CONVERGENCE CRITERIA FOR CONTINUED FRACTIONS*

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1. *Introduction.* The object of this paper is to present two new criteria for the convergence of the continued fraction

$$F(\alpha, z) = \frac{1}{\alpha_1 z} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3 z} + \cdots,$$

in which the numbers α_i are real and different from zero.

Necessary and sufficient conditions for convergence have been discovered for the case that $\alpha_i > 0$ by Stieltjes;† and by Hamburger‡ when $\alpha_{2i+1} > 0$. There seem to be no necessary and sufficient conditions known for the general case, although several sufficient conditions have been found. Van Vleck§ showed that if k is the greatest modulus of the limit points of the numbers $1/(\alpha_i\alpha_{i+1})$, then $F(\alpha, z)$ converges, except for isolated points, within the circle $|z| = 1/(4k)$. Inasmuch as k may be infinite while at the same time $\alpha_i > 0$, $\sum \alpha_i$ diverges, it follows from the work of Stieltjes that $F(\alpha, z)$ may converge to an analytic limit even when the circular region $|z| = 1/(4k)$ vanishes. The theorems which I shall give include certain cases of this sort.

2. *Notation.* Let $a_1, a_2, a_3, \cdots; b_1, b_2, b_3, \cdots$, be two infinite sequences of real non-zero numbers connected by the relations

$$(1) \qquad a_{2i} = b_{2i+1}/(\overset{b}{\delta}_{i-1}\overset{b}{\delta}_i), \qquad a_{2i+1} = b_{2i+2}[\overset{b}{\delta}_i]^2,$$

where

$$\overset{b}{\delta}_i = b_1 + b_3 + \cdots + b_{2i+1}.$$

It is easily seen that if we set

$$\overset{a}{g}_i = a_2 + a_4 + \cdots + a_{2i}, \quad (\overset{a}{g}_0 = 0),$$

* Presented to the Society, April 3, 1931.

† Annales de la Faculté des Sciences de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47.

‡ Mathematische Annalen, vol. 81, pp. 234–319; vol. 82, pp. 120–187.

§ Transactions of this Society, vol. 2, pp. 476–483.

then

$$\text{(2)} \qquad \delta_i^{b} = b_1/(1 - b_1 g_i^{a}),$$

$$\text{(3)} \quad b_{2i+1} = b_1^2 a_{2i}/\left[(1 - b_1 g_i^{a})(1 - b_1 g_{i-1}^{a})\right], b_{2i+2} = a_{2i+1}(1 - b_1 g_i^{a})^2/b_1^2.$$

If we are given a set of non-zero a_i, it is clearly possible to find a set of non-zero b_i such that (1) holds. The formulas (3) effect a transformation,

$$\text{(4)} \qquad b = [a],$$

of the a_i into the b_i. By means of (4) a continued fraction $F(a, z)$ is transformed into another continued fraction $F(b, z)$.

Denote by P_n^α/Q_n^α the nth convergent of $F(\alpha, z)$. Then certain formulas* which I gave in a recent article may be used to connect the polynomials P_n^a, Q_n^a with the P_n^b, Q_n^b. They run as follows:

$$\text{(5)} \qquad \begin{cases} Q_{2n}^{a} = Q_{2n+1}^{b}/(z\delta_n^{b}), \\ P_{2n}^{a} = b_1^{-1} Q_{2n}^{a} - P_{2n+1}^{b}/\delta_n^{b}, \\ Q_{2n-1}^{a} = \delta_n^{b} Q_{2n}^{b} - Q_{2n+1}^{b}/z, \\ P_{2n-1}^{a} = b_1^{-1} Q_{2n-1}^{a} - z\left[\delta_n^{b} P_{2n}^{b} - P_{2n+1}^{b}/z\right]. \end{cases}$$

3. *Convergence of* $F(a, z)$. If a continued fraction $F(a, z)$ is transformed by (4) into a continued fraction $F(b, z)$, the equations (5) serve to connect the convergents of $F(a, z)$ with those of $F(b, z)$.

THEOREM 1. *Let $F(b, z)$ be the continued fraction obtained from $F(a, z)$ by means of the transformation $b = [a]$, and suppose that $\sum |b_i|$ converges. Then there is a value of z for which $F(a, z)$ converges if and only if*

$$\text{(6)} \qquad \lim_{n \to \infty} |g_n^{a}| = \infty .$$

If $F(a, z)$ converges for a single value of z, then there exist two entire functions $p(z)$ and $q(z)$ such that

$$\lim_n z P_{2n-1}^{a} = - \lim_n z\delta_n^{b} P_{2n}^{a} = p(z),$$

$$\lim_n z Q_{2n-1}^{a} = - \lim_n z\delta_n^{b} Q_{2n}^{a} = q(z),$$

* Transactions of this Society, vol. 33 (1931), Theorem 1, p. 514.

uniformly over every finite closed region, and hence $F(a, z)$ converges over the entire plane except at isolated points, and its limit is $p(z)/q(z)$.

In fact, by a theorem of von Koch,* it follows from the convergence of $\sum |b_i|$ that the sequences of polynomials

$$\overset{b}{P}_{2n-1}, \overset{b}{Q}_{2n-1}, \overset{b}{P}_{2n}, \overset{b}{Q}_{2n}$$

converge uniformly over every finite closed region to entire limit functions

$$r^b, s^b, R^b, S^b,$$

respectively; and

$$r^bS^b - R^bs^b \equiv +1. \tag{7}$$

Now by (2) and (6),

$$\lim_n \overset{b}{\delta}_n = \overset{b}{\delta}, \quad \overset{b}{\delta} = 0. \tag{8}$$

We find that if (6) does not hold, the only alternative to (8) is that

$$\lim_n \overset{b}{\delta}_n = \overset{b}{\delta}, \quad \overset{b}{\delta} \neq 0, \tag{9}$$

where δ^b is finite.

It now follows immediately from (5) that

$$\lim_n z\overset{a}{Q}_{2n-1} = z\overset{b}{\delta}\overset{b}{S} - \overset{b}{s} = z\overset{a}{s},$$

$$\lim_n z\overset{a}{P}_{2n-1} = \overset{-1}{b_1} z\overset{a}{s} - z[z\overset{b}{\delta}\overset{b}{R} - \overset{b}{r}] = z\overset{a}{r},$$

$$-\lim_n z\overset{b}{\delta}_n\overset{a}{Q}_{2n} = -\overset{b}{s} = z\overset{a}{S},$$

$$-\lim_n z\overset{b}{\delta}_n\overset{a}{P}_{2n} = \overset{-1}{b_1} z\overset{a}{S} + zr^b = zR^a,$$

uniformly over every finite closed region. When (8) holds we find that $r^a/s^a \equiv R^a/S^a$, and hence $F(a, z)$ converges as stated in the theorem. The entire functions $p(z)$ and $q(z)$ are as follows:

$$p(z) = zr^b - b_1^{-1}s^b, \quad q(z) = -s^b.$$

When, on the other hand, (9) holds, we find with the aid of (7)

* Bulletin de la Société Mathématique, vol. 23, pp. 33–40.

that $R^a s^a - r^a S^a = \delta^b$, and therefore if (6) does not hold, $F(a, z)$ diverges for every value of z.*

4. *Another Theorem on Convergence.* Another theorem may be obtained if we transform $F(a, z)$ into $F(b, z)$, and then $F(b, z)$ into $F(c, z)$ by means of the transformations $b = [a]$, $c = [b]$. We find with the aid of (2) and (3) that

$$\overset{c}{\delta}_i = c_1/(1 - c_1 \overset{b}{g}_i), \tag{10}$$

$$\left\{\begin{aligned} c_{2i+1} &= \frac{c_1^2 a_{2i-1}(1 - b_1 \overset{a}{g}_{i-1})^2}{b_1^2(1 - c_1 g_i^b)(1 - c_1 g_{i-1}^b)}, \\ c_{2i+2} &= \frac{b_1^2 a_{2i}(1 - c_1 \overset{b}{g}_i)^2}{c_1^2(1 - b_1 g_i^a)(1 - b_1 g_{i-1}^a)}. \end{aligned}\right. \tag{11}$$

Also by (5),

$$\begin{aligned} \overset{a}{Q}_{2n} &= (\overset{c}{\delta}_{n+1}\overset{c}{Q}_{2n+2} - \overset{c}{Q}_{2n+3}/z)/(z\overset{b}{\delta}_n), \\ \overset{a}{P}_{2n} &= (b_1^{-1} - c_1^{-1} z)\overset{a}{Q}_{2n} + z(\overset{c}{\delta}_{n+1}\overset{c}{P}_{2n+2} - \overset{c}{P}_{2n+3}/z)/\overset{b}{\delta}_n, \\ \overset{a}{Q}_{2n+1} &= \overset{b}{\delta}_n\overset{c}{Q}_{2n+1}/(z\overset{c}{\delta}_n) - \overset{c}{\delta}_{n+1}\overset{c}{Q}_{2n+2}/z + \overset{c}{Q}_{2n+3}/z^2, \\ \overset{a}{P}_{2n-1} &= (b_1^{-1} - c_1^{-1} z)\overset{a}{Q}_{2n-1} + z^2(\overset{b}{\delta}_n\overset{c}{P}_{2n+1}/(z\overset{c}{\delta}_n) - \overset{c}{\delta}_{n+1}\overset{c}{P}_{2n+2}/z + \overset{c}{P}_{2n+3}/z^2). \end{aligned} \tag{12}$$

THEOREM 2. *Let $F(a, z)$, $F(b, z)$, $F(c, z)$ be connected by the relations*

$$b = [a], \quad c = [b],$$

and suppose $\sum |c_i|$ converges. Then $F(a, z)$ converges over the entire plane except at isolated points in the following cases:

(i) $\sum c_{2i+1} = 0$, (ii) $\sum c_{2i+1} \neq 0$, $\sum b_{2i+1} = 0$.

In every other case $F(a, z)$ diverges for all z. In fact, by (12),

$$\frac{\overset{a}{P}_{2n}}{Q^\alpha_{2n}} = b_1^{-1} - c_1^{-1} z + z^2\left[\frac{z\overset{c}{\delta}_{n+1}\overset{c}{P}_{2n+2} - \overset{c}{P}_{2n+3}}{z\overset{c}{\delta}_{n+1}\overset{c}{Q}_{2n+2} - \overset{c}{Q}_{2+n3}}\right].$$

Now if (i) holds, the numerator and denominator of the fraction on the right converge uniformly over every finite closed region to entire limit functions

$$-r^c, \quad -s^c,$$

* Except possibly at $z=0$. But one may verify directly that, at $z=0$, r^a/s^a has a pole, while R^a/S^a is regular at $z=0$.

respectively. It follows that

$$\lim_n \frac{\overset{a}{P}_{2n}}{\overset{\alpha}{Q}_{2n}} = b_{\bar{1}}^{-1} - c_{\bar{1}}^{-1}z + z^2 \frac{r^c}{s^c},$$

everywhere except for isolated values of z. Again by (12)

$$\frac{\overset{\alpha}{P}_{2n-1}}{\overset{\alpha}{Q}_{2n-1}} = b_{\bar{1}}^{-1} - c_{\bar{1}}^{-1}z$$
$$+ z^2\left[\frac{z\overset{c}{P}_{2n+1} - z\overset{c}{\delta}_{n+1}(\overset{c}{\delta}_n/\overset{b}{\delta}_n)\overset{c}{P}_{2n+2} + (\overset{c}{\delta}_n/\overset{b}{\delta}_n)\overset{c}{P}_{2n+3}}{z\overset{c}{Q}_{2n+1} - z\overset{c}{\delta}_{n+1}(\overset{c}{\delta}_{2n}/\overset{b}{\delta}_n)\overset{c}{Q}_{n+2} + (\overset{c}{\delta}_n/\overset{b}{\delta}_n)\overset{c}{Q}_{2n+3}}\right].$$

Denote by $f_n(z)$ the fraction on the right, and suppose that z is not a root of s^c. We find that the sequence

$$f_1(z),\ f_2(z),\ f_3(z),\ \cdots$$

is compact. Indeed every infinite subsequence contains a subsequence with the limit r^c/s^c, so that the sequence also converges to the same limit. Hence

$$\lim_n \frac{\overset{a}{P}_{2n-1}}{\overset{a}{Q}_{2n-1}} = b_{\bar{1}}^{-1} - c_{\bar{1}}^{-1}z + z^2 \frac{r^c}{s^c},$$

and therefore $F(a, z)$ converges, except at isolated points.

If (ii) holds, then we find that $\sum |b_i|$ converges, and hence by Theorem 1, $F(a, z)$ converges if and only if $\sum b_{2i+1} = 0$.

5. *Example.* Let

$$\alpha_n = \begin{cases} \sigma^i, \text{ if } n = 2i, \\ \rho^i, \text{ if } n = 2i - 1, \end{cases}$$

where ρ, σ are real and not zero. Then we find that $F(\alpha, z)$ is a meromorphic function by Theorem 1 when

$$|\rho| < 1,\ |1/\sigma| < 1,\ |\rho\sigma^2| < 1;$$

and when

$$|\sigma| < 1,\ |1/\rho| < 1,\ |\rho^2\sigma| < 1,$$

by Theorem 2. Here $\lim_n |1/(\alpha_n\alpha_{n+1})| = \infty$, and the continued fraction does not, in general, satisfy any of the criteria mentioned at the beginning of this article.

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