ON UNIFORM CONVERGENCE OF FOURIER SERIES

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1. Introduction. In this section we collect some known concepts and simple facts, pertinent to our subject.

Given a sequence of real numbers s_n, $n \geqq 0$, consider for any $\lambda > 1$

$$\limsup_{n\to\infty} \max_{n<m\leqq\lambda n} (s_m - s_n) = u(\lambda) \leqq +\infty;$$

clearly $u(\lambda)$ decreases as $\lambda \downarrow 1$; if

$$\lim_{\lambda\to 1} u(\lambda) \leqq 0, \tag{1.1}$$

then the sequence $\{s_n\}$ is called slowly oscillating from above; similarly slow oscillation from below is defined by

$$\lim_{\lambda\to 1} \liminf_{n\to\infty} \min_{n<m\leqq\lambda n} (s_m - s_n) \geqq 0. \tag{1.2}$$

If both (1.1) and (1.2) hold, that is if

$$\lim_{\lambda\to 1} \limsup_{n\to\infty} \max_{n<m\leqq\lambda n} | s_m - s_n | = 0, \tag{1.3}$$

then the sequence is called simply slowly oscillating. If $s_n = \sum_0^n a_\nu$ is the nth partial sum of a series $\sum_0^\infty a_\nu$, then (1.3) can be written as

$$\lim_{\lambda\to 1} \limsup_{n\to\infty} \max_{n<m\leqq\lambda n} \left| \sum_{n+1}^{m} a_\nu \right| = 0. \tag{1.4}$$

A more restricted class of series is defined by

$$\lim_{\lambda\to 1} \limsup_{n\to\infty} \sum_{n<\nu\leqq\lambda n} | a_\nu | = 0. \tag{1.5}$$

Special cases: If for some $p > 0$, $n|a_n| < p$ for all n, then

$$\sum_{n<\nu\leqq\lambda n} | a_\nu | < p \sum_{n}^{\lambda n} \frac{1}{\nu} = O(\log \lambda).$$

Hence (1.5) holds.

If only

$$na_n > -p \qquad \text{for all } n,$$

then

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$$n(|a_n| - a_n) < 2p,$$

hence

$$(1.6) \qquad \lim_{\lambda \to 1} \limsup_{n\to\infty} \sum_{n}^{\lambda n} (|a_\nu| - a_\nu) = 0.$$

This relation implies (1.2), but not necessarily (1.4). The following lemma is immediate:

LEMMA 1. *Every convergent series satisfies* (1.3); *furthermore* (1.3) *and* (1.6) *imply* (1.5).

A sequence of functions $s_n(t)$, defined at a point set $\mathcal{E}$ having $t=\tau$ for a limit point, is said to be uniformly convergent at $t=\tau$ if $\lim s_n(t_n)$ exists for any sequence $t_n \to \tau$. It is an immediate consequence of the definition that the limit of $s_n(t_n)$ is then unique.

If for each n, $s_n(t)$ is defined and continuous at $t=\tau$, then clearly a necessary condition for uniform convergence at $t=\tau$ is that $\lim_{n\to\infty} s_n(\tau) = s$ exists.

We restrict ourselves to such sequences; then the following lemma holds:

LEMMA 2. *The following two properties are equivalent*:

(a) $s_n(t_n) \to s$ *as* $t_n \to \tau$;

(b) $s_n(\tau) \to s$, *and* $|s_n(\tau) - s_n(t)| < \epsilon$ *for any* $\epsilon > 0$, *and for* $|\tau - t| < \delta(\epsilon)$, $n > n_0(\delta, \epsilon) = n_0(\epsilon)$.

Thus either (a) or (b) defines uniform convergence at $t=\tau$.

For the proof assume that (a) holds; if (b) would not hold, there would exist an $\epsilon=\epsilon_0$, so that $\limsup_{t_n \to \tau} |s_n(\tau) - s_n(t_n)| > \epsilon_0$. But this contradicts (a). Similarly if (b) holds, then (a) follows.

2. The cosine series. We now prove the following theorem.

THEOREM 1. *Suppose that the coefficients of the Fourier cosine series*

$$(2.1) \qquad \phi(t) \sim a_0/2 + \sum_{1}^{\infty} a_n \cos nt$$

satisfy the condition (1.6), *and that* $\phi(t)$ *is continuous at* $t=0$; *then the series* (2.1) *is uniformly convergent at* $t=0$.

Let

$$s_0 = \frac{a_0}{2}, \quad s_n(t) = \frac{a_0}{2} + \sum_{1}^{n} a_\nu \cos \nu t, \quad \sigma_n(t) = \frac{1}{n+1} \sum_{0}^{n} s_\nu(t).$$

By a theorem of Fejér [1][1]

$$\sigma_n(t_n) \to \phi(0) \qquad \text{as } t_n \to 0; \tag{2.2}$$

in particular

$$\sigma_n(0) \to \phi(0) \qquad \text{as } n \to \infty. \tag{2.3}$$

By a well known theorem of Tauberian type, (2.3) and (1.6) (or only (1.2)) imply that

$$s_n(0) \to \phi(0). \tag{2.4}$$

By Lemma 1, (1.6) and (2.4) imply (1.5).

We next employ the often used identity

$$s_n - \sigma_{n+\nu} = \frac{n}{\nu+1}(\sigma_{n+\nu} - \sigma_{n-1}) - \frac{1}{\nu+1}\sum_{k=1}^{\nu}(\nu - k + 1)c_{n+k}, \qquad n \geqq 1,\ \nu \geqq 1, \tag{2.5}$$

where s_n, σ_n are the partial sums and arithmetical means respectively of the series $\sum c_n$. Thus

$$\begin{aligned} s_n(0) - s_n(t) - \{\sigma_{n+\nu}(0) - \sigma_{n+\nu}(t)\} &= \frac{n}{\nu+1}\{\sigma_{n+\nu}(0) - \sigma_{n+\nu}(t) - [\sigma_{n-1}(0) - \sigma_{n-1}(t)]\} \\ &\quad - \frac{1}{\nu+1}\sum_{1}^{\nu}(\nu - k + 1)[1 - \cos(n+k)t]a_{n+k}. \end{aligned} \tag{2.6}$$

By (2.2) and Lemma 2

$$|\sigma_n(0) - \sigma_n(t)| < \epsilon \quad \text{for} \quad |t| < \delta(\epsilon) \quad \text{and} \quad n \geqq n_0(\epsilon);$$

hence, from (2.6)

$$\begin{aligned} |s_n(0) - s_n(t)| &< \epsilon + \frac{2n\epsilon}{\nu+1} + \frac{2}{\nu+1}\sum_{1}^{\nu}(\nu - k + 1)|a_{n+k}| \\ &< \epsilon + \frac{2n\epsilon}{\nu+1} + 2\sum_{1}^{\nu}|a_{n+k}|, \qquad |t| < \delta(\epsilon),\ n > n_0(\epsilon). \end{aligned} \tag{2.7}$$

Write

$$\limsup_{n\to\infty}\sum_{n}^{\lambda n}|a_\nu| = \omega(\lambda),$$

then

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[1] Numbers in brackets refer to the literature listed at the end of the paper.

$$(2.8) \qquad \sum_{n}^{\lambda n} |a_\nu| < \epsilon + \omega(\lambda) \qquad \text{for } n > n_1(\epsilon, \lambda).$$

Given $\epsilon > 0$, choose $\nu = [n\epsilon^{1/2}]$, and $\lambda = 1 + \epsilon^{1/2}$, then, from (2.7) and (2.8),

$$|s_n(0) - s_n(t)| < \epsilon + 2\epsilon^{1/2} + 2(\epsilon + \omega(1 + \epsilon^{1/2})) \qquad \text{for } n > n_2(\epsilon),$$

when n_2 is the larger of the two numbers n_0, n_1. The theorem now follows from (1.5) and Lemma 2.

The identity

$$s_n(t) - \sigma_n(t) = \frac{1}{n+1} \sum_{1}^{n} \nu a_\nu \cos \nu t$$

yields the corollary:

COROLLARY TO THEOREM 1. *Under the assumptions of Theorem* 1 $n^{-1}\sum_1^n \nu a_\nu \cos \nu t \to 0$ *uniformly at* $t = 0$.

3. **The sine series.** In this case convergence at $t = 0$ is trivial; we introduce two lemmas.

LEMMA 3. *Suppose that the coefficients of the Fourier sine series*

$$(3.1) \qquad \psi(t) \sim \sum_{1}^{\infty} b_n \sin nt$$

satisfy the condition (1.2) *with* $s_n = \sum_1^n b_\nu$, *and that*

$$2h^{-1} \int_0^h \psi(t) dt \to d \qquad \text{as } h \downarrow 0,$$

then

$$n^{-1} \sum_{1}^{n} \nu b_\nu \to \pi^{-1} d.$$

This is Lemma 6 of our paper [6].

LEMMA 4. *If for a sequence* $\{b_n\}$

$$\lim n^{-1} \sum_{1}^{n} \nu b_\nu = l$$

exists, and if

$$(3.2) \qquad \lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} (|b_\nu| - b_\nu) = 0,$$

then

$$\lim_{\lambda\downarrow 1}\limsup_{n\to\infty}\sum_{n}^{\lambda n}|b_\nu| = 0. \tag{3.3}$$

Write

$$\limsup_{n\to\infty}\sum_{n}^{\lambda n}(|b_\nu| - b_\nu) = \xi(\lambda), \qquad \lambda > 1,$$

then by (3.2), $\xi(\lambda)\to 0$ as $\lambda\downarrow 1$. We have

$$\sum_{n}^{\lambda n}\nu(|b_\nu| - b_\nu) \leqq \lambda n\sum_{n}^{\lambda n}(|b_\nu| - b_\nu),$$

hence

$$\limsup n^{-1}\sum_{n}^{\lambda n}\nu(|b_\nu| - b_\nu) \leqq \lambda\xi(\lambda).$$

Furthermore

$$n^{-1}\sum_{n}^{\lambda n}\nu b_\nu = \lambda(\lambda n)^{-1}\sum_{1}^{\lambda n}\nu b_\nu - n^{-1}\sum_{1}^{n-1}\nu b_\nu \to (\lambda - 1)l,$$

hence

$$\limsup n^{-1}\sum_{n}^{\lambda n}\nu|b_\nu| \leqq (\lambda - 1)l + \lambda\xi(\lambda).$$

But

$$\sum_{n}^{\lambda n}|b_\nu| \leqq n^{-1}\sum_{n}^{\lambda n}\nu|b_\nu|,$$

hence

$$\limsup\sum_{n}^{\lambda n}|b_\nu| \leqq (\lambda - 1)l + \lambda\xi(\lambda).$$

Letting $\lambda\downarrow 1$, we get (3.3).

THEOREM 2. *Suppose that the function $\psi(t)$ is continuous at $t=0$, that is $\psi(0)=0$, and that its Fourier coefficients satisfy* (3.2). *Then $\sum_1^n\nu b_\nu = o(n)$, and the series* (3.1) *is uniformly convergent at $t=0$.*

We now write

$$s_n(t) = \sum_{1}^{n} b_\nu \sin \nu t, \qquad \sigma_n(t) = \frac{1}{n+1}\sum_{1}^{n} s_\nu(t);$$

then by the theorem of Fejér

$$\sigma_n(t_n) \to 0 \qquad \text{as } t_n \to 0. \tag{3.4}$$

Also by Lemma 3

$$\sum_{1}^{n}\nu b_\nu = o(n),$$

and Lemma 4 now yields (3.3). Finally from (2.5) with $c_n = b_n \sin nt$, applying (3.4) and Lemma 2,

$$| s_n(t) | < \epsilon + \frac{2n\epsilon}{\nu + 1} + \sum_{n+1}^{n+\nu} | b_k |, \qquad \text{for } | t | < \delta(\epsilon) \text{ and } n > n_0(\epsilon).$$

Write

$$\limsup \sum_{n}^{\lambda n} | b_\nu | = v(\lambda),$$

then by (3.3)

$$v(\lambda) \to 0 \quad \text{as} \quad \lambda \downarrow 1.$$

We now choose $\nu = [n\epsilon^{1/2}]$, then, as in §2,

$$| s_n(t) | < 3\epsilon^{1/2} + 2v(1 + \epsilon^{1/2}) \qquad \text{for } | t | < \delta(\epsilon) \text{ and } n > n_1(\epsilon),$$

which proves the theorem.

COROLLARY. *Under the assumptions of Theorem 2*

$$n^{-1}\sum_{1}^{n} \nu b_\nu \sin \nu t \to 0 \textit{ uniformly at } t = 0.$$

This follows from $s_n(t) - \sigma_n(t) = (n+1)^{-1}\sum_1^n \nu b_\nu \sin \nu t$.

4. A converse theorem. To prove a converse of Theorem 2, we introduce the lemma.

LEMMA 5. *Suppose that* $B_n \geqq 0$, *that for some* $c > 0$

$$B_{n+1} \leqq (1 + c/n)B_n, \qquad n = 1, 2, 3, \cdots, \tag{4.1}$$

and that the sequence $\{B_n\}$ *is Abel summable to* B; *then* $B_n \to B$.

It is known that $B_n \geqq 0$ and Abel summability imply $(C, 1)B_n \to B$, that is

$$B_n' = \sum_{1}^{n} B_\nu \sim nB. \tag{4.2}$$

From (4.1)

$$B_{n+k} \leqq (1 + c/n)^k B_n, \qquad k = 0, 1, 2, \cdots,$$

hence

$$\sum_{k=0}^{\nu} B_{n+k} \leqq B_n \sum_{k=0}^{\nu} (1 + c/n)^k = nB_n c^{-1}\{(1 + c/n)^{\nu+1} - 1\},$$

or $B_n \geqq cn^{-1}\{(1+c/n)^{\nu+1} - 1\}^{-1}(B_{n+\nu}' - B_{n-1}')$. To any given $\delta > 0$ choose $\nu = [\delta n]$, so that $\nu n^{-1} \to \delta$. Then

$$\liminf_{n\to\infty} B_n \geqq c(e^{c\delta} - 1)^{-1}\{(1 + \delta)B - B\} = c\delta B(e^{c\delta} - 1)^{-1};$$

letting $\delta \downarrow 0$, we get

$$\liminf B_n \geqq B. \tag{4.3}$$

Similarly from (4.1) by induction

$$B_{n-k} \geqq (1 + c/(n - k))^{-k-1}B_{n+1} \geqq (1 + c/\nu)^{-k-1}B_{n+1} \qquad \text{for } n - k \geqq \nu > 0,$$

hence

$$\sum_{k=0}^{n-\nu} B_{n-k} \geqq B_{n+1}\sum_{k=0}^{n-\nu}\left(1 + \frac{c}{\nu}\right)^{-k-1} = \nu B_{n+1}c^{-1}\left\{1 - \left(1 + \frac{c}{\nu}\right)^{\nu-n-1}\right\},$$

or

$$B_{n+1} \leqq c\nu^{-1}\{1 - (1 + c/\nu)^{\nu-n-1}\}^{-1}(B'_n - B'_{\nu-1}).$$

Again to any given positive $\delta<1$ choose $\nu = [n\delta]$; then

$$\limsup B_{n+1} \leqq c\{1 - e^{c-c\delta^{-1}}\}^{-1}(B\delta^{-1} - B) = c\delta^{-1}B\frac{1 - \delta}{1 - \exp(c - c\delta^{-1})}.$$

Letting $\delta \uparrow 1$ we find

$$\limsup B_n \leqq B. \tag{4.4}$$

(4.3) and (4.4) prove the lemma.

It is easily seen that the assumption (4.1) is equivalent to saying $n^{-\gamma}B_n$ is decreasing for some γ; our lemma is in close connection to a lemma due to Hardy [**3**, p. 442].

THEOREM 3. *Suppose that* $\psi(t)\sim\sum b_n \sin nt$, *that*

$$\psi(t) \to \pi A/2 \qquad \text{as } t\downarrow 0, \tag{4.5}$$

and that for some constants p *and* c

$$0 \leqq (n + 1)b_{n+1} + p \leqq (1 + c/n)(nb_n + p), \qquad n \geqq 1. \tag{4.6}$$

Then $nb_n \to A$.

Let

$$g(t) = (\pi - t)/2 = \sum_1^\infty n^{-1} \sin nt, \qquad 0 < t \leqq \pi,$$

and

$$\chi(t) = \psi(t) - Ag(t) \sim \sum (b_n - A_n^{-1}) \sin nt \equiv \sum \beta_n \sin nt, \tag{4.7}$$

then, from (4.5),

$$\chi(t) \to 0 \qquad \text{as } t \downarrow 0.$$

Furthermore

$$n\beta_n = nb_n - A \geqq -p - A = -q,$$

say, and

$$(n+1)\beta_{n+1} + q = (n+1)b_{n+1} + p \leqq (1 + c/n)(n\beta_n + q).$$

Thus we need only prove $n\beta_n \to 0$, that is Theorem 3 is reduced to the case $A = 0$. Now for this case Theorem 2 yields $\sum_1^n \nu b_\nu = o(n)$; finally Lemma 5 applied to $B_n = nb_n + p$ gives Theorem 3.

A special case. Let $p = 0$; $c = 1$; then (4.6) reduces to $0 \leqq b_{n+1} \leqq b_n$.

For this case and $A = 0$ the theorem is due to Chaundy and Jolliffe, while for $A \neq 0$ it is due to Hardy [**3**, **4**]. As Hardy remarked, here the case $A \neq 0$ is not immediately reducible to the case $A = 0$. Our generalization has the advantage of such reduction.

5. **On Gibbs' phenomenon.** We shall apply Theorem 2 to the Gibbs' phenomenon (cf. [**7**, p. 181]). Consider again the assumption (4.5); that is $\psi(t)$ has the jump πA, while $\chi(t)$ is continuous at $t = 0$. We assume in addition (3.2); then evidently the β_n satisfy the same assumption, hence by Theorem 2

$$\sum_1^n \nu\beta_\nu = \sum_1^n \nu b_\nu - nA = o(n),$$

and the series (4.7) is uniformly convergent at $t = 0$. On the other hand Fejér proved that

$$\limsup_{t_n \downarrow 0} \sum_1^n \nu^{-1} \sin \nu t_n = \lim_{nt_n \to \pi} \sum_1^n \nu^{-1} \sin \nu t_n = \int_0^\pi t^{-1} \sin t dt,$$

hence assuming, as we may, $A > 0$,

$$\limsup_{t_n \downarrow 0} \sum_1^n b_\nu \sin \nu t_n = \lim_{nt_n \to \pi} \sum_1^n b_\nu \sin \nu t_n = A \int_0^\pi t^{-1} \sin t dt. \tag{5.1}$$

We have thus proved the theorem:

THEOREM 4. *Suppose that* $\psi(t) \sim \sum b_n \sin nt$ *satisfies the conditions* (4.5) *and* (3.2); *then*

$$\sum_1^n \nu b_\nu \sim An,$$

and

$$\sum_{1}^{n} b_\nu \sin \nu t_n - A \sum_{1}^{n} \nu^{-1} \sin \nu t_n \to 0 \qquad \text{as } t_n \to 0;$$

in particular (5.1) *holds, that is the two series of* $\psi(t)$ *and* $Ag(t)$ *present the same phenomenon of Gibbs.*

For the special case $nb_n = O(1)$ Gibbs' phenomenon was observed by Rogosinski [5, pp. 134–135], however it is difficult to follow his argument.

6. **A contre example.** We cannot replace in Theorems 1 and 2 the conditions (1.6) and (3.2) by (1.3) with $s_n = \sum_1^n a_\nu$ or $s_n = \sum_1^n b_\nu$ respectively. This is seen from an example constructed by Fejér [2] for a similar purpose. It is a power series $\sum_{k=1}^{\infty} c_k z^k$ with the following properties [2, pp. 38–46]: The coefficients are all real; the power series is convergent for $|z| \leqq 1$; the function $f(z) = \sum c_k z^k$ is continuous for $|z| \leqq 1$; the power series is uniformly convergent for $z = e^{it}$, $\epsilon \leqq t \leqq 2\pi - \epsilon$, $\epsilon > 0$, but neither of the series $\sum a_k \cos kt$, $\sum a_k \sin kt$ is uniformly convergent for $|t| \leqq \epsilon$. It follows easily that neither series is uniformly convergent at $t = 0$, for this would imply uniform convergence on the entire unit circle.

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