AN APPLICATION OF THE NOTIONS OF "GENERAL ANALYSIS" TO A PROBLEM OF THE CALCULUS OF VARIATIONS.

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THE object of the following note is to give an illustration of the unifying power of Professor E. H. Moore's methods of "General Analysis" * by showing that a certain theorem of the calculus of variations and a certain theorem of analytic geometry are special cases of one and the same theorem of general analysis.

The theorem of the calculus of variations is the so-called fundamental lemma for isoperimetric problems,† viz.,

THEOREM I. "*If*

$$(1) \qquad \mu_0(\eta) \equiv \int_{x_1}^{x_2} [M_0(x)\eta(x) + N_0(x)\eta'(x)]\,dx = 0$$

for all functions $\eta(x)$ *which are* (*a*) *of class* C' *on* $[x_1x_2]$, (*b*) *vanish at* x_1 *and* x_2, *and* (*c*) *satisfy the* m *conditions*

$$(2) \qquad \mu_i(\eta) \equiv \int_{x_1}^{x_2} [M_i(x)\eta(x) + N_i(x)\eta'(x)]\,dx = 0 \qquad (i = 1, 2, \cdots, m),$$

then there exist m *constants* $c_1, c_2, \cdots, c_m$ *such that*

$$(3) \qquad \mu_0(\eta) + c_1\mu_1(\eta) + c_2\mu_2(\eta) + \cdots + c_m\mu_m(\eta) = 0$$

for all functions $\eta(x)$ *satisfying conditions* (*a*) *and* (*b*).

The functions $M(x)$, $N(x)$ *are supposed to be continuous on* $[x_1x_2]$.

The theorem of analytic geometry is the well known

* Compare E. H. Moore, "On a form of General Analysis with applications to linear differential equations and integral equations," Atti del IV congresso internazionale dei Mathematici, vol. 2, p. 98; and "Introduction to a form of General Analysis," in The New Haven Mathematical Colloquium, Yale University Press, New Haven, 1910.

† Compare for instance Bolza, Vorlesungen über Variationsrechnung, p. 462, footnote 1, and the references given there.

THEOREM II. "*If, in a plane and in homogeneous coordinates,*

$$(1') \qquad U_0 \equiv A_0x + B_0y + C_0z = 0$$

is the equation of a straight line passing through the point of intersection of the two non-coinciding lines*

$$(2') \quad U_1 \equiv A_1x + B_1y + C_1z = 0, \quad U_2 \equiv A_2x + B_2y + C_2z = 0,$$

then there exist two constants λ_1, λ_2 *such that*

$$U_0 \equiv \lambda_1 U_1 + \lambda_2 U_2."$$

§ 1. *The General Theorem.*

Let p be a general parameter† ranging over a set $\mathfrak{P}$ of elements; these elements may be any mathematical entities whatever: real or complex numbers, pairs, triples, etc., of such numbers, even infinite sets of numbers; functions of one or several variables; systems of functions; points, curves, surfaces; etc., etc.

Along with the set $\mathfrak{P}$ we consider the set $\mathfrak{Q}$ of all possible systems $(a_1, a_2; p_1, p_2)$ of a pair of real numbers a_1, a_2 and a pair of elements p_1, p_2 of $\mathfrak{P}$, and we suppose that a correspondence has been established by which to every element of $\mathfrak{Q}$ corresponds a unique element of $\mathfrak{P}$ which we denote by ‡

$$F(a_1, a_2; p_1, p_2).$$

We shall then say that a real single-valued function § $\mu(p)$ defined on $\mathfrak{P}$ is "*linear as to F,*" if

$$(4) \qquad \mu[F(a_1, a_2; p_1, p_2)] = a_1\mu(p_1) + a_2\mu(p_2) \quad \text{on } \mathfrak{Q},$$

i. e., for every combination $(a_1, a_2; p_1, p_2)$ of $\mathfrak{Q}$.

Then the following theorem holds: ||

THEOREM III. *If*

$$\mu_0(p), \ \mu_1(p), \ \cdots, \ \mu_m(p)$$

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* We may omit the word "non-coinciding" if we replace "point of intersection of" by "point or points common to."

† Compare Moore, "Introduction etc.," § 1; I use throughout this section Moore's notation.

‡ In Moore's terminology F is a "function on $\mathfrak{Q}$ to $\mathfrak{P}$," "Introduction etc.," § 4.

§ Compare Moore, "Introduction etc.," § 5; if $\mathfrak{A}$ denotes the set of all real numbers, $\mu(p)$ is in Moore's terminology a "function on $\mathfrak{P}$ to $\mathfrak{A}$."

|| This generalization of Theorem I has been suggested to me by a remark in § 177 of Hadamard's Leçons sur le calcul des variations, Paris, 1910.

are $m+1$ *real single-valued functions of* p, *defined on* $\mathfrak{P}$, *which satisfy the following two conditions:*

A) they are linear as to F,

B) the equation

$$(1'') \qquad \mu_0(p) = 0$$

holds for every element of $\mathfrak{P}$ *which satisfies simultaneously the* m *equations*

$$(2'') \qquad \mu_1(p) = 0, \; \mu_2(p) = 0, \; \cdots, \; \mu_m(p) = 0,$$

then there exist m *real numbers* $c_1, c_2, \cdots, c_m$, *independent of* p, *such that*

$$(3'') \qquad \mu_0(p) + c_1\mu_1(p) + \cdots + c_m\mu_m(p) = 0 \quad \textit{on} \;\; \mathfrak{P},$$

i. e., for every element of $\mathfrak{P}$.

Proof: We notice first that there always exist elements of $\mathfrak{P}$ which do satisfy the m equations $(2'')$; for $F(0, 0; p_1, p_2)$ is an element of $\mathfrak{P}$ for any two elements p_1, p_2 of $\mathfrak{P}$, and on account of A)

$$\mu_i[F(0, 0; p_1, p_2)] = 0, \quad (i = 1, 2, \cdots, m).$$

Further we observe that if we define

$$F[1, a_3; F(a_1, a_2; p_1, p_2), p_3] = F(a_1, a_2, a_3; p_1, p_2, p_3)$$

and generally

$$(5) \quad F[1, a_n; F(a_1, a_2, \cdots, a_{n-1}; p_1, p_2, \cdots, p_{n-1}), p_n] = F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n),$$

then $F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n)$ is again an element of $\mathfrak{P}$, and, if (4) is satisfied, then also

$$(6) \quad \mu[F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n)] = a_1\mu(p_1) + a_2\mu(p_2) + \cdots + a_n\mu(p_n).$$

After these preliminary remarks we distinguish two cases:

Case I: The m equations $(2'')$ are satisfied for every p of $\mathfrak{P}$. Then according to B)

$$\mu_0(p) = 0 \;\text{ on }\; \mathfrak{P}.$$

Hence we may write

$$\mu_0(p) + 0 \cdot \mu_1(p) + 0 \cdot \mu_2(p) + \cdots + 0 \cdot \mu_m(p) = 0 \quad \text{on } \mathfrak{P},$$

and the theorem is proved with the particular values $c_1 = 0$, $c_2 = 0, \cdots, c_m = 0$.

Case II: The m equations $(2'')$ are not all satisfied for every p of $\mathfrak{P}$.

Then there exists a definite integer n $(1 \leqq n \leqq m)$ such that in the determinant

$$\Delta = |\mu_i(p_k)| \qquad (i, k = 1, 2, \cdots, m)$$

at least one minor of degree n is different from zero for some special system $p_1, p_2, \cdots, p_m$, whereas (for $n < m$) all minors of degree $n + 1$ vanish identically, that is, for every choice of the m elements $p_1, p_2, \cdots, p_m$. In order to fix the ideas we suppose that the minor

$$(7) \qquad \Delta_0 = |\mu_g(p_h)| \neq 0 \quad (g, h = 1, 2, \cdots, n).$$

Let now p be any element of $\mathfrak{P}$ and $p_1, p_2, \cdots, p_n$ the n special elements for which $\Delta_0 \neq 0$; then

$$q = F(1, a_1, a_2, \cdots, a_n; p, p_1, p_2, \cdots, p_n)$$

is an element of $\mathfrak{P}$, and according to A)

$$(8) \qquad \mu_j(q) = \mu_j(p) + a_1\mu_j(p_1) + \cdots + a_n\mu_j(p_n)$$
$$(j = 0, 1, 2, \cdots, m).$$

On account of (7) we can so determine $a_1, a_2, \cdots, a_n$ that

$$(9) \qquad \mu_1(q) = 0, \ \mu_2(q) = 0, \ \cdots, \ \mu_n(q) = 0.$$

If $n < m$, it follows from the identical vanishing of the minors of degree $n + 1$ of the determinant Δ, p taking the place of p_{n+1}, that also

$$(10) \qquad \mu_{n+1}(q) = 0, \ \mu_{n+2}(q) = 0, \ \cdots, \ \mu_m(q) = 0.$$

Hence for $n < m$ as well as for $n = m$, q is an element of $\mathfrak{P}$ which satisfies the m equations $(2'')$ and therefore it satisfies according to B) also the equation

$$(11) \qquad \mu_0(q) = 0.$$

But from the $n + 1$ equations (9) and (11) it follows, if we write the $\mu_j(q)$'s in their explicit form (8), that the determinant

$$(12) \quad |\mu_j(p), \ \mu_j(p_1), \ \cdots, \ \mu_j(p_n)| = 0 \quad (j = 0, 1, 2, \cdots, n).$$

If now we expand this determinant according to the elements of the first column, the coefficient of $\mu_0(p)$ is the determinant Δ_0 and therefore different from zero, and this determinant as well as the remaining coefficients of the expansion is *independent of* p. Hence if we divide by Δ_0, we obtain equation (3″) with $c_{n+1}=0, c_{n+2}=0, \cdots, c_m=0$, and this equation holds on $\mathfrak{P}$, since p was *any* element of $\mathfrak{P}$. Thus our theorem is proved.*

§ 2. *Theorems I and II as Special Cases of Theorem III.*

In order to obtain Theorem I as a special case of Theorem III, we identify the set $\mathfrak{P}$ with the totality of all functions $\eta(x)$ of class C' on $[x_1x_2]$ which vanish at x_1 and x_2, and define

$$F(a_1, a_2; \eta_1, \eta_2) = a_1\eta_1 + a_2\eta_2. \tag{13}$$

If a_1, a_2 are two constants and $\eta_1(x)$, $\eta_2(x)$ two functions of $\mathfrak{P}$, $a_1\eta_1(x) + a_2\eta_2(x)$ again belongs to $\mathfrak{P}$ and the "functions"

$$\mu_j(\eta) = \int_{x_1}^{x_2} [M_j(x)\eta(x) + N_j(x)\eta'(x)]\,dx \qquad (j=0, 1, \cdots, m)$$

are "linear as to F," since

$$\mu_j(a_1\eta_1 + a_2\eta_2) = a_1\mu_j(\eta_1) + a_2\mu_j(\eta_2). \tag{14}$$

For this special choice of the set $\mathfrak{P}$, the operator F, and the functions μ_j, Theorem III becomes identical with Theorem I.

More generally we may take for $\mathfrak{P}$ the totality of all functions $\eta(x)$ of class C' on $[x_1x_2]$ which satisfy any given system of conditions provided only that these conditions are *linear*, *i. e.*, such that they are satisfied by $a_1\eta_1 + a_2\eta_2$ whenever they are satisfied by η_1 and η_2, two functions of class C' on $[x_1x_2]$. We thus obtain a generalization of Theorem I indicated by Hadamard.†

On the other hand, to obtain Theorem II as a special case of Theorem III, we identify the set $\mathfrak{P}$ with the totality of all triples $p = (x, y, z)$ formed with three independent variables x, y, z,

* I had originally thought it necessary to add to the assumptions A) and B) of the theorem the further assumption that $\Delta \neq 0$ for some system $p_1, p_2, \cdots, p_m$; I am indebted to Professor Moore for calling my attention to the fact that this assumption may be omitted, as well as for other valuable suggestions.

† loc. cit., § 176.

each ranging over all real values, and define, in Cayley's set notation,

$$F(a_1, a_2;\, p_1, p_2) = a_1(x_1, y_1, z_1) + a_2(x_2, y_2, z_2),\ i.\,e., \tag{15}$$
$$\equiv (a_1x_1 + a_2x_2,\ a_1y_1 + a_2y_2,\ a_1z_1 + a_2z_2).$$

$F(a_1, a_2;\, p_1, p_2)$ belongs again to $\mathfrak{P}$, however the numbers a_1, a_2 and the triples $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ may be chosen.

With this definition of F, the functions

$$\mu_j(p) = A_jx + B_jy + C_jz, \quad (j = 0, 1, 2) \tag{19}$$

are "linear as to F."

If $n = 2$, there exists at least one pair of triples (x_1, y_1, z_1), (x_2, y_2, z_2) for which the determinant

$$\begin{vmatrix} A_1x_1 + B_1y_1 + C_1z_1, & A_2x_1 + B_2y_1 + C_2z_1 \\ A_1x_2 + B_1y_2 + C_1z_2, & A_2x_2 + B_2y_2 + C_2z_2 \end{vmatrix} \neq 0.$$

This means geometrically, if we interpret x, y, z as homogeneous coordinates of a point in a plane, that the two lines

$$A_1x + B_1y + C_1z = 0, \quad A_2x + B_2y + C_2z = 0 \tag{20}$$

do not coincide.

Theorem III then specializes into Theorem II.

The assumption $n = 1$ leads to the trivial case alluded to on page 403, footnote *.

In like manner the corresponding theorems on pencils and bundles of planes and their generalizations to spaces of higher dimensions follow immediately as special cases from Theorem III.

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