A THEOREM ON THE ANALYTIC EXTENSION OF POWER SERIES.

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IN a note published in the *Journal de Mathématiques* in 1903 I considered the function $f(z)$ defined by a given power series*

$$(1) \qquad g(0) + g(1)z + g(2)z^2 + \cdots + g(n)z^n + \cdots \qquad (r = rad. \ of \ conv. > 0).$$

The problem was to determine the character of $f(z)$ *outside* the circle of convergence and it was shown that if the coefficient $g(n)$ could be generalized into a function $g(w)$ of the complex variable $w = x + iy$ satisfying certain specified conditions, then $f(z)$ could be extended analytically throughout any (finite) region which did not include the positive half of the real axis, and an explicit form was given defining the function throughout such region. The conditions there imposed upon $g(w)$ were unnecessarily restrictive and, in view of certain analogous but more general theorems of Mellin, Le Roy, and Lindelöf,† it is proposed to show in the present note that my earlier results may be generalized into the following

THEOREM: *If the coefficient $g(n)$ of the power series*

$$(2) \qquad \sum_{n=a}^{\infty} g(n)z^n \qquad \begin{array}{l} a = \textit{integer, positive, negative, or zero} \\ r = \textit{radius of convergence} > 0 \end{array}$$

may be considered as a function $g(w)$ of the complex variable $w = x + iy$ satisfying the following conditions: (a) it is single valued and analytic throughout all portions of the plane lying to the right of (or upon) the vertical line $w = a - \frac{1}{2} + iy$; and (b) it is such that

$$\lim_{y=+\infty} e^{-\epsilon y} g(x \pm iy) = 0 \qquad (x \geqq a - \tfrac{1}{2}),$$

* Subsequently generalized to functions defined by double power series. Cf. *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 260–274.

† For an exposition of this subject with bibliography see Lindelöf, Le Calcul des Residus etc., Chap V. (Paris, Gauthier-Villars, 1905). It is believed that the theorem of the present paper, while equally general with those of Lindelöf and others, has the advantage of furnishing a considerably simpler form for the function $f(z)$.

in which ϵ *represents an arbitrarily small positive quantity, then the function* $f(z)$ *of the complex variable* z *defined by* (1) *when* $|z| < r$ *may be extended analytically throughout the whole (finite)* z *plane with the exception of the positive half of the real axis, and for this region* $f(z)$ *will be defined by the equation*

$$(3)\qquad f(z) = (-1)^a \frac{(-z)^{a-\frac{1}{2}}}{2} \int_{-\infty}^{\infty} \frac{g(a-\frac{1}{2}+iy)(-z)^{iy}}{\cosh \pi y}\, dy$$

in which if we place $z = \rho(\cos\phi + i \sin\phi)$ *it is supposed that we take* $-2\pi < \phi < 0$ *and write* $(-z)^{iy} = e^{iy \log(-z)} = e^{iy[\log \rho + (\phi+\pi)i]}$.

For the proof of this theorem let us consider, as in the former note, the result obtained by integrating the function

$$\frac{\pi g(w)(-z)^w}{\sin \pi w}$$

about a rectangular contour C_n in the w plane. In the present instance let this rectangle be formed by the lines $w = a - \frac{1}{2} + iy$, $w = \frac{1}{2} + 2n + iy$, $w = x \pm ij$, where n is any integer such that $2n > a$ and where j is any positive quantity, arbitrarily large. We thus arrive directly by elementary results in the calculus of residues at the equation

$$\sum_{n=a}^{2n} g(n) z^n = \frac{1}{2i} \int_{C_n} \frac{g(w)(-z)^w}{\sin \pi w}\, dw.$$

We proceed to study the integral here appearing, supposing at first that z is real and negative.

First, along the side upon which $w = x + ij$ we have $dw = dx$ and $\sin \pi w = \sin \pi(x+ij) = \sin \pi j(\sin \pi x \coth \pi j + i \cos \pi x)$ so that, if we call the contribution from the side in question I, we may write

$$I = \frac{(-z)^{ij}}{2i \sinh \pi j} \int_{a-\frac{1}{2}}^{2n+\frac{1}{2}} \frac{g(x+ij)(-z)^x}{\sin \pi x \coth \pi j + i \cos \pi x}\, dx.$$

Whence, $\lim_{j=\infty} I = 0$, provided that

$$(4)\qquad \lim_{j=\infty} e^{-\pi j} g(x+ij) = 0 \qquad (x \geqq a - \tfrac{1}{2}).$$

Similarly, we find the same result for the contribution arising from the side of C_n upon which $w = x - ij$, provided that

$$(5) \qquad \lim_{j=\infty} e^{-\pi j} g(x - ij) = 0 \qquad (x \geqq a - \tfrac{1}{2}).$$

Next, let us consider the side upon which $w = \frac{1}{2} + 2n + iy$. Here we have $dw = idy$, $\sin \pi w = \cos i\pi y = \cosh \pi y$, so that having taken $j = \infty$, the contribution in question becomes

$$J = \frac{(-z)^{\frac{1}{2}+2n}}{2} \int_{-\infty}^{\infty} \frac{g(\frac{1}{2} + 2n + iy)(-z)^{iy}}{\cosh \pi y} dy.$$

If we now substitute for conditions (4) and (5) the single, stronger condition

$$(6) \qquad \lim_{y=+\infty} e^{-\epsilon y} g(x \pm iy) = 0,$$

in which ϵ represents an arbitrarily small positive quantity, it appears directly that the improper integral here occurring has a meaning (z real and negative). If in particular $|z| < 1$ we shall evidently have also $\lim_{n=\infty} J = 0$.

Whence, if we now take account of the contribution arising from the remaining side $w = a - \frac{1}{2} + iy$ of C_n, noting that we here have $\sin \pi w = (-1)^{a-1} \cosh \pi y$ while the integration takes place from $y = +\infty$ to $y = -\infty$, we may write

$$(7) \quad \sum_{n=a}^{\infty} g(n) z^n = (-1)^a \frac{(-z)^{a-\frac{1}{2}}}{2} \int_{-\infty}^{\infty} \frac{g(a - \frac{1}{2} + iy)(-z)^{iy}}{\cosh \pi y} dy.$$

This relation must hold good, as we have indicated, for all values of z which are real and negative and such that $|z| < 1$. But the first member represents a function of the complex variable z which is single valued and analytic throughout the the circle of convergence of (1), while the second member, with proper conventions as regards the meaning of $(-z)^{iy}$, represents, as we shall now show, a function of z which is analytic and single valued throughout the whole z plane except for the positive half of the real axis.

Thus let us place $z = \rho(\cos \phi + i \sin \phi)$ and agree to write

$$\log(-z) = \log \rho + i(\phi + \pi).$$

Then

$$(-z)^{iy} = e^{iy \log(-z)} = e^{iy[\log \rho + i(\phi + \pi)]} = e^{iy \log \rho} e^{-(\phi + \pi) y}.$$

Moreover, for all values of z within a region T which does not cut or touch the positive half of the real axis we shall have $-\pi < \phi + \pi < \pi$, provided we agree to choose ϕ at every point so that $-2\pi < \phi < 0$. It follows, upon introducing (6), that when the above agreements are made we may always choose ϵ so small that the improper integral in (7) will converge *uniformly* for all values of z in T. Whence, the same integral and hence also the second member of (7) will have the analytic properties indicated above.*

Thus we reach in summary the theorem stated at the beginning.

It may be observed that in case the function $g(w)$ satisfies the conditions demanded except that it has a finite number of singularities in the region of the w plane lying to the right of the line $w = a - \frac{1}{2} + iy$ the theorem continues true provided we subtract from the second member of (3) the sum of the residues of the function

$$\frac{\pi g(w)(-z)^w}{\sin \pi w}$$

corresponding to such singularities.

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EXTENSIONS OF TWO THEOREMS DUE TO CAUCHY

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THE last one of the noted series of papers on substitution groups published by Cauchy during 1845–6 in the Paris *Comptes Rendus* is devoted to a simplification of his earlier proof of an important theorem which may be stated as follows: If the symmetric group G of degree n involves at least one substitution which transforms one of its subgroups H_1 into a group having only identity in common with the subgroup H_2, the total number of such substitutions in G is divisible by the product of the orders of H_1 and H_2. The proof given by Cauchy is

*Cf. Osgood, Encyklopädie, II, p. 21.