Asian J. Math.
Vol. 13, No. 2, pp. 191–214, June 2009

003

ON SINGULAR CUBIC SURFACES*

IVAN CHELTSOV†

Abstract. We study global log canonical thresholds of singular cubic surfaces.

Key words. Cubic surfaces, singularities, log canonical thresholds, del Pezzo fibrations, birational maps, Kahler-Einstein metric, alpha-invariant of Tian, orbifolds.

AMS subject classifications. 14J26, 14J45, 14J70, 14Q10, 14B05, 14E05, 32Q20

All varieties are assumed to be defined over $\mathbb{C}$.

1. Introduction. Let X be a variety with at most log terminal singularities, let $Z \subseteq X$ be a closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then the number

$$\mathrm{lct}_Z\big(X, D\big) = \sup\Big\{\lambda \in \mathbb{Q}\ \Big|\ \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical along } Z\Big\}$$

is said to be the log canonical threshold of D along Z (see [8]).

Example 1.1. Let $\phi \in \mathbb{C}[z_1, \cdots, z_n]$ be a nonzero polynomial, let $O \in \mathbb{C}^n$ be the origin. Then

$$\mathrm{lct}_O\Big(\mathbb{C}^n, \big(\phi = 0\big)\Big) = \sup\left\{c \in \mathbb{Q}\ \middle|\ \text{the function } \frac{1}{\big|\phi\big|^{2c}} \text{ is locally integrable near } O\right\}.$$

For the case $Z = X$ we use the notation $\mathrm{lct}(X, D)$ instead of $\mathrm{lct}_X(X, D)$. Then

$$\begin{aligned}\mathrm{lct}\big(X, D\big) &= \inf\Big\{\mathrm{lct}_P\big(X, D\big)\ \Big|\ P \in X\Big\}\\ &= \sup\Big\{\lambda \in \mathbb{Q}\ \Big|\ \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical}\Big\}.\end{aligned}$$

Suppose, in addition, that X is a Fano variety.

Definition 1.2. We define the global log canonical threshold of X by the number

$$\mathrm{lct}\big(X\big) = \inf\Big\{\mathrm{lct}\big(X, D\big)\ \Big|\ D \text{ is effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D\ \equiv -K_X\Big\}.$$

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant introduced in [11].

Example 1.3. Let X be a smooth cubic surface in $\mathbb{P}^3$. Then it follows from [4] that

$$\mathrm{lct}\big(X\big) = \begin{cases} 2/3 \text{ when } X \text{ has an Eckardt point,}\\ 3/4 \text{ when } X \text{ does not have Eckardt points.}\end{cases}$$

*Received October 28, 2008; accepted for publication December 2, 2008.
†School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, UK (I.Cheltsov@ed.ac.uk). The author would like to thank I. Dolgachev, J. Kollár, V. Shokurov, J. Park and the referee for useful comments.

In this paper we prove the following result[1].

THEOREM 1.4. *Let X be a singular cubic surface in $\mathbb{P}^3$ with canonical singularities. Then*

$$\mathrm{lct}\big(X\big)=\begin{cases}2/3 \textit{ when } \mathrm{Sing}\big(X\big)=\big\{\mathbb{A}_1\big\},\\ 1/3 \textit{ when } \mathrm{Sing}\big(X\big)\supseteq\big\{\mathbb{A}_4\big\},\\ 1/3 \textit{ when } \mathrm{Sing}\big(X\big)=\big\{\mathbb{D}_4\big\},\\ 1/3 \textit{ when } \mathrm{Sing}\big(X\big)\supseteq\big\{\mathbb{A}_2,\mathbb{A}_2\big\},\\ 1/4 \textit{ when } \mathrm{Sing}\big(X\big)\supseteq\big\{\mathbb{A}_5\big\},\\ 1/4 \textit{ when } \mathrm{Sing}\big(X\big)=\big\{\mathbb{D}_5\big\},\\ 1/6 \textit{ when } \mathrm{Sing}\big(X\big)=\big\{\mathbb{E}_6\big\},\\ 1/2 \textit{ in other cases.}\end{cases}$$

Let us consider one birational application of Theorem 1.4.

THEOREM 1.5. *Let Z be a smooth curve. Suppose that there is a commutative diagram*

$$\begin{tikzcd} V \arrow[r, dashed, "\rho"] \arrow[d, "\pi"'] & \bar{V} \arrow[d, "\bar{\pi}"] \\ Z \arrow[r, equal] & Z \end{tikzcd} \qquad (1.6)$$

such that π and $\bar{\pi}$ are flat morphisms, and ρ is a birational map that induces an isomorphism

$$\rho\big|_{V\setminus X}\colon V\setminus X\longrightarrow \bar{V}\setminus\bar{X}, \qquad (1.7)$$

where X and $\bar{X}$ are scheme fibers of π and $\bar{\pi}$ over a point $O\in Z$, respectively. Suppose that

- *the varieties V and $\bar{V}$ have terminal $\mathbb{Q}$-factorial singularities,*
- *the divisors $-K_V$ and $-K_{\bar{V}}$ are π-ample and $\bar{\pi}$-ample, respectively,*
- *the fibers X and $\bar{X}$ are irreducible.*

Then ρ is an isomorphism if one of the following conditions hold:

- *the varieties X and $\bar{X}$ have log terminal singularities, and* $\mathrm{lct}(X)+\mathrm{lct}(\bar{X})>1$;
- *the variety X has log terminal singularities, and* $\mathrm{lct}(X)\geqslant 1$.

The assertion of Theorem 1.5 is sharp (see [10, Example 5.2–5.6]).

EXAMPLE 1.8. Let V be $\bar{V}$ subvarieties in $\mathbb{C}^1\times\mathbb{P}^3$ given by the equations

$$x^3+y^3+z^2w+t^6w^3=0 \text{ and } x^3+y^3+z^2w+w^3=0,$$

respectively, where t is a coordinate on $\mathbb{C}^1$, and (x,y,z,w) are coordinates on $\mathbb{P}^3$. The projections

$$\pi\colon V\longrightarrow\mathbb{C}^1 \text{ and } \bar{\pi}\colon\bar{V}\longrightarrow\mathbb{C}^1$$

[1] A cubic surface in $\mathbb{P}^3$ with isolated singularities has canonical singularities $\iff$ it is not a cone.

are fibrations into cubic surfaces. Let O be the point on $\mathbb{C}^1$ given by $t = 0$. Then $\bar{X}$ is smooth, the surface X has one singular point of type $\mathbb{D}_4$. Put $Z = \mathbb{C}^1$. Then the map

$$\big(x, y, z, w\big) \longrightarrow \big(t^2x, t^2y, t^3z, w\big)$$

induces a birational map $\rho\colon V \dashrightarrow \bar{V}$ such that the diagrams 1.6 and isomorphism 1.7 exist, and ρ is not biregular. But $\mathrm{lct}(X) = 1/3$ and $\mathrm{lct}(\bar{X}) = 2/3$ (see Example 1.3 and Theorem 1.4).

Example 1.9. Let V be $\bar{V}$ subvarieties in $\mathbb{C}^1 \times \mathbb{P}^3$ given by the equations

$$x^3 + y^2z + z^2w + t^{12}w^3 = 0 \text{ and } x^3 + y^2z + z^2w + w^3 = 0,$$

respectively, where t is a coordinate on $\mathbb{C}^1$, and (x, y, z, w) are coordinates on $\mathbb{P}^3$. The projections

$$\pi\colon V \longrightarrow \mathbb{C}^1 \text{ and } \bar{\pi}\colon \bar{V} \longrightarrow \mathbb{C}^1$$

are fibrations into cubic surfaces. Let O be the point on $\mathbb{C}^1$ given by $t = 0$. Then $\bar{X}$ is smooth, the surface X has one singular point of type $\mathbb{E}_6$. Put $Z = \mathbb{C}^1$. Then the map

$$\big(x, y, z, w\big) \longrightarrow \big(t^2x, t^3y, z, t^6w\big)$$

induces a birational map $\rho\colon V \dashrightarrow \bar{V}$ such that the diagrams 1.6 and isomorphism 1.7 exist, and ρ is not biregular. But $\mathrm{lct}(X) = 1/6$ and $\mathrm{lct}(\bar{X}) = 2/3$ (see Example 1.3 and Theorem 1.4).

Example 1.10. Let V be $\bar{V}$ subvarieties in $\mathbb{C}^1 \times \mathbb{P}^3$ given by the equations

$$wz^2 + zx^2 + y^2x + t^8w^3 = 0 \text{ and } wz^2 + zx^2 + y^2x + w^3 = 0,$$

respectively, where t is a coordinate on $\mathbb{C}^1$, and (x, y, z, w) are coordinates on $\mathbb{P}^3$. The projections

$$\pi\colon V \longrightarrow \mathbb{C}^1 \text{ and } \bar{\pi}\colon \bar{V} \longrightarrow \mathbb{C}^1$$

are fibrations into cubic surfaces. Let O be the point on $\mathbb{C}^1$ given by $t = 0$. Then $\bar{X}$ is smooth, the surface X has one singular point of type $\mathbb{D}_5$. Put $Z = \mathbb{C}^1$. Then the map

$$\big(x, y, z, w\big) \longrightarrow \big(t^2x, ty, z, t^4w\big)$$

induces a birational map $\rho\colon V \dashrightarrow \bar{V}$ such that the diagrams 1.6 and isomorphism 1.7 exist, and ρ is not biregular. But $\mathrm{lct}(X) = 1/4$ and $\mathrm{lct}(\bar{X}) = 2/3$ (see Example 1.3 and Theorem 1.4).

The number $\mathrm{lct}(X)$ is closely related to the existence of a Kähler–Einstein metric (see [6]), but we can not use Theorem 1.4 to prove the existence of such a metric on singular cubic surfaces.

Remark 1.11. If a singular normal cubic surface in $\mathbb{P}^3$ admits an orbifold Kähler–Einstein metric, then its singular locus must consist of singular points of type $\mathbb{A}_1$ and $\mathbb{A}_2$ (see [7]).

Nevertheless, we can use an equivariant analogue of the number $\mathrm{lct}(X)$ to prove the existence of an orbifold Kähler–Einstein metric on some symmetric singular cubic surfaces.

EXAMPLE 1.12. Let X_1 be the Cayley cubic surface in $\mathbb{P}^3$, i.e. a singular surface given by

$$xyz + xyt + xzt + yzt = 0 \subseteq \mathbb{P}^3 \cong \mathrm{Proj}\Big(\mathbb{C}[x, y, z, t]\Big),$$

and let X_2 be a cubic surface in $\mathbb{P}^3$ that is given by the equation $xyz = t^3$. Put

$$\mathrm{lct}\big(X_1,\ \mathrm{S}_4\big) = \sup\left\{\lambda \in \mathbb{Q}\ \left|\ \begin{array}{l}\text{the log pair } (X_1, \lambda D) \text{ has log canonical singularities}\\ \text{for every } \mathrm{S}_4\text{-invariant effective } \mathbb{Q}\text{-divisor } D \equiv -K_{X_1}\end{array}\right.\right\},$$

where we consider S_4 as a subgroup of $\mathrm{Aut}(X_1)$. Similarly, we define $\mathrm{lct}(X_2, \mathrm{S}_3 \times \mathbb{Z}_3)$. Then

$$\mathrm{lct}\Big(X_1, \mathrm{S}_4\Big) = \mathrm{lct}\Big(X_2, \mathrm{S}_3 \times \mathbb{Z}_3\Big) = 1 > \frac{2}{3}$$

by [4, Lemma 5.1]. Then X_1 and X_2 admit Kähler–Einstein metrics[2] by [6] (cf. [5, Appendix A]).

We prove Theorem 1.4 in Section 3, and we prove Theorem 1.5 in Section 4.

2. Basic tools. Let S be a surface with canonical singularities, and D be an effective $\mathbb{Q}$-divisor on it.

REMARK 2.1. Let B be an effective $\mathbb{Q}$-divisor on S such that (S, B) is log canonical. Then

$$\left(S,\ \frac{1}{1-\alpha}\Big(D - \alpha B\Big)\right)$$

is not log canonical if (S, D) is not log canonical, where $\alpha \in \mathbb{Q}$ such that $0 \leqslant \alpha < 1$.

Let $\mathrm{LCS}(S, D) \subset S$ be a subset such that $P \in \mathrm{LCS}(S, D)$ if and only if (S, D) is not log terminal at the point P. The set $\mathrm{LCS}(S, D)$ is called the locus of log canonical singularities.

LEMMA 2.2. *Suppose that* $-(K_S + D)$ *is ample. Then* $\mathrm{LCS}(S, D)$ *is connected.*

Proof. See Theorem 17.4 in [9]. □

Let P be a point of the surface S such that (S, D) is not log canonical at the point P.

REMARK 2.3. Suppose that S is smooth at P. Then $\mathrm{mult}_P(D) > 1$.

Let C be an irreducible curve on the surface S. Put

$$D = mC + \Omega,$$

where $m \in \mathbb{Q}$ such that $m \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subseteq \mathrm{Supp}(\Omega)$.

[2]The existence of orbifold Kähler–Einstein metrics on X_1 and X_2 is obvious, because both X_1 and X_2 are quotients branched over singular points of smooth Kähler–Einstein del Pezzo surfaces (see [2] and [7, Example 1.4]).

REMARK 2.4. Suppose that $C \subseteq \mathrm{LCS}(S, D)$. Then $m \geqslant 1$.

LEMMA 2.5. *Suppose that $P \in C$, the surface S is smooth at P, and $m \leqslant 1$. Then $C \cdot \Omega > 1$.*

Proof. It follows from Theorem 17.6 in [9] that $C \cdot \Omega \geqslant \mathrm{mult}_P(\Omega|_C) > 1$. □

Let $\pi\colon \bar{S} \to S$ be a birational morphism such that the surface $\bar{S}$ has canonical singularities, and $\bar{D}$ is a proper transform of D via π. Then

$$K_{\bar{S}} + \bar{D} + \sum_{i=1}^{r} a_i E_i \equiv \pi^*\big(K_S + D\big),$$

where E_i is a π-exceptional curve, and a_i is a rational number.

REMARK 2.6. The log pair (S, D) is log canonical if and only if $(\bar{S}, \bar{D}+\sum_{i=1}^{r} a_i E_i)$ is log canonical.

Suppose that $r = 1$, $\pi(E_1) = P$, and P is an ordinary double point.

LEMMA 2.7. *Suppose that $\bar{S}$ is smooth along E_1. Then $a_1 > 1/2$.*

Proof. The inequality $a_1 > 1/2$ follows from Theorem 17.6 in [9]. □

Most of the described results are valid in much more general settings (see [9]).

3. Main result. Let S be a singular cubic surface in $\mathbb{P}^3$ with canonical singularities. Put $\Sigma = \mathrm{Sing}(S)$ and

$$\mathrm{lct}_n\big(S\big) = \sup\Big\{\mu \in \mathbb{Q}\ \Big|\ \text{the log pair } \Big(S, \frac{\mu}{n}D\Big) \text{ is log canonical for every } D \in \big|-nK_X\big|\Big\}$$

for every $n \in \mathbb{N}$. Then it follows from [12] that

$$\mathrm{lct}\big(S\big) = \mathrm{inf}_{n\in\mathbb{N}}\Big(\mathrm{lct}_n\big(S\big)\Big) \leqslant \mathrm{lct}_1\big(S\big) = \begin{cases} 2/3 \text{ when } \Sigma = \big\{\mathbb{A}_1\big\}, \\ 1/3 \text{ when } \Sigma \supseteq \big\{\mathbb{A}_4\big\}, \\ 1/3 \text{ when } \Sigma = \big\{\mathbb{D}_4\big\}, \\ 1/3 \text{ when } \Sigma \supseteq \big\{\mathbb{A}_2, \mathbb{A}_2\big\}, \\ 1/4 \text{ when } \Sigma \supseteq \big\{\mathbb{A}_5\big\}, \\ 1/4 \text{ when } \Sigma = \big\{\mathbb{D}_5\big\}, \\ 1/6 \text{ when } \Sigma = \big\{\mathbb{E}_6\big\}, \\ 1/2 \text{ in other cases.} \end{cases}$$

Let D be an arbitrary effective $\mathbb{Q}$-divisor on the surface S such that

$$D \equiv -K_S \sim \mathcal{O}_{\mathbb{P}^3}\big(1\big)\Big|_S,$$

and let λ be an arbitrary positive rational number such that $\lambda < \mathrm{lct}_1(S)$.

LEMMA 3.1. *Suppose that* $\mathrm{lct}_1(S) \leqslant 1/3$. *Then* $\mathrm{LCS}(S, \lambda D) \subseteq \Sigma$.

Proof. Suppose that $(S, \lambda D)$ is not log terminal at a smooth point $P \in S$. Then

$$3 = -K_S \cdot D \geqslant \mathrm{mult}_P\big(D\big) > 1/\lambda > 3,$$

which is a contradiction. The obtained contradiction implies that $\mathrm{LCS}(S, \lambda D) \subseteq \Sigma$. □

Lemma 3.2. *Suppose that* $|\mathrm{LCS}(S, \lambda D)| < +\infty$. *Then* $\mathrm{LCS}(S, \lambda D) \subseteq \Sigma$.

Proof. The required assertion follows from [4]. □

Let O be a singular point of the surface S, and $\alpha\colon \bar{S} \to S$ be a partial resolution of singularities that contracts smooth rational curves $E_1, \ldots, E_k$ to the point O such that

$$\bar{S} \setminus \left(\bigcup_{i=1}^{k} E_i\right) \cong S \setminus O,$$

the surface $\bar{S}$ is smooth along $\cup_{i=1}^{k} E_i$, and $E_i^2 = -2$ for every $i = 1, \ldots, k$. Then

$$\bar{D} \equiv \alpha^*\big(D\big) - \sum_{i=1}^{k} a_i E_i,$$

where $\bar{D}$ is the proper transform of D on the surface $\bar{S}$, and $a_i \in \mathbb{Q}$. Let $L_1, \ldots, L_r$ be lines on the surface S such that $O \in L_i$, and $\bar{L}_i$ be the proper transform of L_i on the surface $\bar{S}$. Then

$$-K_{\bar{S}} \cdot \bar{L}_1 = \cdots = -K_{\bar{S}} \cdot \bar{L}_r = 1.$$

Remark 3.3. To prove Theorem 1.4, we must show that the equality

$$\mathrm{lct}\big(S\big) = \mathrm{lct}_1\big(S\big)$$

holds. Hence, it follows from the choice of the divisor D and $\lambda \in \mathbb{Q}$ that to prove Theorem 1.4 it is enough to show that the singularities of the log pair $(S, \lambda D)$ are log canonical.

In the rest of the section, we prove Theorem 1.4 case by case using [1].

Lemma 3.4. *Suppose that* $\Sigma = \{\mathbb{A}_1\}$. *Then* $\mathrm{lct}(S) = 2/3$.

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical. Let us derive a contradiction.

Suppose that there is an irreducible curve $Z \subset S$ such that $D = \mu Z + \Omega$, where μ is a rational number such that $\mu \geqslant 1/\lambda$, and Ω is an effective $\mathbb{Q}$-divisor such that $Z \not\subset \mathrm{Supp}(\Omega)$. Then

$$3 = -K_S \cdot D = \mu \mathrm{deg}\big(Z\big) - K_S \cdot \Omega \geqslant \mu \mathrm{deg}\big(Z\big) > 3\mathrm{deg}\big(Z\big)/2,$$

which implies that Z is a line. Let C be a general conic on S such that $-K_S \sim Z + C$. Then

$$2 = C \cdot D = \mu C \cdot Z + C \cdot \Omega \geqslant \mu C \cdot Z \geqslant \frac{3}{2}\mu,$$

which is a contradiction. Then $\mathrm{LCS}(S, \lambda D) = O$ by Lemma 3.2.

We have $3 - 2a_1 = \bar{H} \cdot \bar{D} \geqslant 0$, where $\bar{H}$ is a general curve in $|-K_{\bar{S}} - E_1|$. It follows from

$$K_{\bar{S}} + \lambda \bar{D} + \lambda a_1 E_1 \equiv \alpha^*\big(K_S + \lambda D\big)$$

that there is a point $Q \in E_1$ such that $(\bar{S}, \lambda\bar{D}+\lambda a_1 E_1)$ is not log canonical at the point Q.

It follows from [1] that $r = 6$. Let $\pi\colon \bar{S} \to \mathbb{P}^2$ be a contraction of the curves $\bar{L}_1, \ldots, \bar{L}_6$.

Suppose that $Q \notin \cup_{i=1}^{6} \bar{L}_i$. Then

$$\pi\big(\bar{D} + a_1 E_1\big) \equiv \pi\big(-K_{\bar{S}}\big) \equiv -K_{\mathbb{P}^2},$$

and π is an isomorphism in a neighborhood of Q. Let L be a general line on $\mathbb{P}^2$. Then the locus

$$\mathrm{LCS}\Big(\mathbb{P}^2,\ L + \pi\big(\lambda\bar{D} + \lambda a_1 E_1\big)\Big)$$

is not connected, which is impossible by Lemma 2.2.

Therefore, we may assume that $Q \in \bar{L}_1$. Put $D = aL_1 + \Upsilon$, where a is a non-negative rational number, and Υ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line L_1. Then

$$\bar{\Upsilon} \equiv \alpha^*\big(\Upsilon\big) - \epsilon E_1,$$

where $\epsilon = a_1 - a/2$, and $\bar{\Upsilon}$ is the proper transform of the divisor Υ on the surface $\bar{S}$.

The log pair $(\bar{S}, \lambda a\bar{L}_1 + \lambda\bar{\Upsilon} + \lambda(a/2+\epsilon)E_1)$ is not log canonical at Q. Then

$$1 + a/2 - \epsilon = \bar{L}_1 \cdot \bar{\Upsilon} > 1/\lambda - a/2 - \epsilon$$

by Lemma 2.5, because $\lambda a \leqslant 1$. Hence, we have $a > 1/2$.

It follows from [12] that there is a conic $C_1 \subset S$ such that the log pair

$$\Big(S,\ \mathrm{lct}_1(S)\big(L_1 + C_1\big)\Big)$$

is not log terminal. But it must be log canonical. Therefore, in the case when $C_1 \subseteq \mathrm{Supp}(D)$, we can use Remark 2.1 to find an effective divisor D' on the surface S such that the equivalence

$$D' \equiv -K_S$$

holds, the log pair $(S, \lambda D')$ is not log canonical at the point P, and $C_1 \not\subseteq \mathrm{Supp}(D')$.

To complete the proof, we may assume that $C_1 \not\subseteq \mathrm{Supp}(D)$.

Let $\bar{C}_1$ be the proper transforms of the conic C_1 on the surface $\bar{S}$. Then

$$2 - 3a/2 - \epsilon = \bar{C}_1 \cdot \bar{\Upsilon} \geqslant \mathrm{mult}_Q\big(\bar{\Upsilon}\big) > 1/\lambda - a/2 - \epsilon,$$

which implies that $a < 1/2$. But $a > 1/2$. The obtained contradiction completes the proof. ∎

LEMMA 3.5. *Suppose that* $\Sigma = \{\mathbb{A}_1, \ldots, \mathbb{A}_1\}$ *and* $|\Sigma| \geqslant 2$. *Then* $\mathrm{lct}(S) = 1/2$.

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical. Let us derive a contradiction.

Suppose that there is an irreducible curve Z on the surface S such that

$$D = \mu Z + \Omega,$$

where μ is a rational number such that $\mu \geqslant 1/\lambda$, and Ω is an effective $\mathbb{Q}$-divisor, whose support does not contain the curve Z. Then Z is a line (see the proof of Lemma 3.4). We have

$$2 = C \cdot D = \mu C \cdot Z + C \cdot \Omega \geqslant \mu C \cdot Z \geqslant \mu \geqslant 1/\lambda > 2,$$

where C is a general conic on S that intersects Z in two points.

We may assume that $\mathrm{LCS}(S, \lambda D) = O$ by Lemmas 2.2 and 3.2. Then $a_1 > 1$ by Lemma 2.7.

Arguing as in the proof of Lemma 3.4, we see that there is a point $Q \in E$ such that the singularities of the log pair $(\bar{S}, \lambda\bar{D} + \lambda a_1 E_1)$ are not log canonical at the point Q.

Let P be a point in Σ such that $P \neq O$. We may assume that $P \in L_1$. Then

$$2L_1 + L' \sim -K_S$$

for some line $L' \subset S$.

Suppose that $Q \in \bar{L}_1$. Let a be a non-negative rational number such that

$$D = aL_1 + \Upsilon,$$

where Υ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line L_1. Then

$$\bar{\Upsilon} \equiv \alpha^*\big(\Upsilon\big) - \epsilon E_1,$$

where $\bar{\Upsilon}$ is the proper transforms of Υ on the surface $\bar{S}$, and $\epsilon = a_1 - a/2$. The log pair

$$\Big(\bar{S},\ \lambda a\bar{L}_1 + \lambda\bar{\Upsilon} + \lambda(a/2 + \epsilon)E_1\Big)$$

is not log canonical at the point Q. We have $\bar{L}_1^2 = -1/2$. Then

$$1 - \epsilon = \bar{L}_1 \cdot \bar{\Upsilon} > 1/\lambda - a/2 - \epsilon$$

by Lemma 2.5. We have $a > 1/\lambda$, which is impossible. Hence, we see that $Q \notin \bar{L}_1$.

There is a unique reduced conic $Z \subset S$ such that $O \in Z \ni P$ and $Q \in \bar{Z}$, where $\bar{Z}$ is the proper transform of the conic Z on the surface $\bar{S}$. Then $L_1 \not\subseteq \mathrm{Supp}(Z)$, because $Q \notin \bar{L}_1$.

Suppose that Z is irreducible. Put

$$D = eZ + \Delta,$$

where $e \in \mathbb{Q}$, and Δ is an effective $\mathbb{Q}$-divisor such that $C \not\subseteq \mathrm{Supp}(\Delta)$. Then

$$\bar{\Delta} \equiv \alpha^*\big(\Delta\big) - \delta E_1,$$

where $\bar{\Delta}$ is the proper transforms of Δ on the surface $\bar{S}$, and $\delta = a_1 - e/2$. Then

$$2 - e - \delta = \bar{Z} \cdot \bar{\Delta} > 1/\lambda - e/2 - \delta > 2 - e/2 - \delta$$

by Lemma 2.5, because $\bar{C}^2 = 1/2$. We have $e < 0$, which is impossible.

We see that the conic Z is reducible. Then

$$Z = L_2 + L_2',$$

where L_2' is a line on S such that $P \in L_2'$ and $L_2 \cap L_2' \neq \varnothing$.

The intersection $L_2 \cap L_2'$ consists of a single point. The impossibility of the case $Q \in \bar{L}_1$ implies that the surface S is smooth at the point $L_2 \cap L_2'$. There is a rational number $c \geqslant 0$ such that

$$D = cL_2 + \Xi,$$

where Ξ is an effective $\mathbb{Q}$-divisor, whose support does not contain the line L_2. Then

$$\bar{\Xi} \equiv \alpha^*\big(\Xi\big) - \upsilon E_1,$$

where $\bar{\Xi}$ is the proper transforms of Ξ on the surface $\bar{S}$, and $\upsilon = a_1 - c/2$. The log pair

$$\Big(\bar{S},\ \lambda c\bar{L}_2 + \lambda\bar{\Xi} + \lambda(c/2 + \upsilon)E_1\Big)$$

is not log canonical at Q. We have $Q \in \bar{L}_2$ and $\bar{L}_2^2 = -1$. Then

$$1 + c/2 - \upsilon = \bar{L}_2 \cdot \bar{\Xi} > 1/\lambda - c/2 - \upsilon > 2 - c/2 - \upsilon$$

by Lemma 2.5. Therefore, the inequality $c > 1$ holds.

There is a unique hyperplane section T of the surface S such that $T = C_2 + L_2$ and

$$Q = \bar{C}_2 \cap \bar{L}_2 = O,$$

where C_2 is a conic, and $\bar{C}_2$ is the proper transforms of C_2 on the surface $\bar{S}$.

The conic C_2 is irreducible. We may assume that $C_2 \not\subseteq \mathrm{Supp}(D)$ (see Remark 2.1). Then

$$2 - 3c/2 - \upsilon = \bar{C}_2 \cdot \bar{\Xi} \geqslant \mathrm{mult}_Q\big(\bar{\Xi}\big) > 1/\lambda - c/2 - \upsilon,$$

which implies that $c < 0$. The obtained contradiction completes the proof. □

Lemma 3.6. *Suppose that* $\Sigma = \{\mathbb{D}_4\}$. *Then* $\mathrm{lct}(S) = 1/3$.

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 3$. The lines L_1, L_2, L_3 lie in a single plane. Thus, we may assume that $L_3 \not\subseteq \mathrm{Supp}(D)$ due to Remark 2.1 and Lemma 3.1.

Let $\beta\colon \tilde{S} \to S$ be a birational morphism such that the morphism α contracts one irreducible rational curve E that contains three singular points O_1, O_2, O_3 of type $\mathbb{A}_1$.

Let $\tilde{D}$ and $\tilde{L}_i$ be the proper transforms of D and L_i on the surface $\tilde{S}$, respectively. Then

$$\tilde{D} \equiv \beta^*\big(D\big) - \mu E,$$

where μ is a positive rational number. We have $\tilde{L}_i \equiv \beta^*(L_i) - E$. Then

$$0 \leqslant \tilde{D} \cdot \tilde{L}_3 = \Big(\beta^*\big(D\big) - \mu E\Big) \cdot \tilde{L}_3 = 1 - \mu E \cdot \tilde{L}_3 = 1 - \mu/2,$$

which implies that $\mu \leqslant 2$. Therefore, we may assume that there is a point $Q \in E$ such that the singularities of the log pair $(\tilde{S}, \lambda\tilde{D} + \mu E)$ are not log canonical at the point Q (see Lemma 3.1).

Suppose that $\tilde{S}$ is smooth at Q. The log pair $(\tilde{S}, \lambda\tilde{D}+E)$ is not log canonical at Q. Then

$$1 \geqslant \mu/2 = -\mu E^2 = E\cdot\tilde{D} > 1/\lambda > 3$$

by Lemma 2.5. We see that $Q = O_j$ for some j.

The curves $\tilde{L}_1$, $\tilde{L}_2$ and $\tilde{L}_3$ are disjoined, and each of them passes through a singular point of the surface $\tilde{S}$. Therefore, we may assume that $O_i \in \tilde{L}_i$ for every i.

Let $\gamma\colon \hat{S} \to \tilde{S}$ be a blow up of the point O_j, and G be the exceptional curve of γ. Then

$$\hat{L}_j \equiv \gamma^*\big(\tilde{L}_j\big) - \frac{1}{2}G \equiv \big(\beta\circ\gamma\big)^*\big(L_1\big) - \hat{E} - G,$$

where $\hat{L}_j$ and $\hat{E}$ are proper transforms of the curves $\bar{L}_j$ and E on the surface $\hat{S}$, respectively.

Let $\hat{D}$ be the proper transform of the divisor $\tilde{D}$ on the surface $\hat{S}$. Then

$$\hat{D} \equiv \gamma^*\big(\tilde{D}\big) - \epsilon G \equiv \big(\beta\circ\gamma\big)^*\big(D\big) - \mu\hat{E} - \big(\mu/2+\epsilon\big)G,$$

where ϵ is a rational number, because $2\hat{E} \equiv \gamma^*(2E) - G$. By Lemma 2.7, we have

$$\lambda\epsilon + \lambda\mu/2 > 1/2.$$

Suppose that $j = 3$. Then $1 - \mu/2 - \epsilon = \hat{D}\cdot\hat{L}_3 \geqslant 0$. But $\epsilon + \mu/2 > 3/2$.

We may assume that $Q = O_1$, and the support of the divisor D contains the line L_1. Put

$$D = aL_1 + \Omega,$$

where $a \in \mathbb{Q}$ and $a \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$\hat{\Omega} \equiv \big(\beta\circ\gamma\big)^*\big(\Omega\big) - m\hat{E} - \big(m/2+b\big)G,$$

where $\hat{\Omega}$ is the proper transform of Ω, and m and b are non-negative rational numbers. Then

$$\begin{aligned}&\big(\beta\circ\gamma\big)^*\big(D\big) - \mu\hat{E} - \big(\mu/2+\epsilon\big)G \equiv \hat{D} = a\hat{L}_1 + \hat{\Omega}\\ &\equiv \big(\beta\circ\gamma\big)^*\big(aL_1+\Omega\big) - \big(a+m\big)\hat{E} - \big(a+m/2+b\big)G,\end{aligned}$$

which implies that $\mu = a + m \leqslant 2$ and $\epsilon = a/2 + b$. We have

$$\hat{L}_1^2 = -1,\ \hat{E}^2 = -1,\ G^2 = -2,\ \hat{L}\cdot\hat{E} = 0,\ \hat{L}\cdot G = \hat{E}\cdot G = 1$$

on the surface $\hat{S}$. The surface $\hat{S}$ is smooth along the curve G. Then

$$-a \leqslant -a + \hat{\Omega}\cdot\hat{L}_1 = \Big(a\hat{L}_1 + \hat{\Omega}\Big)\cdot\hat{L}_1 = 1 - a - m/2 - b,$$

which implies that $m/2+b \leqslant 1$ and $a+m/2+b \leqslant 1+a \leqslant 3$. Thus, by the equivalence

$$K_{\hat{S}} + \lambda a\hat{L}_1 + \lambda\hat{\Omega} + \lambda\big(a+m\big)\hat{E} + \lambda\big(a+m/2+b\big)G \equiv \big(\beta\circ\gamma\big)^*\big(K_S + \lambda aL_1 + \lambda\Omega\big),$$

there exists a point $A \in G$ such that the log pair

$$\Big(\hat{S},\ \lambda a\hat{L}_1 + \lambda\hat{\Omega} + \lambda\big(a+m\big)\hat{E} + \lambda\big(a+m/2+b\big)G\Big)$$

is not log canonical at the point A.

Suppose that $A \notin \hat{L}_1 \cup \hat{E}$. Then $(\hat{S}, \lambda\hat{\Omega} + \lambda(a+m/2+b)G)$ is not log canonical at A, and

$$2b + a = \Big(a\hat{L}_1 + \hat{\Omega}\Big)\cdot G = a + \hat{\Omega}\cdot G > a + 3,$$

by Lemma 2.5. We see that $b > 3/2$. But $m/2 + b \leqslant 1$. We see that $A \in \hat{L}_1 \cup \hat{E}$.

Suppose that $A \notin \hat{L}_1$. The log pair

$$\Big(\hat{S},\ \lambda\hat{\Omega} + \lambda\big(a+m\big)\hat{E} + \lambda\big(a+m/2+b\big)G\Big)$$

is not log canonical at the point A. Arguing as in the previous case, we see that

$$m/2 - b = \hat{\Omega}\cdot\hat{E} > 3 - a - m/2 - b,$$

which implies that $a + m > 3$. But $a + m \leqslant 2$. We see that $A \in \hat{L}_1$.

The log pair $(\hat{S}, \lambda a\hat{L}_1 + \lambda\hat{\Omega} + \lambda(a+m/2+b)G)$ is not log canonical at A. Then

$$1 - a - m/2 - b = \Big(a\hat{L}_1 + \hat{\Omega}\Big)\cdot\hat{L}_1 = -a + \hat{\Omega}\cdot\hat{L}_1 > -a + 3 - \big(a + m/2 + b\big)$$

by Lemma 2.5. We have $a > 2$. But $a + m \leqslant 2$, which is a contradiction. □

Lemma 3.7. *Suppose that* $\Sigma = \{\mathbb{D}_5\}$. *Then* $\mathrm{lct}(S) = 1/4$.

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical. Let us derive a contradiction.

We see that $\mathrm{LCS}(S, \lambda D) = \{O\}$ by Lemma 3.1.

It follows from [1] that $r = 2$ and the surface S contains a line L such that $O \notin L$.

Projecting from L, we see that there is a conic $C \subset S$ such that the equivalence

$$-K_S \sim C + L$$

holds, $O \notin C$ and $|C \cap L| = 1$. Put $P = C \cap L$. Then

$$P \cup O \subseteq \mathrm{LCS}\left(S,\ \frac{3}{4}\Big(C + L\Big) + \lambda D\right) \subseteq P \cup O \cup C \cup L,$$

which is impossible by Lemma 2.2. The obtained contradiction completes the proof. □

Lemma 3.8. *Suppose that* $\Sigma = \{\mathbb{E}_6\}$. *Then* $\mathrm{lct}(S) = 1/6$

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 1$. The log pair

$$\Big(S,\ \mathrm{lct}_1(S)L_1\Big)$$

is not log terminal. But it must be log canonical. The surface S contains a plane cuspidal cubic curve C such that $O \notin C$. Arguing as in the proof of Lemma 3.6, we obtain a contradiction. □

Using the classification of possible singularities of the surface S obtained in [1], we see that it follows from Lemmas 3.4, 3.5, 3.6, 3.7 and 3.8 that we may assume that

$$\Sigma = \Big\{\mathbb{A}_{i_1}, \dots, \mathbb{A}_{i_s}\Big\}$$

to complete the proof of Theorem 1.4 . We assume that $i_1 \leqslant \dots \leqslant i_s$ and O is of type $\mathbb{A}_{i_s}$.

Lemma 3.9. *Suppose that* $\Sigma = \{\mathbb{A}_2\}$. *Then* $\mathrm{lct}(S) = 1/2$.

Proof. Suppose that the log pair $(S, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 6$. We may assume that the equivalences

$$-K_S \sim L_1 + L_2 + L_3 \sim L_4 + L_5 + L_6$$

hold. The log pairs $(S, \mathrm{lct}_1(S)(L_1 + L_2 + L_3))$ and $(S, \mathrm{lct}_1(S)(L_4 + L_5 + L_6))$ are log canonical.

Arguing as in the proof of Lemma 3.4, we see that

$$\mathrm{LCS}\Big(S, \lambda D\Big) = O.$$

Let $\bar{H}$ be a proper transform on $\bar{S}$ of a general hyperplane section that contains O. Then

$$0 \leqslant \bar{H} \cdot \bar{D} = 3 - a_1 - a_2,\ 2a_1 - a_2 = E_1 \cdot \bar{D} \geqslant 0,\ 2a_2 - a_1 = E_2 \cdot \bar{D} \geqslant 0,$$

which implies that $a_1 \leqslant 2$ and $a_2 \leqslant 2$. There is a point $Q \in E_1 \cup E_2$ such that the log pair

$$\Big(\bar{S},\ \lambda\big(\bar{D} + a_1E_1 + a_2E_2\big)\Big)$$

is not log canonical at Q. We may assume that $Q \in E_1$, and

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = \bar{L}_6 \cdot E_2 = 1,$$

which implies that $\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = \bar{L}_6 \cdot E_1 = 0$.

It follows from Remark 2.1 that we may assume that $\bar{L}_1 \not\subseteq \mathrm{Supp}(D) \not\supseteq \bar{L}_4$. Then

$$\begin{cases} 1 - a_1 = \bar{D} \cdot \bar{L}_1 \geqslant 0, \\ 1 - a_2 = \bar{D} \cdot \bar{L}_4 \geqslant 0, \end{cases}$$

which implies that $a_1 \leqslant 1$ and $a_2 \leqslant 1$.

Suppose that $Q \notin E_2$. Then $(\bar{S}, \lambda\bar{D} + E_1)$ is not log canonical at Q. We have

$$2a_1 - a_2 = \bar{D} \cdot E_1 > 1/\lambda > 2,$$

by Lemma 2.5. Then $a_1 \geqslant 4/3$, which is impossible, because $a_1 \leqslant 1$. Hence, we see that $Q \in E_2$.

The log pairs $(\bar{S}, \lambda\bar{D} + E_1 + a_2E_2)$ and $(\bar{S}, \lambda\bar{D} + a_1E_1 + E_2)$ are not log canonical at Q. Then

$$\begin{cases} 2a_1 - a_2 = \bar{D} \cdot E_1 > 1/\lambda - a_2 > 2 - a_2, \\ 2a_2 - a_1 = \bar{D} \cdot E_2 > 1/\lambda - a_1 > 2 - a_1, \end{cases}$$

by Lemma 2.5. Then $a_1 > 1$ and $a_2 > 1$. But $a_1 \leqslant 1$ and $a_2 \leqslant 1$, which is a contradiction. □

LEMMA 3.10. *Suppose that* $\Sigma = \{\mathbb{A}_3\}$. *Then* $\mathrm{lct}(S) = 1/2$

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 5$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_3 = \bar{L}_5 \cdot E_3 = 1,$$

which implies that $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = 0$ and

$$\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = 0.$$

The inequalities $\bar{L}_i^2 = -1$ and $\bar{L}_i \cdot \bar{L}_j = 0$ hold for $i \neq j$. We have $-K_S \sim L_1 + L_2 + L_3$.

Suppose that there are a line $L \subset S$ and a rational number $\mu \geqslant 1/\lambda$ such that $D = \mu L + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor, whose support does not contain the line L. Then

$$2 = C \cdot D = \mu C \cdot L + C \cdot \Omega \geqslant \mu C \cdot L > 2C \cdot L,$$

where C is a general conic on the surface S such that the divisor $C + L$ is a hyperplane section of the surface S. Then $|L \cap C| = 1$ and $C \cdot L < 1$, which implies that $L = L_3$. But $L_3 \cdot C = 1$.

Arguing as in the proof of Lemma 3.2, we see that $\mathrm{LCS}(S, \lambda D) = O$ by Lemmas 2.2.

Let $\bar{H}$ be a general curve in $|-K_{\bar{S}} - \sum_{i=1}^{3} E_i|$. Then

$$a_1 + a_3 \leqslant 3,\ 2a_1 \geqslant a_2,\ 2a_2 \geqslant a_1 + a_3,\ 2a_3 \geqslant a_2,$$

because $\bar{H} \cdot \bar{D} \geqslant 0$, $E_1 \cdot \bar{D} \geqslant 0$, $E_2 \cdot \bar{D} \geqslant 0$, $E_3 \cdot \bar{D} \geqslant 0$, respectively.

We may assume that either $L_1 \not\subseteq \mathrm{Supp}(D)$ or $L_3 \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. But

$$\bar{L}_1 \cdot \bar{D} = 1 - a_1,\ \bar{L}_3 \cdot \bar{D} = 1 - a_2,$$

which implies that either $a_1 \leqslant 1$ or $a_2 \leqslant 1$. Similarly, we assume that either $a_3 \leqslant 1$ or $a_2 \leqslant 1$.

We have $a_1 \leqslant 2$, $a_2 \leqslant 2$, $a_3 \leqslant 2$. There is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the log pair

$$\Big(\bar{S},\ \lambda\big(\bar{D} + a_1 E_1 + a_2 E_2 + a_3 E_3\big)\Big)$$

is not log canonical at Q. We may assume that $Q \notin E_3$.

Suppose that $Q \notin E_2$. Then $(\bar{S}, \lambda \bar{D} + E_1)$ is not log canonical at Q, which implies that

$$2a_1 - a_2 = \bar{D} \cdot E_1 > 2$$

by Lemma 2.5. Then $a_1 > 3/2$ and $a_2 > 1$. But either $a_1 \leqslant 1$ or $a_2 \leqslant 1$.

Suppose that $Q \in E_2 \cap E_1$. Arguing as in the proof of of Lemma 3.9, we see that

$$\begin{cases} 2a_1 - a_2 = \bar{D} \cdot E_1 > 1/\lambda - a_2 > 2 - a_2, \\ 2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 > 1/\lambda - a_1 > 2 - a_1, \end{cases}$$

by Lemma 2.5. Then $a_1 > 1$ and $2a_2 > 2 + a_3$, which is impossible.

We see that $Q \in E_2$ and $Q \notin E_1$. Then $(\bar{S}, \lambda\bar{D} + E_2)$ is not log canonical at Q. We have

$$2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 > 1/\lambda > 2,$$

which implies that $a_1 > 3/2$ and $a_2 > 2$. The obtained contradiction completes the proof. □

Lemma 3.11. *Suppose that* $\Sigma = \{\mathbb{A}_4\}$. *Then* $\mathrm{lct}(S) = 1/3$

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 4$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_3 = \bar{L}_4 \cdot E_4 = 1,$$

which implies that $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_4 = 0$ and

$$\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_2 \cdot E_4 = \bar{L}_4 \cdot E_1 = \bar{L}_4 \cdot E_2 = \bar{L}_4 \cdot E_3 = 0.$$

We have $\mathrm{LCS}(S, \lambda D) = O$ by Lemma 3.1. Let $\bar{H}$ be a general curve in $|-K_{\bar{S}} - \sum_{i=1}^{4} E_i|$. Then

$$3 \geqslant a_1 + a_4,\ 2a_1 \geqslant a_2,\ 2a_2 \geqslant a_1 + a_3,\ 2a_3 \geqslant a_2 + a_4,\ 2a_4 \geqslant a_3,$$

because $\bar{H} \cdot \bar{D} \geqslant 0$, $E_1 \cdot \bar{D} \geqslant 0$, $E_2 \cdot \bar{D} \geqslant 0$, $E_3 \cdot \bar{D} \geqslant 0$, $E_4 \cdot \bar{D} \geqslant 0$, respectively.

One can easily check that the equivalences

$$-K_S \sim L_1 + L_2 + L_3 \sim 2L_3 + L_4$$

hold. Therefore, we may assume that either

$$L_1 \not\subseteq \mathrm{Supp}(D) \not\supseteq L_4$$

or $L_3 \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1 and Lemma 3.1. But

$$\bar{L}_3 \cdot \bar{D} = 1 - a_3,\ \bar{L}_1 \cdot \bar{D} = 1 - a_1,\ \bar{L}_4 \cdot \bar{D} = 1 - a_4,$$

which implies that there is a point $Q \in \cup_{i=1}^{4} E_i$ such that the log pair

$$\Big(\bar{S},\ \lambda\big(\bar{D} + \sum_{i=1}^{4} a_i E_i\big)\Big)$$

is not log canonical at the point Q. Arguing as in the proof of Lemma 3.10, we see that

$$\begin{cases} Q \in E_1 \setminus (E_1 \cap E_2) \Rightarrow 2a_1 > a_2 + 3, \\ Q \in E_1 \cap E_2 \Rightarrow 2a_1 > 3 \text{ and } 2a_2 > 3 + a_3, \\ Q \in E_2 \setminus \big((E_1 \cap E_2) \cup (E_2 \cap E_3)\big) \Rightarrow 2a_2 > a_1 + a_3 + 3, \\ Q \in E_2 \cap E_3 \Rightarrow 2a_2 > 3 + a_1 \text{ and } 2a_3 > 3 + a_4, \\ Q \in E_3 \setminus \big((E_2 \cap E_3) \cup (E_3 \cap E_4)\big) \Rightarrow 2a_3 > 3 + a_2 + a_4, \\ Q \in E_3 \cap E_4 \Rightarrow 2a_3 > 3 + a_2 \text{ and } 2a_4 > 3, \\ Q \in E_4 \setminus (E_4 \cap E_3) \Rightarrow 2a_4 > 3, \end{cases}$$

which leads to a contradiction, because either $a_3 \leqslant 1$ or $a_1 \leqslant 1$ and $a_4 \leqslant 1$. □

LEMMA 3.12. *Suppose that* $\Sigma = \mathbb{A}_5$. *Then* $\mathrm{lct}(S) = 1/4$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 3$. We may assume that $\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_4 = 1$ and

$$\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_2 \cdot E_4 = \bar{L}_1 \cdot E_5 = \bar{L}_2 \cdot E_3 = 0$$

and $\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_3 = \bar{L}_3 \cdot E_5 = 0$. Then $\mathrm{LCS}(S, \lambda D) = O$ by Lemma 3.1.

Let $\bar{H}$ be a proper transform on $\bar{S}$ of a general hyperplane section that contains O. Then

$$3 \geqslant a_1 + a_5,\ 2a_1 \geqslant a_2,\ 2a_2 \geqslant a_1 + a_3,\ 2a_3 \geqslant a_2 + a_4,\ 2a_4 \geqslant a_3 + a_5,\ 2a_5 \geqslant a_4, \tag{3.13}$$

because $\bar{H} \cdot \bar{D} \geqslant 0$, $E_1 \cdot \bar{D} \geqslant 0$, $E_2 \cdot \bar{D} \geqslant 0$, $E_3 \cdot \bar{D} \geqslant 0$, $E_4 \cdot \bar{D} \geqslant 0$, $E_5 \cdot \bar{D} \geqslant 0$, respectively.

We have $-K_S \sim 3L_3$. Thus, we may assume that $L_3 \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. Then

$$a_1 \leqslant 5/2,\ a_2 \leqslant 2,\ a_3 \leqslant 3/2,\ a_4 \leqslant 1,\ a_5 \leqslant 5/4,$$

because $1 - a_4 = \bar{L}_3 \cdot \bar{D} \geqslant 0$.

Arguing as in the proof of Lemma 3.10, we see that there is a point $Q \in \cup_{i=1}^{5} E_i$ such that

$$\begin{cases} Q \in E_1 \setminus (E_1 \cap E_2) \Rightarrow 2a_1 > a_2 + 4, \\ Q \in E_1 \cap E_2 \Rightarrow 2a_1 > 4 \text{ and } 2a_2 > 4 + a_3, \\ Q \in E_2 \setminus \big((E_1 \cap E_2) \cup (E_2 \cap E_3)\big) \Rightarrow 2a_2 > a_1 + a_3 + 4, \\ Q \in E_2 \cap E_3 \Rightarrow 2a_2 > 4 + a_1 \text{ and } 2a_3 > 4 + a_4, \\ Q \in E_3 \setminus \big((E_2 \cap E_3) \cup (E_3 \cap E_4)\big) \Rightarrow 2a_3 > 4 + a_2 + a_4, \\ Q \in E_3 \cap E_4 \Rightarrow 2a_3 > 4 + a_2 \text{ and } 2a_4 > 4 + a_5, \\ Q \in E_4 \setminus \big((E_3 \cap E_4) \cup (E_4 \cap E_5)\big) \Rightarrow 2a_4 > 4 + a_3 + a_5, \\ Q \in E_4 \cap E_5 \Rightarrow 2a_4 > 4 + a_3 \text{ and } 2a_5 > 4, \\ Q \in E_5 \setminus (E_4 \cap E_5) \Rightarrow 2a_5 > a_4 + 4. \end{cases} \tag{3.14}$$

The inequalities 3.13 and 3.14 imply that either $Q = E_3 \cap E_4$ or $Q = E_4 \cap E_5$, because $a_4 \leqslant 1$.

Let H_1 and H_3 be general divisors in $|-K_S|$ that contain L_1 and L_3, respectively. Then

$$H_1 = L_1 + C_1,\ H_3 = L_3 + C_3,$$

where C_1 and C_3 are irreducible conics such that $C_1 \not\subseteq \mathrm{Supp}(D) \not\supseteq C_3$.

Let $\bar{C}_1$ and $\bar{C}_3$ be the proper transforms of C_1 and C_3 on the surface $\bar{S}$, respectively. Then

$$\begin{cases} 2 - a_5 = \bar{C}_1 \cdot \bar{D} \geqslant 0, \\ 2 - a_2 = \bar{C}_3 \cdot \bar{D} \geqslant 0, \end{cases}$$

which is impossible due to the inequalities 3.13 and 3.14. □

LEMMA 3.15. *Suppose that* $\Sigma = \{\mathbb{A}_1, \mathbb{A}_5\}$. *Then* $\mathrm{lct}(S) = 1/4$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 2$. We have $\mathrm{LCS}(S, \lambda D) \subseteq \Sigma$ by Lemma 3.1.

Let P be a point in Σ of type $\mathbb{A}_1$. We may assume that $P \in L_1$. Then

$$\bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_2 \cdot E_3 = \bar{L}_2 \cdot E_5 = \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_1 \cdot E_4 = \bar{L}_1 \cdot E_5 = 0,$$

and $\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_4 = 1$. The equivalence $-K_S \sim 3L_2$ holds.

Suppose that $(S, \lambda D)$ is not log canonical at P. Let $\beta\colon \tilde{S} \to S$ be a blow up of P. Then

$$\tilde{D} \equiv \beta^*\big(-K_S\big) - mF,$$

where F is the β-exceptional curve, $\tilde{D}$ is the proper transform of the divisor D, and $m \in \mathbb{Q}$. Then

$$0 \leqslant \tilde{H} \cdot \tilde{D} = \Big(\beta^*\big(-K_S\big) - mF\Big) \cdot \Big(\beta^*\big(-K_S\big) - F\Big) = 3 - 2m,$$

where $\tilde{H}$ is general curve in $|-K_{\tilde{S}} - F|$. Thus, we have $m \leqslant 3/2$. But $m > 2$ by Lemma 2.7.

We see that $\mathrm{LCS}(S, \lambda D) = O$. Let C_1 and C_2 be general conics on the surface S such that

$$L_1 + C_1 \sim L_2 + C_2 \sim -K_S,$$

and let $\bar{C}_1$ and $\bar{C}_2$ be the proper transforms of C_1 and C_2 on the surface $\bar{S}$, respectively. Then

$$\begin{cases} 2 - a_1 = \bar{C}_1 \cdot \bar{D} \geqslant 0, \\ 2 - a_5 = \bar{C}_2 \cdot \bar{D} \geqslant 0, \end{cases}$$

because $C_1 \not\subseteq \mathrm{Supp}(D) \not\supseteq C_2$. We may assume that $L_2 \not\subseteq \mathrm{Supp}(D)$ due to Remark 2.1.

Arguing as in the proof of Lemma 3.12, we obtain the inequalities

$$\begin{gathered} 3 \geqslant a_1 + a_5,\ 2a_1 \geqslant a_2,\ 2a_2 \geqslant a_1 + a_3,\ 2a_3 \geqslant a_2 + a_4, \\ 2a_4 \geqslant a_3 + a_5,\ 2a_5 \geqslant a_4,\ 2 \geqslant a_2,\ 2 \geqslant a_5,\ 1 \geqslant a_4, \end{gathered}$$

which imply that there is a point $Q \in \cup_{i=1}^{5} E_i$ such that the log pair

$$\Big(\bar{S},\ \lambda\big(\bar{D} + \sum_{i=1}^{5} a_i E_i\big)\Big)$$

is not log canonical at Q. Arguing as in the proof of Lemma 3.10, we obtain a contradiction. □

LEMMA 3.16. *Suppose that* $\Sigma = \{\mathbb{A}_1, \mathbb{A}_4\}$. *Then* $\mathrm{lct}(S) = 1/3$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

Let P be a point in Σ of type $\mathbb{A}_1$. We may assume that $P \in L_1$.

It follows from [1] that $r = 3$. Then

$$\bar{L}_1 \cdot E_1 = 1,\ \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_1 \cdot E_4 = 0,$$

and we may assume that $\bar{L}_3 \cdot E_3 = \bar{L}_2 \cdot E_4 = 1$. Then $-K_S \sim L_2 + 2L_3$ and

$$\bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_3 \cdot E_4 = \bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_2 \cdot E_3 = 0.$$

We may assume that either $L_3 \not\subseteq \mathrm{Supp}(D)$ or $L_1 \not\subseteq \mathrm{Supp}(D) \not\supseteq L_2$ (see Remark 2.1).

Arguing as in the proof of Lemma 3.15, we see that

$$\mathrm{LCS}\Big(S,\ \lambda D\Big) = O,$$

and arguing as in the proof of Lemma 3.11, we obtain a contradiction. □

Lemma 3.17. *Suppose that* $\Sigma = \{\mathbb{A}_1, \mathbb{A}_3\}$. *Then* $\mathrm{lct}(S) = 1/2$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

Let P be a point in Σ of type $\mathbb{A}_1$. We may assume that $P \in L_1$.

It follows from [1] that $r = 4$ and S contains lines L_5, L_6, L_7 such that

$$L_5 \ni P \in L_6,\ O \notin L_7 \not\ni P,\ L_3 \cap L_5 \neq \varnothing,\ L_4 \cap L_6 \neq \varnothing,$$
$$L_7 \cap L_2 \neq \varnothing,\ L_7 \cap L_5 \neq \varnothing,\ L_7 \cap L_6 \neq \varnothing,$$

which implies that $L_7 \cap L_1 = L_7 \cap L_3 = L_7 \cap L_4 = \varnothing$. Then

$$L_1 + L_3 + L_5 \sim L_1 + L_4 + L_6 \sim L_5 + L_6 + L_7 \sim L_2 + 2L_1 \sim L_2 + L_3 + L_4 \sim 2L_2 + L_7$$

and $-K_S \sim L_1 + L_3 + L_5$. Put

$$D = \mu_i L_i + \Omega_i,$$

where μ_i is a non-negative rational number, and Ω_i is an effective $\mathbb{Q}$-divisor, whose support does not contain the line L_i. Let us show that that $\mu_i < 1/\lambda$ for $i = 1, \ldots, 7$.

Suppose that $\mu_2 \geqslant 1/\lambda$. We may assume that $L_1 \not\subseteq \mathrm{Supp}(D)$ by Remark 2.1. Then

$$1 = L_1 \cdot D = L_1 \cdot \big(\mu_2 L_2 + \Omega_2\big) \geqslant \mu_2 L_1 \cdot L_2 = \mu_2/2 > 1,$$

which is a contradiction. Similarly, we see that $\mu_i < 1/\lambda$ for $i = 1, \ldots, 7$.

Arguing as in the proof of Lemma 3.4, we see that

$$\mathrm{LCS}\Big(S,\ \lambda D\Big) \subseteq \Sigma,$$

which implies that $\mathrm{LCS}(S, \lambda D) = O$ or $\mathrm{LCS}(S, \lambda D) = P$ by Lemma 2.2.

Suppose that $\mathrm{LCS}(S, \lambda D) = P$. Put

$$D = \mu_5 L_5 + \mu_6 L_6 + \Upsilon,$$

where Υ is an effective $\mathbb{Q}$-divisor such that $L_5 \not\subseteq \mathrm{Supp}(\Upsilon) \not\supseteq L_6$. Then $\mu_5 > 0$ and $\mu_6 > 0$. But

$$1 = L_7 \cdot D = L_7 \cdot \big(\mu_5 L_5 + \mu_6 L_6 + \Upsilon\big) \geqslant L_7 \cdot \big(\mu_5 L_5 + \mu_6 L_6\big) = \mu_5 + \mu_6,$$

because we may assume that $L_7 \not\subseteq \mathrm{Supp}(\Upsilon)$. Let $\beta\colon \tilde{S} \to S$ be a blow up of the point P. Then

$$\mu_5\tilde{L}_5 + \mu_6\tilde{L}_6 + \tilde{\Upsilon} \equiv \beta^*\big(\mu_5L_5 + \mu_6L_6 + \Upsilon\big) - \big(\mu_5/2 + \mu_6/2 + \epsilon\big)G,$$

where ϵ is a rational number, G is the exceptional curve of β, and $\tilde{L}_5$, $\tilde{L}_6$, $\tilde{\Upsilon}$ are proper transforms of the divisors L_5, L_6, Υ on the surface $\tilde{S}$, respectively. Then

$$0 \leqslant \Big(\mu_5\tilde{L}_5 + \mu_6\tilde{L}_6 + \tilde{\Upsilon}\Big)\tilde{H} = 3 - \mu_5 - \mu_6 - 2\epsilon,$$

where $\tilde{H}$ is a general curve in $|-K_{\tilde{S}} - G|$. There is a point $Q \in G$ such that the log pair

$$\Big(\tilde{S},\ \lambda\big(\mu_5\tilde{L}_5 + \mu_6\tilde{L}_6 + \tilde{\Upsilon}\big) + \lambda\big(\mu_5/2 + \mu_6/2 + \epsilon\big)G\Big)$$

are not log canonical at Q. We have

$$2 - 2\epsilon = \tilde{\Upsilon} \cdot \big(\tilde{L}_5 + \tilde{L}_6\big) \geqslant 0,$$

which implies that $\epsilon \leqslant 1$. Then it follows from Lemma 2.5 that

$$2\epsilon = \tilde{\Omega} \cdot G > 2$$

if $Q \notin \tilde{L}_5 \cup \tilde{L}_6$, which implies that we may assume that $Q \in \tilde{L}_5$. Then

$$1 + \mu_5/2 - \mu_6 - \epsilon = \tilde{\Omega} \cdot \tilde{L}_5 > 2 - \mu_5/2 - \mu_6/2 - \epsilon,$$

by Lemma 2.5. Thus, we see that $\mu_5 > 1$. But

$$\mu_5 \leqslant \mu_5 + \mu_6 \leqslant 1,$$

which is a contradiction. The obtained contradiction shows that $\mathrm{LCS}(S, \lambda D) \neq P$.

We see that $\mathrm{LCS}(S, \lambda D) = O$. We may assume that

$$\bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_3 = \bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_1 = \bar{L}_4 \cdot E_2 = 0$$

and $\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_3 = \bar{L}_4 \cdot E_3 = 1$. But the log pair

$$\Big(S,\ \mathrm{lct}_1(S)\big(2L_1 + L_2\big)\Big)$$

has log canonical singularities. Similarly, the log pair

$$\Big(S,\ \mathrm{lct}_1(S)\big(L_2 + L_3 + L_3\big)\Big)$$

is log canonical. By Remark 2.1 and Lemma 3.1, we may assume that either

$$L_1 \not\subseteq \mathrm{Supp}\big(D\big) \not\supseteq L_3$$

or $L_2 \not\subseteq \mathrm{Supp}(D)$. Arguing as in the proof of Lemma 3.10, we obtain a contradiction. ∎

LEMMA 3.18. *Suppose that* $\Sigma = \{\mathbb{A}_1, \mathbb{A}_2\}$. *Then* $\mathrm{lct}(S) = 1/2$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 5$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_2 = \bar{L}_5 \cdot E_2 = 1$$

and $\bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_1 = \bar{L}_5 \cdot E_1 = 0$.

Let P be a point in Σ of type $\mathbb{A}_1$. We may assume that $P \in L_1$.

It follows from [1] that S contains lines $L_6, L_7, L_8, L_9, L_{10}, L_{11}$ such that

$$P = L_1 \cap L_6 \cap L_7 \cap L_8,\ L_9 \cap L_6 \neq \varnothing,\ L_9 \cap L_7 \neq \varnothing, L_9 \cap L_6 \neq \varnothing$$

and $L_9 \cap L_7 \neq \varnothing$, $L_{10} \cap L_7 \neq \varnothing$, $L_{10} \cap L_8 \neq \varnothing$, $L_{11} \cap L_6 \neq \varnothing$, $L_{11} \cap L_8 \neq \varnothing$. Then

$$L_2 \not\ni P \notin L_3,\ L_4 \not\ni P \notin L_5,\ L_6 \not\ni O \notin L_7,\ L_8 \not\ni O \notin L_9,\ L_{10} \not\ni O \notin L_{11},$$

which implies that $-K_S \sim L_3 + L_4 + L_5 \sim 2L_1 + L_2 \sim L_3 + L_4 + L_5$ and

$$-K_S \sim 2L_1 + L_2 \sim L_1 + L_3 + L_6 \sim L_1 + L_4 + L_7 \sim L_1 + L_5 + L_8 \sim L_6 + L_7 + L_9$$

and $-K_S \sim L_7 + L_8 + L_{10} \sim L_6 + L_8 + L_{11}$.

Arguing as in the proof of Lemma 3.17, we see that

$$\mathrm{LCS}\Big(S,\ \lambda D\Big) = O.$$

By Remark 2.1, we may assume that either $L_1 \not\subseteq \mathrm{Supp}(D)$ or $L_2 \not\subseteq \mathrm{Supp}(D)$, because

$$2L_1 + L_2 \sim -K_S$$

and the log pair $(S, \mathrm{lct}_1(S)(2L_1 + L_2))$ has log canonical singularities. Similarly, we may assume that $\mathrm{Supp}(D)$ does not contain at least one of the lines L_3, L_4, L_5, because the equivalence

$$L_3 + L_4 + L_5 \sim -K_S$$

holds. Arguing as in the proof of Lemma 3.9, we obtain a contradiction. $\square$

Lemma 3.19. *Suppose that* $\Sigma = \{\mathbb{A}_2, \ldots, \mathbb{A}_2\}$ *and* $|\Sigma| \geqslant 2$. *Then* $\mathrm{lct}(S) = 1/3$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

Let P be a point in Σ such that $P \neq O$. We may assume that $P \in L_1$. Then

$$-K_S \sim 3L_1.$$

We may assume that $(S, \lambda D)$ is not log canonical at O by Lemma 3.1, and we assume that

$$L_1 \not\subseteq \mathrm{Supp}(D)$$

by Remark 2.1 and Lemma 3.1.

We may assume that $\bar{L}_1 \cap E_2 \neq \varnothing$. Then $a_2 \leqslant 1$, because $\bar{D} \cdot \bar{L}_1 \geqslant 0$.

Arguing as in the proof of Lemma 3.9, we see that $3 \geqslant a_1 + a_2, 2a_1 \geqslant a_2, 2a_2 \geqslant a_1, 1 \geqslant a_2$.

There is a point $Q \in E_1 \cup E_2$ such that the log pair

$$\Big(\bar{S},\ \lambda\big(\bar{D} + a_1E_1 + a_2E_2\big)\Big)$$

is not log canonical at the point Q. Arguing as in the proof of Lemma 3.9, we see that

$$\begin{cases} Q \in E_1 \setminus (E_1 \cap E_2) \Rightarrow 2a_1 > a_2 + 3, \\ Q \in E_1 \cap E_2 \Rightarrow 2a_1 > 3 \text{ and } 2a_2 > 3, \\ Q \in E_2 \setminus (E_2 \cap E_1) \Rightarrow 2a_2 > a_1 + 3, \end{cases}$$

which easily leads to a contradiction, because $3 \geqslant a_1+a_2, 2a_1 \geqslant a_2, 2a_2 \geqslant a_1, 1 \geqslant a_2$. ◻

Lemma 3.20. *Suppose that* $\Sigma = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_2\}$. *Then* $\mathrm{lct}(S) = 1/3$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from Lemma 3.1 that $\mathrm{LCS}(S, \lambda D) \subseteq \Sigma$.

Let $P \neq O$ be a point in Σ of type $\mathbb{A}_2$. We may assume that $P \in L_1$. Then

$$-K_S \sim 3L_1,$$

which implies that we may assume that $L_1 \not\subseteq \mathrm{Supp}(D)$ due to Remark 2.1 and Lemma 3.1.

Arguing as in the proof of Lemma 3.15, we see that

$$\mathrm{LCS}\Big(S,\ \lambda D\Big) \subseteq O \cup P,$$

which easily leads to a contradiction (see the proof of Lemma 3.19). ◻

Lemma 3.21. *Suppose that* $\Sigma = \{\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_3\}$. *Then* $\mathrm{lct}(S) = 1/2$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 3$.

Let P_1 and P_2 be points in Σ of type $\mathbb{A}_1$. Then we may assume that $P_1 \in L_1$ and $P_2 \in L_2$.

It follows from [1] that S contains lines L_4 and L_5 such that

$$P_1 \in L_4 \ni P_2,\ O \notin L_4,\ P_1 \notin L_3 \not\ni P_2,\ L_5 \cap \Sigma = \varnothing,$$

which implies that $L_5 \cap L_3 \neq \varnothing$, $L_5 \cap L_4 \neq \varnothing$, $L_5 \cap L_1 = \varnothing$, $L_5 \cap L_2 = \varnothing$. Then

$$-K_S \sim L_1 + L_2 + L_4 \sim L_3 + 2L_1 \sim L_3 + 2L_2 \sim 2L_3 + L_5 \sim 2L_4 + L_5. \tag{3.22}$$

Let us show that $\mathrm{LCS}(S, \lambda D)$ does not contains the lines $L_1, \ldots, L_5$. Put

$$D = \mu_i L_i + \Omega_i,$$

where $\mu_i \in \mathbb{Q}$, and Ω_i is an effective $\mathbb{Q}$-divisor such that $L_i \not\subseteq \mathrm{Supp}(\Omega_i)$.

Suppose that $\mu_1 \geqslant 1/\lambda$. Then it follows from the equivalence 3.22 and Remark 2.1 that we may assume that $L_3 \not\subseteq \mathrm{Supp}(D)$. Therefore, we have

$$1 = L_3 \cdot D = L_3 \cdot \big(\mu_1 L_1 + \Omega_1\big) \geqslant \mu_1 L_3 \cdot L_1 = \mu_1/2 > 1,$$

which is a contradiction. Similarly, we see that $\mu_2 < 1/\lambda$, $\mu_3 < 1/\lambda$, $\mu_4 < 1/\lambda$, $\mu_5 < 1/\lambda$.

Arguing as in the proof of Lemma 3.4, we see that $|\mathrm{LCS}(S, \lambda D)| = 1$ and

$$\mathrm{LCS}\Big(S,\ \lambda D\Big) \subsetneq \Sigma.$$

Suppose that $\mathrm{LCS}(S, \lambda D) = P_1$. Let $\beta\colon \tilde{S} \to S$ be a blow up of the point P_1. Then

$$\mu_4 \tilde{L}_4 + \tilde{\Omega} \equiv \beta^*\big(\mu_4 L_4 + \Omega\big) - \big(\mu_4/2 + \epsilon\big)G,$$

where G is the exceptional curve of the birational morphism β, $\tilde{L}_4$ and $\tilde{\Omega}$ are proper transforms of the divisors L_4 and Ω on the surface $\tilde{S}$, respectively, and ϵ is a positive rational number. Then

$$0 \leqslant \Big(\mu_4\tilde{L}_4 + \tilde{\Omega}\Big)\tilde{H} = \Big(\beta^*\big(\mu_4 L_4 + \Omega\big) - \big(\mu_4/2 + \epsilon\big)G\Big) \cdot \Big(\beta^*\big(-K_S\big) - G\Big) = 3 - \mu_4 - 2\epsilon,$$

where $\tilde{H}$ is a general curve in $|-K_{\tilde{S}} - G|$. Thus, there is a point $P \in G$ such that the log pair

$$\Big(\tilde{S},\ \mu_4\tilde{L}_4 + \tilde{\Omega} + \big(\mu_4/2 + \epsilon\big)G\Big)$$

is not log canonical at P. Then $1 - \epsilon = \tilde{\Omega} \cdot \tilde{L}_4 \geqslant 0$. It follows from Lemma 2.5 that

$$2\epsilon = \tilde{\Omega} \cdot G > 2$$

in the case when $P \notin \tilde{L}_4$. Therefore, we see that $P \in \tilde{L}_4$. Then

$$1 - \epsilon = \tilde{\Omega} \cdot \tilde{L}_4 > 2 - \mu_4/2 - \epsilon$$

by Lemma 2.5. Thus, we see that $\mu_4 > 2$, which is a contradiction.

Similarly, we see that $P_2 \notin \mathrm{LCS}(S, \lambda D)$. Then $\mathrm{LCS}(S, \lambda D) = O$. We may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_3 = \bar{L}_3 \cdot E_2 = 1,\ \bar{L}_1 \cdot E_2 = \bar{L}_1 \cdot E_3 = \bar{L}_2 \cdot E_1 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_1 = \bar{L}_3 \cdot E_3 = 0.$$

It follows from the equivalences 3.22 that we may assume that either $L_3 \not\subseteq \mathrm{Supp}(D)$ or

$$L_1 \not\subseteq \mathrm{Supp}(D) \not\supseteq L_2$$

by Remark 2.1. Arguing as in the proof of Lemma 3.10, we obtain a contradiction. ∎

Lemma 3.23. *Suppose that* $\Sigma = \{\mathbb{A}_1, \mathbb{A}_1, \mathbb{A}_2\}$. *Then* $\mathrm{lct}(S) = 1/2$.

Proof. Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let us derive a contradiction.

It follows from [1] that $r = 4$.

Let $P_1 \neq P_2$ be points in Σ of type $\mathbb{A}_1$. Then we may assume that $P_1 \in L_1$ and $P_2 \in L_4$.

It follows from [1] that S contains lines L_5, L_6, L_7, L_8 such that

$$P_1 \in L_5,\ P_2 \in L_6,\ P_1 \in L_7 \ni P_2,\ O \notin L_8,\ P_1 \notin L_8 \not\ni P_2,$$

which implies that $L_8 \cap L_7 \neq \varnothing$, $L_8 \cap L_2 \neq \varnothing$, $L_8 \cap L_3 \neq \varnothing$, $L_2 \cap L_7 = \varnothing$, $L_3 \cap L_7 = \varnothing$. Then

$$\begin{aligned} & L_1 + L_4 + L_7 \sim L_2 + 2L_1 \sim L_3 + 2L_4 \sim 2L_7 + L_8 \\ & \sim L_2 + L_3 + L_8 \sim L_1 + L_3 + L_5 \sim L_4 + L_2 + L_6, \end{aligned}$$

and $-K_S \sim L_1 + L_4 + L_7$. Without loss of generality, we may assume that

$$\bar{L}_1 \cdot E_1 = \bar{L}_2 \cdot E_1 = \bar{L}_3 \cdot E_2 = \bar{L}_4 \cdot E_2 = 1,\ \bar{L}_1 \cdot E_2 = \bar{L}_2 \cdot E_2 = \bar{L}_3 \cdot E_1 = \bar{L}_4 \cdot E_1 = 0.$$

Arguing as in the proof of Lemma 3.21, we see that $\mathrm{LCS}(S, \lambda D) = O$.

By Remark 2.1, we may assume that either $L_1 \not\subseteq \mathrm{Supp}(D)$ or $L_2 \not\subseteq \mathrm{Supp}(D)$, because

$$2L_1 + L_2 \sim -K_S \sim \mathcal{O}_{\mathbb{P}^3}(1)\Big|_S$$

and the log pair $(X, \mathrm{lct}_1(S)(2L_1+L_2))$ is log canonical, where $\mathrm{lct}_1(S) = 1/2$. Similarly, we may assume that either $L_3 \not\subseteq \mathrm{Supp}(D)$ or $L_4 \not\subseteq \mathrm{Supp}(D)$, because $-K_S \sim L_3+2L_4$.

Arguing as in the proof of Lemma 3.9, we obtain a contradiction. $\square$

It follows from [1], that the equalities

$$\mathrm{lct}\big(S\big) = \mathrm{lct}_1\big(S\big) = \begin{cases} 2/3 \text{ when } \Sigma = \big\{\mathbb{A}_1\big\}, \\ 1/3 \text{ when } \Sigma \supseteq \big\{\mathbb{A}_4\big\}, \\ 1/3 \text{ when } \Sigma = \big\{\mathbb{D}_4\big\}, \\ 1/3 \text{ when } \Sigma \supseteq \big\{\mathbb{A}_2, \mathbb{A}_2\big\}, \\ 1/4 \text{ when } \Sigma \supseteq \big\{\mathbb{A}_5\big\}, \\ 1/4 \text{ when } \Sigma = \big\{\mathbb{D}_5\big\}, \\ 1/6 \text{ when } \Sigma = \big\{\mathbb{E}_6\big\}, \\ 1/2 \text{ in other cases.} \end{cases}$$

are proved for all possible values of the set Σ. Hence, the assertion of Theorem 1.4 is proved.

4. Fiberwise maps. Let us use the assumptions and notation of Theorem 1.5.

Proof of Theorem 1.5. Suppose that X is log terminal and $\mathrm{lct}(X) \geqslant 1$, but ρ is not an isomorphism. Let D be a general very ample divisor on Z. Put

$$\Lambda = \big| - nK_V + \pi^*(nD)\big|,\ \Gamma = \big| - nK_{\bar{V}} + \bar{\pi}^*(nD)\big|,\ \bar{\Lambda} = \rho(\Lambda),\ \bar{\Gamma} = \rho^{-1}(\Gamma),$$

where n is a natural number such that Λ and Γ have no base points. Put

$$M_V = \frac{2\varepsilon}{n}\,\Lambda + \frac{1-\varepsilon}{n}\,\overline{\Gamma},\ M_{\bar{V}} = \frac{2\varepsilon}{n}\,\bar{\Lambda} + \frac{1-\varepsilon}{n}\,\Gamma,$$

where ε is a positive rational number.

The log pairs (V, M_V) and $(\bar{V}, M_{\bar{V}})$ are birationally equivalent, and $K_V + M_V$ and $K_{\bar{V}} + M_{\bar{V}}$ are ample. The uniqueness of canonical model (see [3, Theorem 1.3.20]) implies that ρ is biregular if the singularities of both log pairs (V, M_V) and $(V, M_{\bar{V}})$ are canonical.

The linear system Γ does not have base points. Thus, there is a rational number ε such that the log pair $(\bar{V}, M_{\bar{V}})$ is canonical. So, the log pair (V, M_V) is not canonical. Then the log pair

$$\Big(V,\ X+\frac{1-\varepsilon}{n}\,\bar{\Gamma}\Big)$$

is not log canonical, because Λ does not have not base points, and $\bar{\Gamma}$ does not have base points outside of the fiber X, which is a Cartier divisor on the variety V. The log pair

$$\Big(X,\ \frac{1-\varepsilon}{n}\,\bar{\Gamma}\Big|_X\Big)$$

is not log canonical by Theorem 17.6 in [9], which is impossible, because $\mathrm{lct}(X) \geqslant 1$.

To conclude the proof we may assume that the varieties X and $\bar{X}$ have log terminal singularities, the inequality $\mathrm{lct}(X)+\mathrm{lct}(\bar{X}) > 1$ holds, and ρ is not an isomorphism.

Let Λ, Γ, $\bar{\Lambda}$, $\bar{\Gamma}$ and n be the same as in the previous case. Put

$$M_V=\frac{\mathrm{lct}(\bar{X})-\varepsilon}{n}\,\Lambda+\frac{\mathrm{lct}(X)-\varepsilon}{n}\,\overline{\Gamma},\ M_{\bar{V}}=\frac{\mathrm{lct}(\bar{X})-\varepsilon}{n}\,\bar{\Lambda}+\frac{\mathrm{lct}(X)-\varepsilon}{n}\,\Gamma,$$

where ε is a sufficiently small positive rational number. Then it follows from the uniqueness of canonical model that ρ is biregular if both log pair (V, M_V) and $(V, M_{\bar{V}})$ are canonical.

Without loss of generality, we may assume that the singularities of the log pair (V, M_V) are not canonical. Arguing as in the previous case, we see that the log pair

$$\Big(X,\ \frac{\mathrm{lct}(X)-\varepsilon}{n}\,\bar{\Gamma}\Big|_X\Big)$$

is not log canonical, which is impossible, because $\bar{\Gamma}|_X \equiv -nK_X$. ◻

The assertion of Theorem 1.5 is a generalization of the Main Theorem in [10].

REFERENCES

[1] J.W. Bruce and C.T.C. Wall, *On the classification of cubic surfaces,* Journal of the London Mathematical Society, 19 (1979), pp. 245–256.

[2] A. del Centina and S. Recillas, *Some projective geometry associated with unramified double covers of curves of genus* 4*,* Annali di Matematica Pura ed Applicata, 133 (1983), pp. 125–140.

[3] I. Cheltsov, *Birationally rigid Fano varieties*, Russian Mathematical Surveys, 60 (2005), pp. 875–965

[4] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geometric and Functional Analysis, 18 (2008), pp. 1118–1144.

[5] I. Cheltsov and C. Shramov, *Log canonical thresholds of smooth Fano threefolds. With an appendix by Jean-Pierre Demailly*, Russian Mathematical Surveys, 63 (2008), pp. 73–180.

[6] J.-P. Demailly and J. Kollár, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Annales Scientifiques de l'École Normale Supérieure, 34 (2001), pp. 525–556.

[7] W. Ding and G. Tian, *Kähler–Einstein metrics and the generalized Futaki invariant*, Inventiones Mathematicae, 110 (1992), pp. 315–335.

[8] J. Kollár, *Singularities of pairs*, Proceedings of Symposia in Pure Mathematics, 62 (1997), pp. 221–287.

[9] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Astérisque, 211 (1992).

[10] J. Park, *Birational maps of del Pezzo fibrations*, Journal fur die Reine und Angewandte Mathematik, 538 (2001), pp. 213–221.

[11] G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with* $c_1(M) > 0$, Inventiones Mathematicae, 89 (1987), pp. 225–246.
[12] J. Park and J. Won, *Log canonical thresholds on Gorenstein canonical del Pezzo surfaces*, arXiv:0904.4513.