A GEOMETRIC CONSTRUCTION FOR QUATERNION PRODUCTS.

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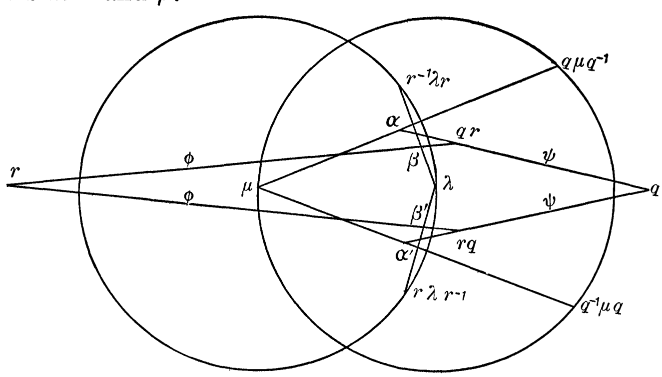
IN the construction here given the lines actually inserted in the figure are supposed to lie on a hypersphere (spherical space of three dimensions) having a radius equal to the linear unit, and in particular all of the lines except the four marked ϕ, ψ, lie on a spherical surface passing through the extremities of the quaternion unit vectors i, j, k. The Greek letters that indicate the position of points are vectors drawn from the centre of the sphere to its surface and q, r, qr, rq are quaternion directors to points on the hypersphere. The centers of the sphere and hypersphere are coincident and are at the origin of vectors and of quaternion directors. The two circles of the figure have their centers at λ and μ.



The quaternions whose products are to be constructed are given in the form

$$q = \cos\phi + \lambda \sin\phi,$$
$$r = \cos\psi + \mu \sin\psi \qquad [Tq = Tr = 1].$$

It is well known that with a pair of quaternions q, r, there is associated a definite plane, whose equation is

$$\lambda t - tr^{-1}\lambda r = 0,*$$

* *Transactions Amer. Math. Soc.*, vol. 2, p. 194.

in which r, through progressive multiplication by q, is turned about the origin into the position qr.* It is easily verified that r, qr and $\lambda + r^{-1}\lambda r$ satisfy the above equation for all values of ϕ and ψ. A plane in which qr must lie is therefore completely determined by the lines r and $\lambda + r^{-1}\lambda r$, while $r^{-1}\lambda r$ is known to be that vector which is got by rotating λ about μ as an axis through the angle 2ψ.†

Similarly, another plane, in which qr must lie, has for its equation

$$q\mu q^{-1}t - t\mu = 0,$$

which is satisfied by q, qr and $\mu + q\mu q^{-1}$, and is therefore completely determined by the lines q and $\mu + q\mu q^{-1}$. The two planes meet in a straight line, for

$$S\lambda q\mu q^{-1} = S\mu r^{-1}\lambda r,$$

since the radii of the two circles are equal. The construction of qr is now determined as the intersection of the two planes whose equations are above written. It is the line drawn from the origin to the point marked qr.

In the diagram, α and β are the points on the spherical surface where the vectors $\mu + q\mu q^{-1}$ and $\lambda + r^{-1}\lambda r$ meet it, that is, the mid-points of the (curvilinear) chords on which they lie.

The operator $q(\ )q^{-1}$ turns μ about λ as an axis through the angle 2ϕ into the position $q\mu q^{-1}$, and $r^{-1}(\ )r$ turns λ about μ as an axis through the angle 2ψ into the position $r^{-1}\lambda r$. These are Hamilton's rotational operators in three dimensional space.

The cosines of the angles between r, qr and q, qr are respectively

$$Sqr/r = \cos\phi \text{ and } Sqr/q = \cos\psi.$$

Hence, the operators $q(\ )1$ and $1(\ )r$, applied to r and q respectively, turn r into qr through the angle ϕ, and q into qr through the angle ψ. These are Hathaway's right and left turns.‡

The construction for rq is indicated in the lower part of the figure, where $r\lambda r^{-1}$ and $q^{-1}\mu q$ take the places of $r^{-1}\lambda r$ and $q\mu q^{-1}$

* Hathaway, BULLETIN, vol. 4, p. 55.
† Tait's Treatise on Quaternions, 3d edition, p. 75.
‡ *Transactions Amer. Math. Soc.*, vol. 3, p. 51.

respectively and represents rotations about μ and λ in the reverse directions. The changes in the figure required by this reversal are obvious.

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SHORTER NOTICES.

Introduction a la Géométrie Générale. By GEORGES LECHALAS. Paris, Gauthier-Villars. Pp. ix + 58.

THE point of view of M. Lechalas is different from that of most recent workers on the foundations of geometry. Ordinary usage at present applies the term mathematical science to any body of propositions deducible from a set of postulates. The term geometry is applied with various degrees of freedom to a large number of sciences, all however characterized by similar types of relation and modes of study. These abstract geometries are "represented concretely" in many different ways, as is illustrated for example, by the beautiful theorem of M. Barbarin :*

"Each of the three spaces, Euclidean, Lobachevskian, Riemannian, contains surfaces of constant curvature of which the geodesic lines have the metric properties of the straight lines of the three spaces."

To Lechalas, on the other hand, a geometry is a "form of externality." Under the title General Geometry he includes only the geometries of Euclid, Lobachevsky and Riemann (double elliptic or spherical geometry), thus excluding from consideration not only the current "bizarre geometries" but even the symmetric non-archimedean geometries of Hilbert and Veronese as well as the classical "single-elliptic" geometry. In no place do we find clear distinctions between metaphysical and mathematical questions, in a book where both are considered. On the contrary we find the following statement (page 16), which reads rather strangely in view of the vast number of different ways of representing an abstract science to the imagination :

* Quoted thus in a translation by G. B. Halsted of a report by P. Mansion on the non-euclidean researches of P. Barbarin, *Science*, n. s., vol. 20, No. 507 (September 16, 1904).