APRIL MEETING OF THE CHICAGO SECTION.

THE twenty-fifth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, April 9–10, 1909. The extensive programme required four full half-day sessions. Professor G. A. Miller, Chairman of the Section presided except on Friday afternoon when Professor E. B. Van Vleck, Vice-President of the Society, occupied the chair. The attendance at the various sessions included sixty-two persons among whom were the following forty-six members of the Society:

Mr. R. P. Baker, Mr. W. H. Bates, Dr. G. D. Birkhoff, Professor G. A. Bliss, Mr. Thomas Buck, Mr. H. E. Buchanan, Professor D. F. Campbell, Professor H. E. Cobb, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor J. C. Fields, Professor W. B. Ford, Professor C. N. Haskins, Mr. T. H. Hildebrandt, Professor T. F. Holgate, Professor Kurt Laves, Dr. A. C. Lunn, Dr. E. B. Lytle, Professor Max Mason, Professor Malcolm McNeill, Mr. E. J. Miles, Dr. W. D. MacMillan, Professor G. A. Miller, Professor E. H. Moore, Dr. R. L. Moore, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Mrs. Anna J. Pell, Professor Alexander Pell, Professor C. A. Proctor, Professor H. L. Rietz, Mr. A. R. Schweitzer, Professor C. S. Slichter, Professor G. T. Sellew, Professor J. B. Shaw, Professor H. E. Slaught, Professor E. J. Townsend, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor A. E. Young, Professor J. W. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening forty members of the Society dined together in the café of the university commons and discussed informally various topics of interest, including the plans for the summer, 1909, meeting of the British association for the advancement of science to be held at Winnipeg, and the next International congress of mathematicians to be held in England in 1912. Professor J. C. Fields, of Toronto, extended a cordial invitation to the members to attend the former, and Professor E. H. Moore urged upon all the desirability of laying plans now to attend the latter.

Particular interest centered in the report of the committee appointed at the January, 1909, meeting to consider plans for improvement in the methods of making mathematical appointments in this country. In case a suitable plan had approved itself to the committee and to the Section, it was the intention to present this plan to the Council for its consideration. The committee, after careful deliberation, reached the conclusion that, while such improvement is manifestly desirable in many American colleges and universities, they were unable to recommend for consideration any plan for such improvement by action of the Society. The Section accepted the report and requested the Secretary to arrange for the publication of a plan which was proposed tentatively by Professor Wilczynski, chairman of the committee, and which he described in some detail. This will appear later in the BULLETIN.

The following papers were read at this meeting:

(1) Professor C. N. HASKINS: "On a class of discontinuous functions of two variables."

(2) Mr. A. E. WESTERN and Dr. J. C. MOREHEAD: "The Fermat number $2^{256}+1$."

(3) Dr. J. C. MOREHEAD: "A simplification of Lagrange's method for the solution of numerical equations."

(4) Mr. THOMAS BUCK: "Oscillations near Lagrange's equilateral triangle points in the problem of three bodies."

(5) Professor L. E. DICKSON: "On the equivalence of pairs of quadratic forms under rational transformation."

(6) Dr. ARTHUR RANUM: "The group of classes of quadratic integers with respect to a composite ideal as modulus."

(7) Professor A. E. YOUNG: "On surfaces having isotherm-conjugate lines of curvature."

(8) Professor W. B. FORD: "A set of criteria for the summability of divergent series."

(9) Professor W. B. FORD: "On the determination of the asymptotic developments of a given function."

(10) Mr. H. E. BUCHANAN: "A class of periodic orbits of three finite bodies."

(11) Professor G. A. MILLER: "Automorphisms of order two."

(12) Mr. R. P. BAKER: "The attack on the space and time concepts by Einstein and Minkowski."

(13) Professor E. W. DAVIS: "An imaginary conic."

(14) Dr. L. I. NEIKIRK: "Groups of rational integral transformations in a general field."

(15) Professor L. E. DICKSON: "A theory of invariants."

(16) Professor L. E. DICKSON: "Combinants."

(17) Mr. W. H. BATES: "Maschke's symbolic method applied to some questions in geometry of hyperspace."

(18) Professor L. D. AMES: "A simpler proof of Lie's theorem for ordinary differential equations."

(19) Dr. W. D. MACMILLAN: "Periodic orbits about an oblate spheroid."

(20) Professor H. L. RIETZ: "On the effect of types of correspondence on Bravais's coefficient of correlation."

(21) Professor F. R. MOULTON: "Oscillating satellites when the finite bodies describe elliptic orbits."

(22) Professor E. J. WILCZYNSKI: "Projective differential geometry of developables."

(23) Mr. ARTHUR PITCHER: "Complete elementary theory of certain properties of classes of functions."

(24) Mr. T. H. HILDEBRANDT: "Remarks on the general theory of point sets."

(25) Mrs. ANNA J. PELL: "Biorthogonal systems."

(26) Professor F. R. MOULTON and Dr. W. D. MACMILLAN: "On the solution of linear differential equations with periodic orbits."

Mr. Pitcher was introduced by Professor Moore. In the absence of the authors the papers of Professors Ames and Davis and Dr. Ranum were read by title. Abstracts of the papers follow below. The numbering corresponds to the titles in the above list.

1. The note of Professor Haskins calls attention to functions of the types

$$Z(x, y) = \frac{yu(x)}{y^2 + [u(x)]^2}, \quad \text{and} \quad Z(x, y) = \frac{ye^{-\frac{1}{[u(x)]^2}}}{y^2 + e^{-\frac{2}{[u(x)]^2}}},$$

where $\underset{x \doteq 0}{\mathrm{L}}\, u(x) = 0$. The first of these functions has the property that

$$\underset{x \doteq 0}{\mathrm{L}}\, Z(x, mv(x)) = 0 \quad \text{if} \quad \underset{x \doteq 0}{\mathrm{L}} \frac{u(x)}{v(x)} = 0 \quad \text{or} \quad \underset{x \doteq 0}{\mathrm{L}} \frac{v(x)}{u(x)} = 0,$$

and

$$\mathop{\mathrm{L}}_{x \doteq 0} Z(x, mv(x)) = \frac{m\alpha}{m^2 + \alpha^2} \quad \text{if} \quad \mathop{\mathrm{L}}_{x \doteq 0} \frac{u(x)}{v(x)} = \alpha \neq 0.$$

This gives rise to a set of functions which show the discontinuity in question along systems of curves having osculation with $y = 0$ of continually increasing order at the origin.

2. This paper describes the method of computation followed by Mr. Western and Dr. Morehead which resulted in showing that the Fermat number $F_8 = 2^{256} + 1$ fails to satisfy the criterion for Fermat primes,*

$$3^{2^{(2^n - n - 2)}} \equiv \pm 2^x(2^{2^{n-1}} \pm 1) \mod F_n, \quad (F_n = 2^{2^n} + 1),$$

and is therefore composite. The result is especially interesting as completing a chain of five composite Fermat numbers, F_5, F_6, F_7, F_8, F_9, following the first five, and only known Fermat primes, 3, 5, 17, 257, 65537.

3. In this paper Dr. Morehead explains the application of binomial synthetic division to the approximate expression of an incommensurable real root of a numerical equation,

$$f_1(x) = c_0 x^m + c_1 x^{m-1} + \cdots + c_m = 0,$$

in terms of a continued fraction of the form

$$(a_1, a_2, a_3, \cdots a_n) = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \frac{1}{+ a_n}.$$

The quotients $a_1, a_2, a_3, \cdots, a_n$ are the integral parts of corresponding roots of the equations

$$f_1(x_1) = c_1 x_1^m + c_1 x_1^{m-1} + \cdots + c_m = 0,$$
$$f_2(x_2) = c_0'' x_2^m + c_1'' x_2^{m-1} + \cdots + c_m'' = 0,$$
$$f_3(x_3) = c_0''' x_3^m + c_1''' x_3^{m-1} + \cdots + c_m''' = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$f_n(x_n) = c_0^{(n)} x_n^m + c_1^{(n)} x_n^{m-1} + \cdots + c_m^{(n)} = 0,$$

respectively, where each equation $f_{k+1}(x_{k+1}) = 0$ is connected with the preceding $f_k(x_k) = 0$ by the relation $x_k = a_k + 1/x_{k+1}$, or $x_{k+1} = 1/(x_k - a_k)$. From the latter relation it is easily seen that the coefficients of $f_{k+1}(x_{k+1})$ may be obtained from

* A modification, due to A. E. Western, of the Pepin criterion for the base 3 : $3^{2^{(2^n - 1)}} \equiv -1 \mod F_n$.

those of $f_k(x_k)$ if we diminish the roots of $f_k(x_k)$ by a_k, using the binomial synthetic division process as in Horner's method, and reverse the order of the coefficients so obtained.

Approximations for the roots of numerical equations may be obtained by this means more easily and more rapidly, in general, than by Horner's method. Thus for example, without using more than six figures in any of the coefficients, we obtain for the root lying between 1 and 2 of the equation,

$$2x^3 - x^2 + 2x - 4 = 0,$$

the approximation (1, 6, 1, 3, 1, 1, 3, 2, 3, 1, 4, 1, 1, 3, 1, 1, 5, 2, 1, 2, 2, 3) = 82994039/72327612, which is correct to 2 in the sixteenth decimal place.

4. In the first part of this paper Mr. Buck considers certain periodic orbits of an infinitesimal body attracted by two finite bodies revolving in circles about their center of mass. Convenient parameters are introduced and by a discussion of the linear terms the generating solutions are found. All the terms of the differential equations are then considered and the question of the existence of the continuations of the generating solutions is discussed.

In the second part of the paper certain periodic orbits of three finite bodies near Lagrange's equilateral triangle points are considered. Parameters are introduced as before and the linear terms discussed. The discussion when the higher terms of the differential equations are considered has not yet been completed.

5. The first paper by Professor Dickson considers the equivalence of two pairs of quadratic forms ϕ, ψ and ϕ', ψ' under linear transformation in a field F. As in the algebraic theory, the elementary divisors of $\phi - \lambda\psi$ must be the same as those of $\phi' - \lambda\psi'$. But this condition is not in general sufficient for equivalence in F. We extend the field F to a field F' by adjoining the roots λ_i of $|\phi - \lambda\psi| = 0$. If ϕ_1, ψ_1 is a canonical type of ϕ, ψ within F', each transformation effecting the reduction is the product of one with conjugate variables by an automorph of ϕ_1, ψ_1. A discussion of the transformations of ϕ_1, ψ_1 into ϕ_1', ψ_1' leads to conditions for their equivalence within F'. The conjugacy of the variables enables us to decide upon the equivalence of the initial pairs within F. Use

is made of the work of Kronecker and a modified form of that of Weierstrass. Both in the singular and in the non-singular cases, simple necessary and sufficient conditions are obtained for the equivalence of two pairs of forms under linear transformation in the given field. This paper has been offered for publication in the *Transactions*.

6. In the ordinary theory of rational numbers the group of classes of congruent integers with respect to a composite modulus is well known and is called by Weber is his Algebra the most important example of a finite abelian group. In the more general theory of numbers in an algebraic field the corresponding group of classes of integers with respect to a composite ideal does not seem to have been studied heretofore. In this paper a beginning is made by Dr. Ranum. For the special case of a quadratic field $(\sqrt{m})$ he determines the structure of the group, and finds a system of independent generators. If the modulus is a power of a prime ideal p, the latter being a factor of the principal ideal (p), the group is one of nine distinct types, depending on the value of m, the grade of p, and the value of the rational prime p ($p = 2$, $p = 3$, or $p > 3$). If the modulus contains as factors two or more distinct prime ideals, the group is a direct product of two or more groups of these nine types. The cases in which the group is cyclic, and therefore primitive roots exist, are enumerated.

7. In a paper published in the January issue of the *Transactions*, Professor Young discussed a class of surfaces characterized by the two properties that their lines of curvature form a network of infinitesimal squares, and their asymptotic lines form a network of rhombuses. In the present paper he discusses the more general class, characterized by the latter of these two properties. The general problem of determining these surfaces is reduced to the simultaneous solution of two partial differential equations of the second order which involve two functions. Particular solutions are found and the corresponding surfaces determined. It is shown that these surfaces compose the class which have isotherm-conjugate lines of curvature, using this term in the sense defined by Bianchi.

8. Professor Ford's first paper appears in full in the present number of the BULLETIN.

9. The object of Professor Ford's second paper is to prove, and show the range of applicability of, a general formula employed extensively of late years by Barnes and made basal by him in a series of memoirs upon the determination of the asymptotic developments of a given function in the neighborhood of the point ∞. The formula in question is *

$$\sum_{n=1}^{m-1} \phi(n) \sim Z + \int \phi(m)dm - \tfrac{1}{2}\phi(m) + \sum_{n=0}^{n=\infty} \frac{(-1)^n B_{n+1}}{(2n+2)!} \frac{d^{2n+1}\phi(m)}{dm^{2n+1}},$$

where Z is a certain constant and B_n the nth Bernoulli number. It is shown that the class of functions $\phi(n)$ to which the formula applies is well defined in that any such function must be developable in a prescribed way about the point $n = \infty$.

10. At the January meeting of the Section, Mr. Buchanan read a preliminary report on a certain class of periodic orbits of three finite bodies. He showed how to expand the right members of the differential equations and discussed the linear terms. In the present paper it is proposed to introduce certain convenient parameters, prove the existence of five different types of orbits, show how to construct these orbits, and state certain theorems concerning their geometrical properties.

11. The necessary and sufficient condition that an automorphism is of order 2 is that every commutator arising from the automorphism corresponds to its inverse under the automorphism. Hence it results that in every automorphism of order 2 some operator besides identity must correspond to its inverse. The independent generators of an abelian group of odd order may be so chosen that in any given automorphism of order 2 each of these generators corresponds either to itself or to its inverse. If all the commutators (besides identity) of a group are of order 2, every complete set of conjugate operators of order 2 generates an abelian group of type $(1, 1, 1, \cdots)$, and such a group involves an abelian subgroup which is the direct product of Sylow subgroups of all odd orders. Every group in which the orders of all the commutators are powers of the same prime number is solvable, and such a group involves only one Sylow subgroup whose order is a power of this prime.

* *Philosophical Transactions*, vol. 199 A (1902), p. 444.

12. The validity of the time and space concepts has been challenged by Einstein * and Minkowski.† Mr. Baker's paper is a critical examination of their theory.

Einstein's definition of time is insufficient; *fluit* must be added. Velocities in (ξ, η, ζ, τ) are non-associative in Einstein's system. Minkowski avoids this by his definition of "Eigenzeit."

The usual treatment of relativity depends on a transformation and its inverse, and on the ordinary dualism. T moves the object and T^{-1} the observer. If the dualism is not abandoned and O represents the ordinary transformation for uniform motion, the selection of OT^{-1} is only possible on a strict definition of *object* as a totality C_3 of light sources which act on moving according to Einstein's hypothesis, that is, are each a totality of pulses (C_1), the pulse being invariant under T and Einstein's source remaining a C_1 of pulses under O. Cross selection in the multiple infinities must be barred and each time t must be associated with its place.

With these restrictions the system becomes a single sense philosophy, the "sense organ" accomplishing the accurate timing of light signals by "synchronized clocks."

The selection of TT^{-1} as the pair is without special significance for relativity, though T of course has its very important significance for the problem of relative motion.

Since O does not preserve the equations of electrodynamics it is probable that Einstein's success with OT^{-1} is due to abstraction.

At any rate "absolut ruhender Raum" as applied to (x, y, z, t) is erroneous. When the transformation is obtained it turns out that there is no analytical distinction between this system and (ξ, η, ζ, τ) beyond the sign of v. Minkowski postulates a world. He asks for (1) continuity in four dimensions (x, y, z, t), (2) each point to be a substantial point, with assigned mass, (3) $ds/dt < V$, (4) a "Geschenk von oben," the contraction of moving bodies in the direction of their motion according to Einstein's law.

In the actual work "each" in (2) is defined by a continuous function, thus avoiding an obvious difficulty with the point set theory.

The contraction in (4) is not an affine transformation, the

* *Annalen der Physik*, vol. 17 (1905).

† *Physikalische Zeitschrift*, Feb., 1909. *Göttinger Nachrichten*, Dec., 1908.

factor for the inverse transformation being the same as for the direct, and not the reciprocal of it. An ordinary hydrodynamic universe in (x, y, z, t) with an affine contraction for (4) has for its only rotating molecules those with fixed axis, size, and angular velocity, posited as eternal.

The erroneous interpretation of the contraction is however a matter of indifference; satisfaction of the equations is all that is necessary.

13. In this paper Professor Davis points out that the conic $\tilde{x}^2 + \tilde{y}^2 e^{2i\phi} = 1$ can be gotten by revolving the intersection of the cylinder $x^2 + y^2 = 1$ with $z = iy$ about the x-axis through an angle ϕ, and then projecting the intersection upon the xy-plane. The distribution of the imaginary elements is indicated by a diagram.

14. In this paper Dr. Neikirk considers the totality of transformations of the type

$$T_{m,i} \equiv [x : \phi_{m,i}(x)],$$

where

$$\phi_{m,i}(x) = \sum_{j=0}^{j=m} \alpha_{j,i} x^{m-j} \qquad (\alpha_{o,i} \neq 0),$$

and the coefficients range independently over the elements or marks of a general field F.

The successive application of two of these transformations is a definite third one in the set. Symbolically $T_{m,i}T_{m',i'} = T_{mm',i''}$. They possess the group property. They also obey the associative law, if x is unrestricted. If x is restricted in range to a certain set, then the result of the action of $T_{m,i}$ on x, *i. e.*, on $\phi_{m,i}(x)$ must be in this same set for associativity.

Not all the transformations in this set have an inverse in the set. If $T_{m,i}$ $(m > 1)$ has $T_{m',i'}$ as an inverse, then x is restricted in range to the roots of an algebraic equation with coefficients in F, and $T_{m,i}$ gives a substitution on the roots of this equation. These transformations on the roots of an equation are a two-fold generalization of the substitution quantics given by Hermite, Dickson, and others. Several theorems are proved and several examples are given of groups generated by these transformations.

15. In the second paper by Professor Dickson, the invariants of a system of forms are investigated from the standpoint

of the classes of forms under the group of transformations of determinant unity. Whereas the discussion in the April number of the *Transactions* was limited to the case of a finite field, the present paper deals with an arbitrary field. For the case of the field of all real and complex numbers, the invariants considered are the rational integral invariants in the ordinary algebraic theory, as well as other single-valued invariants such as the rank of a quadratic form. Since all the single-valued invariants appear from a common standpoint, the present theory presents a synthesis of the special theories. The paper has been offered to the *American Journal of Mathematics.*

16. In the third paper by Professor Dickson, s-fold families $\Sigma x_i q_i$ of forms q_i on m variables ξ_j are separated into classes under the group Γ_1 of m-ary linear transformations on the ξ_j and the group G_1 of s-ary linear transformations on the x_i, each type of transformations having determinant unity. A function of the coefficients of the q_i is invariant under Γ_1, G_1 if and only if it takes the same value for all forms $\Sigma x_i q_i$ in any class. For a finite field the number of the resulting linearly independent invariants equals the number of classes. By an extension of a method given elsewhere,* it is determined which of these invariants are combinants; the number of linearly independent combinants is shown to equal the number of classes under Γ_1, G_1. For the practical determination of combinants various methods are developed. The general theory is applied to determine a fundamental system of five (ten) combinants of two binary (ternary) quadratic forms in the field of order p^n, $p > 2$. That the combinants actually form a fundamental system follows from a general theorem for finite and infinite fields (cf. preceding abstract). The paper has been offered to the *Quarterly Journal of Mathematics.*

17. In this paper, Mr. Bates uses Maschke's symbolic method to derive some of the formulas relating to angles between curves, isometric systems, and lines of curvature in hyperspace.

18. Professor Ames's paper appeared in full in the May BULLETIN.

19. The motion of a particle about an oblate spheroid is not, in general, periodic, but by virtue of the existence of one area

* *Transactions*, April, 1909, § i.

integral it is possible to reduce the problem from three dimensions to two. In the particular case of motion in the equatorial plane Dr. Macmillan shows that, within a certain range of variation of the parameters, the expression for the radius vector is always periodic. The expression for the rotation of the line of apsides has an interesting application to the motion of Jupiter's fifth satellite.

When the motion is not in the equatorial plane it is possible always to consider the motion in a revolving meridian plane. An orbit is considered to be periodic when its motion in this plane is periodic. It is found that the inclination can be taken arbitrarily and the initial velocity so determined that the orbit in the revolving plane is closed after the first revolution. These are the simplest types of orbits. Let us say that they have the period 2π. On varying the initial distance slightly it is found that a new period $2\pi/\lambda$ is introduced. If λ is rational, it is possible to determine the change in the initial velocity in terms of the initial displacement so that the orbit is reentrant after many revolutions. The integration of non-homogeneous linear differential equations with periodic coefficients is involved in every step. These last solutions include five constants of integration out of the six necessary for a complete solution, with the condition that λ must be rational.

20. In this paper, Professor Rietz considers the effect of some different types of correspondence on results obtained by Bravais's formula for correlation. When a one to one correspondence exists for variates of the subject and relative classes, in treating the correlation of two attributes by statistical methods, the formula $r = \sum xy/n\sigma_x\sigma_y$ is uniquely defined. But a more general type of correspondence exists in some practical problems of statistics. A μ to ν correspondence occurs in which μ and ν are constant, but the μ or ν variates are not identical. In this case, it is a somewhat arbitrary matter as to which one of three or more distinct methods is used to obtain a one to one correspondence for application of the formula. It turns out that, with three different methods, which give significant measures of correlation, the values of r are in two of them different; and their numerical order is determined. In particular, if $\mu = 1$, although the correlation coefficients differ, the regression coefficients of the relative on the subject class are equal for two of the methods of correspondence; and this value

of the regression coefficient is the most probable value for the third method if the frequency distributions are symmetrical.

21. It is possible to project any three bodies so that they shall always be collinear and describe conic sections. Suppose one of the three is infinitesimal and that the orbits are ellipses. Then Professor Moulton shows that it is possible to displace the infinitesimal body from its position of equilibrium in an infinity of different ways, both in the plane of motion of the finite bodies and in three dimensions, so that its motion shall be periodic. When the finite bodies describe circles the orbit of the infinitesimal body, referred to rotating axes, re-enters after one revolution, but in the present case it re-enters only after many revolutions. The orbits depend upon a certain parameter and they are periodic for an infinite number of values of it, but these values do not constitute the continuous point set, for they must be such that certain commensurability conditions are satisfied.

22. In this paper Professor Wilczynski indicates a method for the development of a projective differential geometry of developable surfaces. These surfaces cannot be treated by the general methods of his previous papers, based upon the consideration of two linear homogeneous partial differential equations of the second order, because for a developable the corresponding system would become involutory, so that the most general integral would not be a projective transformation of any one. The difficulty is solved by adjoining an independent partial differential equation of the third order, to the involutory system of two equations of the second order. Mr. W. W. Denton is at present developing the details of this theory. It should be mentioned, however, that the dual theory, in which a developable is considered as an envelope of ∞^1 planes, is already contained in Professor Wilczynski's previous investigations.

23. The classes of functions considered by Mr. Pitcher are classes $\mathfrak{M}$ on a general class $\mathfrak{P}$ to $\mathfrak{A}$, where $\mathfrak{A}$ is the class of all real numbers. Certain ten properties of classes of functions have been found by Professor Moore to be very important in his work in general analysis. Four of these properties have to do with systems $(\mathfrak{A}, \mathfrak{P}, \mathfrak{M})$, while the others have to do with systems $(\mathfrak{A}, \mathfrak{P}, \Delta, \mathfrak{M})$ where Δ is a so-called development of $\mathfrak{P}$.

The objects of the present study are: (1) To investigate the elementary relations of dependence and independence of the ten properties mentioned above, (2) to give characterizations in terms of these properties of certain classes of elements $\mathfrak{P}$ and of functions $\mathfrak{M}$. So far the study has been confined to seven of the ten properties. These seven are not completely independent. Various characterizations for $\mathfrak{P}$ singular, $\mathfrak{P}$ dual, $\mathfrak{P}$ finite, and for certain classes of functions have been found.

24. A critical study of Fréchet's thesis * has revealed the fact that in order to obtain a theory of continuous functions, in particular the theory obtained by him, it is not necessary to condition the notion of limit further than to suppose it to be a relation between sequences of elements and single elements. In order to obtain a more general setting for the second part of his study, which concerns a distance function of pairs of elements, *voisinage*, a relation between pairs of elements and integers has been introduced. By the use of suitable conditions on this relation, Mr. Hildebrandt is able to obtain the entire theory of Fréchet and also the extension made by Hahn.† Other questions of independence are being investigated.

25. In Mrs. Pell's paper, necessary and sufficient conditions are found for the existence of the adjoint system of a given system of functions. The system is shown to be denumerable if the adjoint system exists. Conditions for the convergence of

$$\sum_{i=1}^{\infty} \int_a^b f(s) X_i(s) ds \int_a^b g(s) Y_i(s) ds$$

are deduced and it is shown that this sum is equal to

$$\int_a^b f(s) g(s) ds$$

when the given system is complete.

26. Special cases of the differential equations treated in this paper have arisen in the investigations of Professor Moulton and Dr. MacMillan on periodic orbits, and the methods of solving them were first developed in connection with these

* "Sur quelques points du calcul fonctionnel," *Rend. del Cir. di Palermo*, vol. 22 (1906).

† Bemerkungen zu den Untersuchungen des Herrn Fréchet, *Monatshefte f. Math. u. Phys.*, vol. 19 (1908).

practical problems. This discussion is, in fact, reduced to that of periodic solutions of an associated set of differential equations. After giving a new derivation of the properties of the solutions in the general case, the class of special equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \left[a_{ij} + \sum_{\kappa=1}^{\infty} \theta_{ij}^{(\kappa)}(t)\mu^{\kappa} \right] x_j \qquad (i = 1, \cdots, n) \tag{1}$$

is treated. The θ_{ij} are periodic in t with the period 2π. These equations include those which generally arise in practical problems.

When $\mu = 0$ let the roots of the characteristic equation of the resulting differential equations be $\alpha_1^{(0)}, \cdots, \alpha_n^{(0)}$. It is proved that if $\alpha_1^{(0)}, \cdots, \alpha_n^{(0)}$ are distinct and if no two of them differ by an imaginary integer, then the general solution of (1) is

$$x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t} y_{ij}(t) \qquad (i = 1, \cdots, n), \tag{2}$$

where the A_j are arbitrary constants, the y_{ij} are periodic with the period 2π, and

$$\alpha_j = \alpha_j^{(0)} + \sum_{\kappa=1}^{\infty} \alpha_j^{(\kappa)}\mu^{\kappa}, \quad y_{ij} = \sum_{\kappa=0}^{\infty} y_{ij}^{(\kappa)}\mu^{\kappa}. \tag{3}$$

If two or more $\alpha_j^{(0)}$ differ by imaginary integers the solutions have *in general* the form (3), but there are interesting peculiarities both in the proof of their existence and in their construction.

If p of the $\alpha_j^{(0)}$ are equal, then in general the coefficients of $e^{\alpha_j t} y_{ij}(t)$ in (2) are polynomials in t and

$$\alpha_j = \alpha_j^{(0)} + \sum_{\kappa=1}^{\infty} \alpha_j^{(\kappa)}\mu^{\kappa/p}, \quad y_{ij} = \sum_{\kappa=0}^{\infty} y_{ij}^{(\kappa)}\mu^{\kappa/p}. \tag{4}$$

In all the special cases which can arise convenient methods for constructing the solutions follow from the mode of treatment.

H. E. SLAUGHT,
Secretary of the Section.