# ON THE INTERSECTIONS OF PLANE CURVES.

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IN a review of the theory of the "Intersections of plane curves" in the March number of the BULLETIN (pp. 260–273), Professor Charlotte A. Scott has included a full and appreciative criticism of my paper on "Point-Groups in relation to curves" (*Proceedings of the London Mathematical Society* vol. 26 (1895), pp. 495–544). I am exceptionally fortunate in having my work in this subject so clearly described and explained; and I hope I may be allowed to discuss further some interesting points raised in Miss Scott's paper.

For the sake of clearness it may be well to repeat what is meant by excess and defect. If a $C_6$ and $C_7$ are drawn through 4 points on a straight line they intersect again in 38 points. These 38 points are such that a $C_8$ through 35 of them necessarily passes through the remaining 3, and a $C_9$ through 37 of them necessarily passes through the last, so that the 38 points supply only 35 independent conditions for a $C_8$, and 37 for a $C_9$; these properties are expressed by saying that the 8-ic excess of the group of 38 points is 3, and the 9-ic excess is 1. So, in general the $n$-ic excess $r_n$ of a group of $N$ points is the excess of $N$ over the number of independent conditions that the point group $N$ supplies for $n$-ics. This number of conditions is therefore $N - r_n$. So also the $n$-ic defect $q_n$ of the same point group $N$ is the number of independent conditions by which the group falls short in determining an $n$-ic; in other words, it is the degree of freedom of the general $n$-ic through the point group $N$. Hence the formula

$$N - r_n + q_n = \tfrac{1}{2}n(n + 3). \qquad (1)$$

Of course $r_n$ may be zero; but it is important to bear in mind, if $N$ is a point group derived in some way from the intersection of curves, that $r_n$ is just as likely to be greater than zero as to be zero.

The terms excess and defect are nearly equivalent to the terms suggested by Cayley, viz., postulation for the number $N - r_n$, and postulandum for $q_n$; but on account of the fact that the numbers $N - r_n$ and $q_n$ are not so convenient for dealing with as $r_n$ and $q_n$ it is preferable to have simple terms for the latter. Excess and defect are complementary numbers, of equal importance, having an intimate reciprocal relation which is most clearly seen in the light of the Riemann-Roch theorem.

We shall use another formula below, which requires some explanation, for proof of which I refer to my paper. Let a $C_l$ and $C_m$ be drawn through the point group $N$, intersecting again in a finite point group $N' = lm - N$, $l$ and $m$ being great enough for this to be possible. Then, if $q_{n'}'$ denote the $n'$-ic defect of $N'$, we shall have

$$\begin{aligned} & q_{n'}' = r_n - 1, \\ \text{where} \qquad & n + n' = l + m - 3, \end{aligned} \tag{2}$$

provided $n$ is not less than $l-2$ or $m-2$. Formula (2) implies that $N'$ lies on an $n'$-ic if $r_n \geq 1$, since then $q_{n'}' \geq 0$; and that $N'$ does not lie on an $n'$-ic if $r_n = 0$. These properties are both true.

This formula, which is given by Miss Scott, easily supplies the answer to her question on p. 270: "Having found in any given case that the $N$ points which form the partial intersection of $C_l$ and $C_m$ have an $n$-ic excess $r_n$, is there any way of deciding whether a $C_n$ through $N - r_n$ of these necessarily passes through the remainder?" In hazarding an answer she speaks of *the* $n'$-ic through the $N' = lm - N$ points, apparently not noticing that if $r_n > 1$ formula (2) shows that there is a system of $n'$-ics through $N'$ with freedom $r_n - 1$. "An $n$-ic through $N - r_n$ of the $N$ points passes necessarily through the remainder $r_n$, if the $N - r_n$ points supply $N - r_n$ independent conditions for $n$-ics" (Art. 2, $h$ of my paper); but it is quite possible that this condition should not be fulfilled. In order that the $n$-ic excess of $N$ may be $r_n$ it is necessary and sufficient that the $n'$-ic defect of $N'$ should be $r_n - 1$; and in order that the $n$-ic excess of $N - r_n$ may at the same time be 1, it is necessary and sufficient that the $n'$-ic defect of $N' + r_n$ should be 0, or that the $r_n$ points should lie on an $n'$-ic through the $N'$ points. And as the $n'$-ics through $N'$ have a degree of freedom $r_n - 1$, this only imposes one additional interconnection between the $r_n$ or the $N$ points. For example, let $C_7$ and $C_8$ intersect in 56 points, made up of $N' = 9$ points forming the base of a pencil of cubics, $r = 2$ more points lying on a cubic through the 9, and $N - r = 45$ others. The 9-ic excess of the $N = 47$ points is 2, and the 9-ic excess of the $N - r = 45$ points is 1, so that a 9-ic through the $N - r = 45$ points does not necessarily pass through the remaining 2 of the $N = 47$ points.

Miss Scott justly points out (p. 268) the obscurity of the terms complete, incomplete, and redundant, as applied in my paper to point groups. But these terms, or some equivalent ones, are useful; and a great deal depends on recognizing the distinctions involved in them. If we take away 2 points from the group of 38, mentioned in the second paragraph above, we have left a group of 36 points whose 8-ic excess is 1, while every 8-ic through the 36 points passes through 2 more fixed points. The 38 points form a complete group which cannot, in general, be decomposed into simpler groups; while the 36 points are said to form an incomplete group, because they form part of the complete group 38. But 7-ics through the 36 points pass through the 2 fixed points and 4 others, forming a group of 42 points (the total intersection of $C_6$ and $C_7$). This group of points is also complete; but it is not only complete, it is composite, since it can be decomposed into a group of 4 points on a straight line and the group 38. So also the 47 points mentioned above, which are made up of 45 and 2, form a composite group. A redundant point group, consisting of a non-composite group together with an additional number of general points, is the simplest kind of a composite point group. In order to reduce a point group, by passing two curves through it to intersect again in a less complex group, it is, in general, essential to recognize, and separate out, its constituent groups, if it happens to be composite.

The method of reduction in my paper includes the reduction of redundant but not of other composite point groups. Some notes are given explaining how the appearance of composite point groups may be avoided in the course of reduction; but these are not so much "limitations" as extensions of the method. This is exemplified in the reduction of the point group given below. The object is to find the simplest reduction, and consequent construction, for a point group of assigned characterization, *i. e.*, a point group of which the number of points and the excesses for curves of all orders are given. It seems probable that the simplest point group with an assigned characterization is a non-composite one if such exists, and that, in any case, it is the least composite. I may add here that in my paper I give formulæ for easily calculating the number of independent interconnections of the points of a group if its construction is known. This number is an important one, and serves as an index of the complexity of the group.

On p. 272 Miss Scott gives an example of what she considers to be an impossible point group, viz., $N = 369$ with excesses 48, 28, 16, 7, 2 for curves of order 24, 25, 26, 27, 28. I proceed to show that a point group with this characterization can be constructed. We begin by supposing that the point group is redundant, *i. e.*, that it consists of a point group $N_0 = 367$, and 2 general points in the plane. We also suppose $N_0$ to lie on a $C_{23}$, and that its 23-ic defect is 1, so that its 23-ic excess is 69. This would still leave it impossible for a $C_{23}$ to pass through $N$. The excesses of $N_0$ for curves of order 24 to 28 will clearly be the same as those of $N$. For the reduction we modify formula (2) by substituting for $q_{n'}'$ its value in terms of $r_{n'}'$ from (1); we then have

$$r_{n'}' = N' + r_n - \tfrac{1}{2}(n' + 1)(n' + 2), \tag{3}$$

where $n + n' = l + m - 3$, $N + N' = lm$, $r_n > 0$, and $n$ is not less than $l - 2$ or $m - 2$. The reduction may be exhibited as follows, the explanation being given after:

$N_0 = 367,\ r_{28} = 2,\ r_{27} = 7,\ r_{26} = 16,\ r_{25} = 28,\ r_{24} = 48,\ r_{23} = 69;$

$N' = 162,\ r_{15}' = 28,\ r_{16}' = 16,\ r_{17}' = 7,\ r_{18}' = 0,\ r_{19}' = 0,\ r_{20}' = 0;$

$N'' = 63,\quad r_{12}'' = 0,\quad r_{11}'' = 1,\quad r_{10}'' = 4;$

$N_0'' = 65,\quad r_{12}'' = 1,\quad r_{11}'' = 3,\quad r_{10}'' = 6;$

$N''' = 35,\quad r_5''' = 15,\quad r_6''' = 10,\quad r_7''' = 5.$

The point group $N'$ is derived from $N_0$ by passing two curves $C_{23}$ through $N_0$ ($q_{23} = 1$). Hence $N' = 23^2 - 367 = 162$; and in applying (3) to find the excesses $r_{n'}'$ of $N'$, we have $l = m = 23$, $n + n' = 43$, and the values corresponding to $n = 28, 27, \cdots, 23$ in the first line are $n' = 15, 16, \cdots, 20$, in the second line. Similarly $N'' = 63$ is derived by passing two curves $C_{15}$ through $N'$ ($q_{15} = 1$), and (3) is again applied for finding the excesses of $N''$. Since $r_{18}' = 0$, $N''$ does not lie on a $C_9$. We have supposed $N''$ to be incomplete, forming part of a complete $N_0'' = 65$. The 10-ic and 11-ic excesses of $N_0''$ will exceed those of $N''$ by 2, and we have supposed the 12-ic excess to be 1. Finally $N''' = 35$ is derived by passing two curves $C_{10}$ through $N_0''$ ($q_{10}'' = 6$). The last point-group $N'''$ is a recognizable one, although, as Miss Scott says, it must be examined with care. $N'''$ consists of 35 general points on a $C_5$, for the excesses of 35 such points for curves of order 5, 6, 7, 8 are 15, 10, 5, 0.

All the steps are reversible. Since two curves $C_{10}$ through $N_0''$ determine $N'''$, so two curves $C_{10}$ through $N'''$ determine $N_0''$; $N_0''$ deprived of two of its points gives $N''$; two curves $C_{15}$ through $N''$ determine $N'$; and two curves $C_{23}$ through $N'$ determine $N_0$, to which is added any two general points in the plane, giving $N$.

The number of independent interconnections of the $N = 369$, or the $N_0 = 367$, points, when constructed in this way, is 217. The least possible number of interconnections for a point group with the assigned characterization is 215.

If two curves $C_{24}$ are passed through the $N = 369$ points, constructed as above, they determine a group of 207 points which is composite, being made up of 45 points on a conic and the $N' = 162$ points found above. In the actual reduction we have, so to speak, eliminated the 45 points.

If I omitted all reference to the criticism in Miss Scott's last paragraph but one (p. 273) I might be taken as acquiescing in it. The whole question resolves itself into this. Having given a $C_n$ with any number and kind of multiple points can we always find a curve $C_{n'}'$ ($n'$ being greater than $n$ if necessary) whose coefficients differ from those of $C_n$ only by infinitely small amounts, and such that at each and every multiple point $A$, of order $p$ on $C_n$ the curve $C_{n'}'$ passes through $\frac{1}{2}p(p+1)$ points arbitrarily but generally chosen about and infinitely near to $A$? This is not so much a doubtful matter of opinion as a matter of fact which can be proved or disproved analytically. The convention, which I adopt, of replacing $C_n$ by $C_{n'}'$ is an extremely convenient one for the purpose of reasoning geometrically about the intersections of curves, since, for one thing, it enables us to consider the intersection of two curves at a common multiple point as being made up of separate instead of coincident points, just as we consider a tangent to a curve as meeting it in two points at the point of contact. It does not claim to have any other merit or application.

I may add that, since writing the above, I have succeeded in proving that a point group is a possible one if the second differences of its excesses for descending orders of curves are all positive integers, among which zeros may be included; otherwise the point group is impossible.

LONDON,
*June* 6, 1898.