293–309.

6. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 222–224.

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