SURFACES OF REVOLUTION IN THE THEORY OF LAMÉ'S PRODUCTS.

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THE present paper is a review of an article by Haentzschel * in which he criticises certain results obtained by Wangerin.† The problem treated by Wangerin is to obtain the most general orthogonal surfaces of revolution, such that, if Laplace's equation be written in coördinates corresponding to these surfaces, a solution may be obtained in the form of a Lamé's product with an extraneous factor, *i. e.*,

$$V = \lambda . R . R_1 . \theta. \tag{1}$$

R, R_1, θ are functions respectively of the parameters of the two families of surfaces and the meridian planes, while λ may contain all three parameters. Wangerin shows that λ is $\frac{1}{\sqrt{r}}$, where r is the distance from the axis of revolution to the intersection of the three surfaces. His principal result is that the meridian curves are of the fourth degree and are cyclic curves, while the surfaces are of the same degree. Haentzschel states that the most general surfaces are of the thirty-second degree, the meridian curves being of the sixteenth degree.

Both writers give the following equations :

$$\frac{F'(t+iu) \cdot F_1'(t-iu)}{[F(t+iu) - F_1(t-iu)]^2} = H(t) + H_1(u), \tag{2}$$

$$x + ir = F(t+iu), \quad x - ir = F_1(t-iu) \quad (r = \sqrt{y^2+z^2}). \tag{3}$$

By (2) two conjugate imaginary functions F and F_1 are to be determined such that the first member shall be the sum of two functions, one of t alone, the other of u alone. From (3) the two families of meridian curves are to be obtained by elimination of t and u respectively.

After differentiating (2) successively with respect to t and u, the result is

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* Reduction der Potentialgleichung, E. Haentzschel, Berlin, 1893.

† *Berliner Monatsberichte*, Feb., 1878.

$$(F-F_1)\,[(F-F_1)\,(F'''F_1'-F'F_1''')$$
$$+\,(F''F_1'-F'F_1'')\,(F'-F_1')-2\,F'F_1'\,(F''+F_1'')$$
$$-\,2\,(F'+F_1')\,(F'F_1''+F''F_1')]$$
$$-\,3\,(F'-F_1')\,[(F-F_1)\,(F''F_1'-F'F_1'')$$
$$-\,2\,F'F_1'\,(F'+F_1')]=0. \qquad (4)$$

This expression is next differentiated three times with respect to the argument of F and then the derivatives of F_1 are eliminated. From this result, after integration, comes the following equation *defining* F:

$$(F')^2=AF^4+4BF^3+6CF^2+4\,B'F+A'=R(F). \quad (5)$$

Corresponding to (5),

$$(F_1')^2=\bar{A}F_1^4+4\bar{B}F_1^3+6\bar{C}F_1^2+4\bar{B}_1'F+\bar{A}'. \qquad (6)$$

The conjugate imaginary constants in (5) and (6) come from integration but must be taken *real*, since (4) is satisfied by F, F_1, and their derivatives from (5) and (6) when and only when the following condition holds:

$$A-\bar{A}=B-\bar{B}=C-\bar{C}=B'-\bar{B}'=A'-\bar{A}'=0. \quad (7)$$

Haentzschel gives, as the general integral of (5)

$$F\,(t+iu)\,*=z_0+$$
$$\{\sqrt{R(z_0)}\,\sqrt{4s^3-g_2s-g_3}+\tfrac{1}{2}R'(z_0)\,[s-\tfrac{1}{24}R''(z_0)] \qquad (8)$$
$$+\,\tfrac{1}{24}R(z_0)\,R'''(z_0)\}\div\{2\,[s-\tfrac{1}{24}R''(z_0)]^2-\tfrac{1}{2}AR(z_0)\},$$

$$g_2=AR(z_0)-\frac{R'(z_0)R'''(z_0)}{24}+\frac{R''^2(z_0)}{48}$$
$$=-4(\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1), \qquad (9)$$

$$g_3=\frac{AR(z_0)R''(z_0)}{12}+\frac{R'(z_0)R''(z_0)R'''(z_0)}{24^2}-\frac{R(z_0)R'''^2(z_0)}{24^2}$$
$$-\frac{AR'^2(z_0)}{16}-\frac{R''^3(z_0)}{12^3}=4\varepsilon_1\varepsilon_2\varepsilon_3, \qquad (10)$$

$$s=\wp(t+iu), \quad \sigma=\wp(t-iu). \qquad (11)$$

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* Enneper, Elliptische Functionen, Halle, 1890, pp. 27–30.

The constant z_0 is at present unrestricted. When z_0 is a root of the equation

$$R(z) = 0, \tag{12}$$

(8) reduces to

$$F(t + iu) = z_0 + \frac{\frac{1}{4}R'(z_0)}{s - \frac{1}{24}R''(z_0)}. \tag{13}$$

This result is used by Haentzschel to obtain meridian curves of the fourth degree and surfaces of both the fourth and eighth degrees, depending upon the nature of the root of (12) which is used in (13). The writer* has discussed a linear transformation of (13) and obtained restrictions analogous to (7), giving results identical with Wangerin's, viz., surfaces of the fourth degree only.

In the general case, where (12) no longer holds, it is convenient to change (5) to a form similar to Legendre's normal form, by using

$$F = \frac{\alpha\overline{\overline{F}} + \beta}{\gamma\overline{\overline{F}} + \delta}, \tag{14}$$

Hence $$(\overline{\overline{F}}')^2 = \overline{A}_0[\underset{1}{\pm}\overline{\overline{F}}^4 \underset{2}{\pm} \overline{c}\overline{\overline{F}}^2 \underset{3}{\pm} 1] \tag{15}$$

The three independent double signs in (15) correspond to the eight possible cases. It may be shown possible to choose real coefficients in (14) which will make $\overline{A}_0$ positive and $\overline{c}$ real in (15), while (2) is unchanged in form. Next, assume

$$\overline{\overline{F}} = \overline{F} - z_0 = v = \overline{x} + i\overline{r} - z_0. \tag{16}$$

From (15) and (16)

$$(\overline{F}')^2 = \overline{A}_0 [\underset{1}{\pm} (\overline{F} - z_0)^4 \underset{2}{\pm} \overline{c}(\overline{F} - z_0)^2 \underset{3}{\pm} 1] = \overline{R}(\overline{F}). \tag{17}$$

Hence $$\overline{R}(z_0) = \underset{3}{\pm}\overline{A}_0, \quad \overline{R}'(z_0) = 0,$$

$$\frac{\overline{R}''(z_0)}{2\,!} = \underset{2}{\pm}\overline{c}\overline{A}_0, \quad \frac{\overline{R}'''(z_0)}{3\,!} = 0, \quad \frac{\overline{R}^{\text{IV}}(z_0)}{4\,!} = \underset{1}{\pm}\overline{A}_0. \tag{18}$$

Reference to the deduction of (7) will show that z_0 must be real. Equations (17) (18) may be improved in form by the following changes:

$$\underset{3}{\pm}\overline{A}_0 = A_0, \quad (\underset{2}{\pm})(\underset{3}{\pm})\overline{c} = a, \quad (\underset{1}{\pm})(\underset{3}{\pm}) = \pm. \tag{19}$$

* *Amer. Jour. of Math.*, vol. 21, no. 1, pp. 11–23.

Equation (8), which is the solution of (17), is next simplified by the use of (16) and (18) and then solved for s, giving

$$s = A_0 \frac{\left(1 + \frac{av^2}{6} + \sqrt{\pm v^4 + av^2 + 1}\right)}{2v^2} \tag{20}$$

Similarly for σ, the conjugate of s,

$$\sigma = A_0 \frac{\left(1 + \frac{aw^2}{6} \sqrt{\pm w^4 + aw^2 + 1}\right)}{2w^2}. \tag{21}$$

From (9) and (10), corresponding to the *upper signs* in (20) and (21),

$$g_2 = \frac{A_0^2}{2}(a^2+12), \quad g_3 = \frac{A_0^3 a}{6^3}(-a^2+36), \tag{22}$$

$$\varepsilon_1 = \frac{A_0}{12}(a+6), \quad \varepsilon_2 = -\frac{A_0 a}{6}, \quad \varepsilon_3 = \frac{A_0}{12}(a-6). \tag{23}$$

The equations of the two families of meridian curves are to be obtained by eliminating t and u respectively from (11) and then substituting the values s and σ from (20) and (21) in those results. Haentzschel has performed the elimination and substitution but his final result is of the sixteenth degree and he was probably unaware that it is reducible to four curves of the fourth degree. From (11) Haentzschel has obtained on page 17

$$s^2\sigma^2 + \alpha + (s+\sigma)^2\beta + s\sigma(s+\sigma)\gamma + s\sigma\delta + (s+\sigma)\varepsilon = 0. \tag{24}$$

The values of the coefficients are given below and are functions of the parameter ρ, equal respectively to $\wp(2iu)$ and $\wp(2t)$.

Let ρ be replaced by $A_0\varphi$, then by the aid of (22),

$$\left.\begin{aligned}
\alpha &= g_3\rho + \frac{g_2^2}{16} = A_0^4\left(\frac{a^4}{48^2} - \frac{\varphi a^3}{6^3} + \frac{a^2}{96} + \frac{\varphi a}{6} + \frac{1}{16}\right),\\
\beta &= A_0^2\varphi^2, \qquad \gamma = -2A_0\varphi,\\
\delta &= -4\left(\rho^2 - \frac{g_2}{8}\right) = A_0^2\left(\frac{a^2}{24} + \frac{1}{2} - 4\varphi^2\right),\\
\varepsilon &= \frac{g_2\rho}{2} + g_3 = A_0^3\left(-\frac{a^3}{6^3} + \frac{\varphi a^2}{24} + \frac{a}{6} + \frac{\varphi}{2}\right).
\end{aligned}\right\} \tag{25}$$

From (20), (21), (24), (25) comes the following equation of the curves desired:

$$
\begin{aligned}
& v^4w^4\left(2+\frac{13a^2}{18}+\frac{8\varphi a}{3}+8\varphi^2\right) \\
& +(v^4w^2+v^2w^4)\left(3a+\frac{a^3}{36}-\frac{2\varphi a^2}{3}+4\varphi^2 a\right) \\
& +v^2w^2\left(2+\frac{35a^2}{18}-\frac{32\varphi a}{3}-8\varphi^2\right) \\
& +(v^4+w^4)\left(2+\frac{a^2}{18}-\frac{4\varphi a}{3}+8\varphi^2\right) \\
& +(v^2+w^2)\left(\frac{8a}{3}-8\varphi\right)+4 \\
& +\sqrt{v_0}\sqrt{w_0}\left[v^2w^2\left(2+\frac{5a^2}{18}-\frac{8\varphi a}{3}-8\varphi^2\right)\right. \\
& \left.+(v^2+w^2)\left(\frac{2a}{3}-8\varphi\right)+4\right] \\
= & -\sqrt{v_0}\left[v^2w^4(2a)+v^2w^2\left(2+\frac{11a^2}{18}-\frac{20\varphi a}{3}-8\varphi^2\right)\right. \\
& +w^4\left(2+\frac{a^2}{18}-\frac{4\varphi a}{3}+8\varphi^2\right)+w^2\left(\frac{8a}{3}-8\varphi\right) \\
& \left.+v^2\left(\frac{2a}{3}-8\varphi\right)+4\right] \\
& -\sqrt{w_0}\left[v^4w^2(2a)+v^2w^2\left(2+\frac{11a^2}{18}-\frac{20\varphi a}{3}-8\varphi^2\right)\right. \\
& +v^4\left(2+\frac{a^2}{18}-\frac{4\varphi a}{3}+8\varphi^2\right)+v^2\left(\frac{8a}{3}-8\varphi\right) \\
& \left.+w^2\left(\frac{2a}{3}-8\varphi\right)+4\right],
\end{aligned}
\tag{26}
$$

$$v_0=v^4+av^2+1, \quad w_0=w^4+aw^2+1. \tag{27}$$

From (26), by squaring both members, comes

$$A_1(v^4w^4+1)+B_1(v^4w^2+v^2w^4+v^2+w^2)+C_1(v^4+w^4)$$
$$+D_1v^2w^2=-\sqrt{v_0}\sqrt{w_0}\,[E_1(v^2w^2+1)+H_1(v^2+w^2)]. \quad (28)$$

The coefficients A_1, B_1, etc., are functions of φ and a of degree not exceeding the sixth and containing from seven to fourteen terms each. From (28) by squaring both members the rational equation of the curves is obtained, containing more than eleven thousand terms, if expanded completely. The coefficients are functions of a, A_1, B_1, etc., and may be expressed in more convenient form by the following substitutions,

$$b=\frac{2a}{3}+4\varphi, \quad c=1-\frac{a^2}{6^2}+\frac{2\varphi a}{3}-4\varphi^2, \quad (29)$$

$$k=(a-2)^2(a+2)^2, \quad (30)$$

Equation (17) would degenerate if k were to vanish. The equation obtained from (28) is, after division by k,

$$(v^8w^8+1)b^4+(v^8+w^8)c^4+4(v^8w^6+v^6w^8+v^2+w^2)b^3c$$
$$+4(v^8w^2+v^2w^8+v^6+w^6)bc^3+6(v^8w^4+v^4w^8+v^4+w^4)b^2c$$
$$+2v^4w^4(3b^4+3c^4+12b^2c^2+8a^2b^2c^2-8ab^3c-8abc^3)$$
$$+4(v^6w^6+v^2w^2)(c^2-b^2+2abc)b^2$$
$$+4(v^6w^4+v^4w^6+v^4w^2+v^2w^4)(4abc-b^2-c^2)bc$$
$$+4(v^6w^2+v^2w^6)(b^2-c^2+2abc)c^2=0. \quad (31)$$

Equation (31) may be factored, giving

$$\left[(v^2w^2+1)\left(4\varphi+\frac{2a}{3}\right)+(v^2+w^2)\left(1-\frac{a^2}{6^2}+\frac{2\varphi a}{3}-4\varphi^2\right)\right.$$
$$\left.+vw\left(2-\frac{5a^2}{18}+\frac{8\varphi a}{3}+8\varphi^2\right)\right]^2$$
$$\left[(v^2w^2+1)\left(4\varphi+\frac{2a}{3}\right)+(v^2+w^2)\left(1-\frac{a^2}{6^2}+\frac{2\varphi a}{3}-4\varphi^2\right)\right.$$
$$\left.-vw\left(2-\frac{5a^2}{18}+\frac{8\varphi a}{3}+8\varphi^2\right)\right]^2=0. \quad (32)$$

Finally from (32) after introducing ρ again in place of φ and making use of (23).

$$\left[\frac{(v+w)^2}{\rho-\varepsilon_2}+\frac{(vw-1)^2}{\rho-\varepsilon_3}-\frac{(vw+1)^2}{\rho-\varepsilon_1}\right]^2 \cdot$$

$$\left[-\frac{(v-w)^2}{\rho-\varepsilon_2}+\frac{(vw-1)^2}{\rho-\varepsilon_1}-\frac{(vw+1)^2}{\rho-\varepsilon_3}\right]^2=0. \quad (33)$$

These are the familiar cyclic curves and need no further discussion.

If in (20) and (21) the lower signs had been taken the computation would have been identical, for the following changes lead to an equation of the form of (26)

$$\overline{a}=-ai, \quad \overline{v}^2=v^2i, \quad \overline{w}^2=w^2i, \quad \overline{\rho}=-\rho i, \text{ etc.}$$

The result is

$$\left[\frac{(v+w)^2i}{\rho-\varepsilon_2}+\frac{(vw-i)^2}{\rho-\varepsilon_3}-\frac{(vw+i)^2}{\rho-\varepsilon_1}\right]^2$$

$$\left[-\frac{(v-w)^2i}{\rho-\varepsilon_2}+\frac{(vw-i)^2}{\rho-\varepsilon_1}-\frac{(vw+i)^2}{\rho-\varepsilon_3}\right]^2=0. \quad (34)$$

Haentzschel notices the fact that (3) may be replaced by the following:

$$r+ix=F(t+iu), \quad r-ix=F_1(t-iu). \quad (35)$$

There will be changes in several equations, especially in (7), but, as Haentzschel admits, nothing new will result.

From the forms of the cyclic curve equations above it appears that the surfaces corresponding will be of the fourth degree, thus proving Wangerin's assertion, and invalidating Haentzschel's criticisms.

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