ON THE SERIAL RELATION IN BOOLEAN ALGEBRAS*

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The elements $x, y, z, \cdots$ of a class K are said to form a *series* with respect to a dyadic relation S, if they satisfy the following postulates:†

P_1. If $x \neq y$, then either xSy or ySx.

P_2. If xSy, then $x \neq y$.

P_3. If xSy and ySz, then xSz.

It is my object to determine all serial relations in Boolean algebras given by universal propositions expressible in the fundamental Boolean operations of addition, multiplication, and negation.

All the desired serial relations S must be of the form

$$axy + bxy' + cx'y + dx'y' = 0, \tag{1}$$

where x' is the negative of x. Our problem then reduces itself to finding the conditions imposed on the coefficients of (1) by P_1-P_3. We proceed to determine these conditions.

If in (1) $x=0$ and $y=1$, then $c=0$; if $x=1$ and $y=0$, then $b=0$. Hence the condition imposed on (1) by P_1 is

$$b = 0 \quad \text{or} \quad c = 0. \tag{2}$$

The condition imposed on (1) by P_2 is that the equation $axx + bxx' + cx'x + dx'x' = 0$ have no solution. This condition is

$$ad \neq 0. \tag{3}$$

The condition that (1) satisfies P_3 non-vacuously is (this BULLETIN, vol. 30, p. 127)

$$a + d < b + c, \quad ad = 0, \tag{4}$$

which contradicts (3). Hence, *there are no non-vacuous serial relations* (1) *in any Boolean algebra.*

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† See E. V. Huntington, *The Continuum*, 2d ed., p. 10.

But there may possibly be *vacuous* serial relations (1). To determine these, let us first find the condition that there be no x, y, z such that both xSy and ySz hold, i. e., the condition that there be no x, y, z such that simultaneously

$$\text{(i)} \qquad axy + bxy' + cx'y + dx'y' = 0,$$

$$\text{(ii)} \qquad ayz + byz' + cy'z + dy'z' = 0.$$

Since (i) and (ii) together are equivalent to the single equation

$$\text{(iii)} \quad axyz + (b+c)xy'z + (b+c)x'yz + (c+d)x'y'z + (a+b)xyz' + (b+d)xy'z' + (b+c)x'yz' + dx'y'z' = 0,$$

our condition is found to be

$$\text{(5)} \qquad ad(b+c) \neq 0.$$

Now a in (1) must be either 0 or 1, for otherwise neither $1Sa$ nor $aS1$, contrary to P_1. Similarly b, c, d must be either 0 or 1. Hence, taking account of (2) and (5), we find that all vacuous serial relations (1) must be among the relations

$$\text{(6)} \qquad xy + xy' + x'y' = 0, \qquad xy + x'y + x'y' = 0.$$

Since for $e \neq 0, 1$ neither equation of (6) is satisfied by $x = 1$ and $y = e$, or by $x = e$ and $y = 1$, we see that *there are no vacuous serial relations* (1) *in a Boolean algebra consisting of more than two elements.*

However, equations (6) are serial relations in a *two-element* Boolean algebra, since they produce respectively the relation-tables (this BULLETIN vol. 30, p. 27):

(7)

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & - \end{array}, \qquad \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & + & - \end{array}$$

We may sum up our results as follows: *Relations* (6) *are the only serial relations of form* (1) *in Boolean algebras; both of these relations are vacuous and both are confined to the Boolean algebra of order two.*

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