NOTE ON INTERPOLATION FOR A FUNCTION OF SEVERAL VARIABLES

HERBERT E. SALZER

The simplest interpolation formula for a function of ω variables $x, y, \cdots, z$ is the multiple Gregory-Newton formula, which approximates the function by a polynomial in $p, q, \cdots, r$ of total degree n, namely,

$$
\begin{aligned}
&f(x + ph_1, y + qh_2, \cdots, z + rh_\omega) \\
(1) \qquad &= \sum_{i+j+\cdots+k=0}^{n} \binom{p}{i}\binom{q}{j}\cdots\binom{r}{k} \Delta_{x^i y^j \cdots z^k}^{i+j+\cdots+k} f(x, y, \cdots, z),
\end{aligned}
$$

where $x, y, \cdots, z$ denote the independent variables, h_m denotes the tabular intervals,

$$\binom{p}{i} \text{ denotes } \frac{p(p-1)\cdots(p-i+1)}{i!}, \text{ with } \binom{p}{0} = 1,$$

and $\Delta_{x^i y^j \cdots z^k}^{i+j+\cdots+k} f(x, y, \cdots, z)$ denotes the mixed partial advancing difference of $f(x, y, \cdots, z)$, of order i with respect to x, j with respect to y, and so on. The summation is for all sets of values of $i, j, \cdots, k$ such that $i+j+\cdots+k$ goes from 0 to n. Using the notation $f_{s,t,\cdots,u}$ to denote $f(x+sh_1, y+th_2, \cdots, z+uh_\omega)$, it is apparent that the multiple Gregory-Newton formula involves all values $f_{s,t,\cdots,u}$ such that $s+t+\cdots+u=0, 1, 2, \cdots, n$. Thus for the case of 2 dimensions the arguments are the $(n+1)(n+2)/2$ points forming a right triangle, vertex at (x, y), and for 3 dimensions the arguments are the $(n+1)(n+2)(n+3)/6$ points forming a solid tetrahedron, vertex at (x, y, z).

The purpose of the present note is to show that when (1) is expressed in the simpler form

$$(2) \quad f(x + ph_1, y + qh_2, \cdots, z + rh_\omega) = \sum_{s+t+\cdots+u=0}^{n} C_{s,t,\cdots,u} f_{s,t,\cdots,u},$$

then we have

$$(3) \quad C_{s,t,\cdots,u} = \binom{n-p-q-\cdots-r}{n-s-t-\cdots-u}\binom{p}{s}\binom{q}{t}\cdots\binom{r}{u}.$$

Thus (1) can be employed without the labor of finding all the mixed

Received by the editors September 9, 1944.

partial differences, which represents a very convenient simplification in the use of the multiple Gregory-Newton formula.

To prove[1] (3) consider the function

$$\binom{n - x - y - \cdots - z}{n - s_1 - t_1 - \cdots - u_1}\binom{x}{s_1}\binom{y}{t_1}\cdots\binom{z}{u_1},$$

where $s_1, t_1, \cdots, u_1$ are any set of non-negative integers whose sum is not greater than n. This function is a polynomial in $x, y, \cdots, z$ of total degree n and (2) holds exactly. Applying (2) for $x=y=\cdots=z=0$, $h_1=h_2=\cdots=h_\omega=1$, it is apparent that with the exception of $f_{s_1,t_1,\cdots,u_1}=1$, all the other quantities $f_{s,t,\cdots,u}$ vanish, because if some $s, t, \cdots$, or u is less than a respective $s_1, t_1, \cdots$, or u_1, or if every $s, t, \cdots, u$ is greater than or equal to a respective $s_1, t_1, \cdots, u_1$ with at least one greater than, then $f_{s,t,\cdots,u}$ will have a factor

$$\binom{a}{b}, \quad a \text{ and } b \text{ integers,}$$

$b>a$, which is 0. This establishes (3).

MATHEMATICAL TABLES PROJECT,
NATIONAL BUREAU OF STANDARDS

[1] This line of proof was suggested by Professor W. E. Milne. Another longer proof is by induction, making use of the properties of $\binom{p}{i}$ and Newton's backward-difference interpolation formula.