African Diaspora Journal of Mathematics
Volume 17, Number 1, pp. 93–105 (2014)
ISSN 1539-854X
www.math-res-pub.org/adjm

CR-Submanifolds of a Nearly Trans-Hyperbolic Sasakian Manifold with a Semi-Symmetric Non-Metric Connection

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Abstract. In this paper, *CR*-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection are studied. The parallel distributions relating to ξ-vertical and ξ-horizontal *CR*-submanifolds of nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection are obtained. Moreover, Nijenhuis tensor is calculated and integrability conditions of the distributions on *CR*-submanifolds of nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection are discussed.

AMS Subject Classification: 53C40, 53C05.

Keywords: *CR*-submanifolds, nearly trans-hyperbolic Sasakian manifolds, semi-symmetric non-metric connection.

Received: November 5, 2013 || Accepted: June 12, 2014

1 Introduction

In 1978, Aurel Bejancu initiated the study of *CR*-submanifolds of Kaehler manifold [5]. Later on, many geometers (see [13], [21], [14]) studied *CR*-submanifolds of different ambient spaces.

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In 1985, Oubina [15] introduced a new class of almost contact metric manifold known as trans-Sasakian manifold. M. H. Shahid [18] and Al-Solamy [4] studied the geometry of CR-submanifolds of trans-Sasakian and nearly trans-Sasakian manifold [12] respectively. In 1976, Upadhyay and Dube [20] introduced the notion of almost contact hyperbolic (f, g, η, ξ)-structure. Some properties of CR-submanifolds of trans-hyperbolic Sasakian manifold were studied in [7] and [11]. CR-submanifolds of nearly trans-hyperbolic Sasakian manifold is a more general concept. In 2010, Cihan Özgür [16] studied the submanifolds of Riemannian manifold with semi-symmetric non-metric connection. Moreover, Özgür et al. also studied the different structures with semi-symmetric non-metric connection in [17] and [2]. Some properties of semi-invariant submanifolds, hypersurfaces and submanifolds with semi-symmetric non-metric connection were studied in [1], [2] and [9]. Thus motivated sufficiently from the studies referred above in the present paper, we plan to study the CR-submanifolds of nearly trans-hyperbolic Sasakian manifolds with a semi-symmetric non-metric connection.

We know that a linear connection ∇ on a manifold M is called metric connection if $\nabla g = 0$, otherwise, it is non-metric. Further it is said to be a semi-symmetric linear connection [10] if its torsion tensor $T(X, Y)$, is

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. Further, the study of semi-symmetric connection on Riemannian manifold was enriched by K. Yano [22], Agashe and Chaffle [3]. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

Semi-symmetric metric connection plays an important role in the study of Riemannaian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the Earth always facing one definite point, say Mekka or Jaruselam or the North pole, then this displacement is semi-symmetric and metric [19].

This paper is organized as follows. In section 2, we give a brief introduction of nearly trans-hyperbolic Sasakian manifold. In section 3, we have prove some basic lemmas on nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection. In section 4, we discuss parallel distributions and in section 5, we obtain the integrability conditions of distributions on CR-submanifolds.

2 Preliminaries

Let $\bar{M}$ be an almost hyperbolic contact metric manifold [8] with an almost hyperbolic contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 = I - \eta \otimes \xi, \ \phi\xi = 0, \ \eta o \phi = 0, \ \eta(\xi) = -1 \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X). \tag{2.3}$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called trans-hyperbolic Sasakian [7] if and only if

$$(\bar{\bar{\nabla}}_X \phi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.4}$$

for all X, Y tangent to $\bar{M}$, α and β are smooth functions on $\bar{M}$. On a trans-hyperbolic Sasakian manifold $\bar{M}$, we have

$$\bar{\bar{\nabla}}_X\xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \tag{2.5}$$

where g is the Riemannian metric and $\bar{\bar{\nabla}}$ is the Riemannian connection. Further, an almost hyperbolic contact metric manifold $\bar{M}$ on structure (ϕ, ξ, η, g), is called nearly trans-hyperbolic Sasakian if [14]

$$(\bar{\bar{\nabla}}_X\phi)(Y) + (\bar{\bar{\nabla}}_Y\phi)(X) = \alpha(2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y). \tag{2.6}$$

Let M be an m-dimensional isometrically immersed submanifold of nearly-hyperbolic Sasakian manifold $\bar{M}$. We denote by g the Riemannian metric tensor field on M as well as on $\bar{M}$.

Definition 2.1. [7] An m-dimensional Riemannian submanifold M of an almost trans-hyperbolic Sasakian manifold $\bar{M}$ is called a CR-submanifold if ξ is tangent to M and there exists differentiable distribution $D : x \in M \to D_x \subset T_x(M)$ such that

(i) the distribution D_x in invariant under ϕ, that is $\phi D_x \subset D_x$, for each $x \in M$,

(ii) the complementary orthogonal distribution $D^\perp : x \to D_x^\perp \subset T_x(M)$ of the distribution D on M is anti-invariant under ϕ, that is $\phi D_x^\perp(M) \subset T_x^\perp(M)$ for all $x \in M$, where $T_x(M)$ and $T_x^\perp(M)$ are tangent space and normal space of M at $x \in M$ respectively.

If $dimD^\perp = 0$ (resp. $dimD_x = 0$), then CR-submanifold is called an invariant (resp. anti-invariant). The distribution D (resp. $D^\perp$) is called horizontal (resp. vertical) distribution. The pair $(D, D^\perp)$ is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D^\perp$) for $x \in M$.

For any vector field X tangent to M, we write

$$X = PX + QX, \tag{2.7}$$

where PX and QX belong to the distributions D and $D^\perp$ respectively.

For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{2.8}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN. Now, we remark that owing to the existence of the 1-form η, we can define a semi-symmetric non-metric connection $\bar{\nabla}$ in almost hyperbolic contact metric manifold by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)X, \tag{2.9}$$

where $\bar{\bar{\nabla}}$ is the Riemannian connection with respect to g on n-dimensional Riemannian manifold and η is a 1-form associated with the vector field ξ on M defined by

$$\eta(X) = g(X, \xi). \tag{2.10}$$

The torsion tensor T of the connection $\bar{\nabla}$ is given by [3]

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]. \tag{2.11}$$

Also, we have

$$T(X,Y) = \eta(Y)X - \eta(X)Y. \tag{2.12}$$

A linear connection $\bar{\nabla}$, satisfying (2.12) is called a semi-symmetric connection. $\bar{\nabla}$ is called a metric connection if $\bar{\nabla}g = 0$, otherwise, it is said to be non-metric connection. Furthermore, from (2.9), it is easy to see that

$$\bar{\nabla}_X g(Y,Z) = (\bar{\nabla}_X g)(Y,Z) + g(\bar{\nabla}_X Y, Z) + g(Y, \bar{\nabla}_X Z)$$
$$= (\bar{\nabla}_X g)(Y,Z) + \bar{\nabla}_X g(Y,Z) + \eta(Y)g(X,Z) + \eta(Z)g(X,Y)$$

which implies

$$(\bar{\nabla}_X g)(Y,Z) = \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \tag{2.13}$$

for all vector fields X, Y, Z, on M. Therefore, in view of (2.12) and (2.13), $\bar{\nabla}$ is a semi-symmetric non-metric connection. Using (2.4) and (2.9), we get

$$(\bar{\nabla}_X\phi)(Y) = \alpha(g(X,Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) - \eta(Y)\phi X. \tag{2.14}$$

Similarly, we have

$$(\bar{\nabla}_Y\phi)(X) = \alpha(g(Y,X)\xi - \eta(X)\phi Y) + \beta(g(\phi Y, X)\xi - \eta(X)\phi Y) - \eta(X)\phi Y. \tag{2.15}$$

On adding the two equations above, we obtain

$$(\bar{\nabla}_X\phi)(Y) + (\bar{\nabla}_Y\phi)(X) = \alpha(2g(X,Y)\xi - (\alpha + \beta + 1)(\eta(Y)\phi X + \eta(X)\phi Y) \tag{2.16}$$

This is the condition for an almost hyperbolic contact structure (ϕ, ξ, η, g) with a semi-symmetric non-metric connection to be nearly trans-hyperbolic Sasakian manifold. From (2.9) and (2.5), we get

$$\bar{\nabla}_X\xi = -\alpha(\phi X) - \beta(\eta(X)\xi) + (\beta - 1)X. \tag{2.17}$$

Let $\bar{\nabla}$ be the semi-symmetric non-metric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to the unit normal N. Now we have the following theorem:

Theorem 2.2. *The connection induced on CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.*

Proof. Let $\bar{\nabla}$ be the induced connection with respect to the unit normal N on a CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with semi-symmetric non-metric connection $\bar{\nabla}$. Then

$$\bar{\nabla}_X Y = \nabla_X Y + m(X,Y), \tag{2.18}$$

where m is a tensor field of type (0, 2) on the CR-submanifold M. Let ∇^* be the induced connection on CR-submanifolds from Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \nabla^*_X Y + h(X,Y), \tag{2.19}$$

where h is a second fundamental tensor. By the definition of the semi-symmetric non-metric connection, we have

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)X.$$

Using (2.18) and (2.19), we get

$$\nabla_X Y + m(X, Y) = \nabla^*_X Y + h(X, Y) + \eta(Y)X.$$

Equating the tangential and normal components from both sides of the above equation, we obtain

$$m(X, Y) = h(X, Y)$$

and consequently, we have

$$\nabla_X Y = \nabla^*_X Y + \eta(Y)X.$$

Thus ∇ is also a semi-symmetric non-metric connection. □

Now, the Gauss formula for a *CR*-submanifold of a nearly trans-hyperbolic Sasakian manifold with semi-symmetric non-metric connection is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.20}$$

and the Weingarten formula for M is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.21}$$

for any $X, Y \in TM$, $N \in T^{\perp}M$ and h (resp. A_N) is the second fundamental form (resp. tensor) of M in $\bar{M}$ and $\nabla^{\perp}$ denotes the normal connection. Moreover, we have [6]

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.22}$$

3 Some Basic Lemmas

First we prove the following lemmas.

Lemma 3.1. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QX} Y) - P(A_{\phi QY} X) = \phi P \nabla_X Y + \phi P \nabla_Y \tag{3.1}$$

$$+2\alpha g(X, Y)P\xi - (\alpha + \beta + 1)\eta(X)\phi Y - (\alpha + \beta + 1)\eta(Y)\phi PX,$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QX} Y) - Q(A_{\phi QY} X) = 2Bh(X, Y) + 2\alpha g(X, Y)Q\xi, \tag{3.2}$$

$$h(X, \phi PY) + h(Y, \phi PX) + \nabla^{\perp}_X \phi QY + \nabla^{\perp}_{\phi} QX = \phi Q \nabla_Y X + \phi Q \nabla_X Y \tag{3.3}$$

$$+2Ch(X, Y) - (\alpha + \beta + 1)\eta(X)\phi QY - (\alpha + \beta + 1)\eta(Y)\phi QX$$

for any $X, Y \in TM$.

Proof. By direct covariant differentiation, we have

$$\bar{\nabla}_X \phi Y = (\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y).$$

By the virtue of (2.7), (2.10),(2.20) and (2.21), we get

$$\nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY} X + \nabla^{\perp}_X \phi QY - \phi(\nabla_X Y + h(X, Y))$$

$$+\nabla_Y \phi PX + h(Y, \phi PX) - A_{\phi QX} Y - \nabla^{\perp}_Y \phi QX - \phi(\nabla_Y X + h(X, Y))$$

$$= \alpha(2g(X,Y)\xi - (\alpha+\beta+1)(\eta(X)\phi Y + \eta(Y)\phi X).$$

Again using (2.7), we get

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_\phi QXY) - P(A_\phi QYX) \tag{3.4}$$

$$\phi P\nabla_X Y - \phi P\nabla_Y X + Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_\phi QXY)$$

$$-Q(A_\phi QYX) - 2Bh(X,Y) + h(X,\phi PY) + h(Y,\phi PX) + \nabla^\perp_X \phi QY + \nabla^\perp_\phi QX$$

$$-\phi Q\nabla_Y X - \phi\nabla_X Y - 2Ch(X,Y) = 2\alpha((g(X,Y))P\xi + 2\alpha((g(X,Y))Q\xi$$

$$(\alpha+\beta+1)\eta(X)\phi QY - (\alpha+\beta+1)\eta(Y)\phi QX - (\alpha+\beta+1)\eta(X)\phi PY - (\alpha+\beta+1)\eta(Y)\phi PX$$

for $X, Y \in TM$.
Now equating horizontal, vertical and normal components in (3.4), we get the desired results. □

Lemma 3.2. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \tag{3.5}$$

$$+2\alpha(g(X,Y)\xi) - (\alpha+\beta+1)(\eta(Y)\phi X + \eta(X)\phi Y)$$

for $X, Y \in D$.

Proof. From Gauss formula (2.20), we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X - h(Y,\phi X). \tag{3.6}$$

Also, we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X,Y]. \tag{3.7}$$

From (3.6) and (3.7), we get

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X \tag{3.8}$$

$$-h(Y,\phi X) - \phi[X,Y].$$

Also for nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection, we have

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi X) = 2\alpha(g(X,Y)\xi) \tag{3.9}$$

$$-(\alpha+\beta+1)(\eta(X)\phi Y + \eta(Y)\phi X).$$

Adding (3.8) and (3.9), we get

$$2(\bar{\nabla}_X \phi)Y = \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]$$

$$+2\alpha(g(X,Y)\xi) - (\alpha+\beta+1)(\eta(Y)\phi X + \eta(X)\phi Y).$$

Subtracting (3.8) from (3.9), we get

$$2(\bar{\nabla}_Y \phi)X = 2\alpha(g(X,Y)\xi) - (\alpha+\beta+1)(\eta(Y)\phi X + \eta(X)\phi Y) \tag{3.10}$$

$$-\nabla_X \phi Y + \nabla_Y \phi X - h(X,\phi Y) + h(Y,\phi X) + \phi[X,Y].$$

Hence lemma is proved. □

Lemma 3.3. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

$$2(\bar{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_\phi ZY + \nabla^\perp_Y\phi Z - \nabla^\perp_Y\phi Y - \phi[X,Y] \tag{3.11}$$

$$+2\alpha(g(Y,Z)\xi) - (\alpha+\beta+1)(\eta(Y)\phi Z + \eta(Z)\phi Y)$$

for any $X, Y \in D^\perp$.

Proof. From Weingarten fromula (2.21), we have

$$\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = A_\phi YZ - A_\phi ZY + \nabla^\perp_Y\phi Z - \nabla^\perp_Z\phi Y. \tag{3.12}$$

Also we have

$$\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = (\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y + \phi[Y,Z]. \tag{3.13}$$

From (3.11) and (3.12), we get

$$(\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y = A_\phi YZ - A_\phi ZY + \nabla^\perp_Y\phi Z - \nabla^\perp_Z\phi Y - \phi[Y,Z]. \tag{3.14}$$

Also for nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection, we have

$$(\bar{\nabla}_Y\phi)Z + (\bar{\nabla}_Z\phi)Y = 2\alpha(g(Y,Z)\xi) - (\alpha+\beta+1)(\eta(Y)\phi Z + \eta(Z)\phi Y). \tag{3.15}$$

Adding (3.13) and (3.14), we get

$$2(\bar{\nabla}_Y\phi)Z = A_\phi YZ - A_\phi ZY + \nabla^\perp_Y\phi Z - \nabla^\perp_Z\phi Y - \phi[Y,Z]$$

$$+2\alpha(g(Y,Z)\xi) - (\alpha+\beta+1)(\eta(Y)\phi Z + \eta(Z)\phi Y).$$

Subtracting (3.13) and (3.14), we obtain

$$2(\bar{\nabla}_Z\phi)Y = -A_\phi YZ - A_\phi ZY - \nabla^\perp_Y\phi Z + \nabla^\perp_Z\phi Y + \phi[Y,Z]$$

$$+2\alpha(g(Y,Z)\xi) - (\alpha+\beta+1)(\eta(Y)\phi Z + \eta(Z)\phi Y).$$

This proves our assertion. □

Lemma 3.4. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y = -A_\phi YX + \nabla^\perp_X\phi Y + \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

$$+2\alpha(g(X,Y)\xi) - (\alpha+\beta+1)(\eta(X)\phi Y + \eta(Y)\phi X),$$

$$2(\bar{\nabla}_Y\phi)X = A_\phi YX - \nabla^\perp_X\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y]$$

$$+2\alpha(g(X,Y)\xi) - (\alpha+\beta+1)(\eta(X)\phi Y + \eta(Y)\phi X)$$

for any $X \in D$ and $Y \in D^\perp$.

Proof. Using Gauss and Weingarten eqautions for $X \in D$ and $Y \in D^{\perp}$ respectively, we get

$$\nabla_X \phi Y - \bar{\nabla}_Y \phi X = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X). \tag{3.16}$$

Also, we have

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi) Y - (\bar{\nabla}_Y \phi) X + \phi[X, Y]. \tag{3.17}$$

From (3.15) and (3.16), we get

$$(\bar{\nabla}_X \phi) Y - (\bar{\nabla}_Y \phi) X = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X \tag{3.18}$$

$$-h(Y, \phi X) - \phi[X, Y].$$

Moreover, for nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection, we have

$$(\bar{\nabla}_X \phi) Y + (\bar{\nabla}_Y \phi) X = 2\alpha(g(X, Y)\xi) - (\alpha + \beta + 1)(\eta(Y)\phi X + \eta(X)\phi Y). \tag{3.19}$$

Adding (3.17) and (3.18), we obtain

$$2(\bar{\nabla}_X \phi) Y = 2\alpha(g(X, Y)\xi) - (\alpha + \beta + 1)(\eta(X)\phi Y + \eta(Y)\phi X) \tag{3.20}$$

$$-A_{\phi} Y X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y].$$

Subtracting (3.16) and (3.18), we get

$$2(\bar{\nabla}_Y \phi) X = 2\alpha(g(X, Y)\xi) - (\alpha + \beta + 1)(\eta(X)\phi Y + \eta(Y)\phi X) \tag{3.21}$$

$$+A_{\phi} Y X - \nabla^{\perp}_X \phi Y + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y].$$

Hence Lemma is proved. □

4 Parallel Distributions

Definition 4.1. [7] The horizontal (resp. vertical) distribution D (resp. $D^{\perp}$) is said to be parallel with respect to the semi-symmetric non-metric connection on M, if $\nabla_X Y \in D$ ($\nabla_Z W \in D^{\perp}$) for any vector field $X, Y \in D$ (resp. $W, Z \in D^{\perp}$).

Proposition 4.2. *[14] Let M be a ξ-vertical CR-submanifold of neraly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. If the horizontal distribution D is parallel, then*

$$h(X, \phi Y) = h(Y, \phi X) \tag{4.1}$$

for all $X,\ Y \in D$.

Proof. From equation (2.9) and using the parallelism of horizontal distribution D, as

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y) X, \tag{4.2}$$

$\bar{\nabla}_X Y \in D$ if and only if $\bar{\bar{\nabla}}_X Y \in D$, and from [14] Proposition 1, this happens if and only if $h(X, \phi Y) = h(Y, \phi X)$ for any $X, Y \in D$. □

Proposition 4.3. *[14] Let M be a ξ-vertical CR-submanifold of nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection ∇ on M, then*

$$A_{\phi}YZ + A_{\phi}ZY \in D^{\perp}, \tag{4.3}$$

for any $Y, Z \in D^{\perp}$.

Proof. From equation (2.9) and using the parallelism of horizontal distribution $D^{\perp}$, as

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)X,$$

$\bar{\nabla}_X Y \in D^{\perp}$, if and only if $\bar{\bar{\nabla}}_X Y \in D^{\perp}$, that is $D^{\perp}$ is parallel for $\bar{\nabla}$ and $\bar{\bar{\nabla}}$, is also hold from [14] Proposition 1, then $A_{\phi}YZ + A_{\phi}ZY \in D^{\perp}$, for any $Y,\ Z \in D^{\perp}$.

□

Definition 4.4. [7] A *CR*-submanifold of a manifold with a semi-symmetric non-metric connection is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^{\perp}$.

The following lemma is an easy consequence of (2.21).

Lemma 4.5. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.*

Definition 4.6. A normal vector field $N \neq 0$ with semi-symmetric non-metric connection is called D-parallel if normal section $\nabla^{\perp}_X N = 0$ for all $X \in D$.

Now, we have the following proposition.

Proposition 4.7. *Let M be a mixed totally geodesic ξ-vertical CR-submanifold of nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then the normal section $N \in \phi D^{\perp}$ is D-parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.*

Proof. The proof of this proposition can be directly deduced from [14] Proposition 3. □

5 Integrability Conditions of Distributions

In this part of the paper, we calculate the Nijenhuis tensor $N(X, Y)$ on nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. For this, first we need the following lemmas.

Lemma 5.1. *In an almost contact metric manifold, we have*

$$(\bar{\nabla}_Y \phi)\phi X = -\phi(\bar{\nabla}_Y \phi)X + ((\bar{\nabla}_Y \eta)X)\xi + \eta(X)\bar{\nabla}_Y \xi. \tag{5.1}$$

Proof. For $X,\ Y \in T\bar{M}$, we have

$$\begin{aligned}(\bar{\nabla}_Y \phi)\phi X &= \bar{\nabla}_Y(\phi^2 X) - \phi(\bar{\nabla}_Y \phi X) + \phi(\phi \bar{\nabla}_Y X) - \phi^2 \bar{\nabla}_Y X \\ &= \bar{\nabla}_Y(-X + \eta(X)\xi) - \phi(\bar{\nabla}_Y \phi X) \\ &\quad +\phi(\phi \bar{\nabla}_Y X)(-\bar{\nabla}_Y X + \eta(\bar{\nabla}_Y X)\xi),\end{aligned}$$

which gives the equation (5.1).

Lemma 5.2. *Let $\bar{M}$ be a nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection, then*

$$(\bar{\nabla}_{\phi X}\phi)Y = 2\alpha(g(\phi X, Y)\xi) - (\alpha+\beta+1)\eta(Y)X \tag{5.2}$$

$$+(\alpha+\beta+1)\eta(X)\eta(Y)\xi - \eta(Y)\bar{\nabla}_Y\xi + \phi(\bar{\nabla}_Y\phi)X + \eta(\bar{\nabla}_Y X)\xi$$

for any $X, Y \in T\bar{M}$.

Proof. From the definition of nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection, we have

$$(\bar{\nabla}_X\phi)Y = 2\alpha(g(X, Y)\xi) - (\alpha+\beta+1)(\eta(Y)\phi X + \eta X\phi Y).$$

Replacing X by ϕX, in above equation, we get

$$(\bar{\nabla}_\phi X\phi)Y = 2\alpha(g(\phi X, Y)\xi) + (\alpha+\beta+1)\eta(Y)X \tag{5.3}$$

$$+(\alpha+\beta+1)\eta(Y)\eta(X)\xi - (\bar{\nabla}_Y\phi)\phi X.$$

Using Lemma (5.1) and (5.3), we obtain

$$(\bar{\nabla}_\phi X\phi)Y = 2\alpha(g(\phi X, Y)\xi) + (\alpha+\beta+1)\eta(Y)X + (\alpha+\beta+1)\eta(Y)\eta(X)\xi$$

$$+\phi(\bar{\nabla}_Y\phi)X + ((\bar{\nabla}_Y\eta)X)\xi - \eta(X)\bar{\nabla}_Y\xi$$

for all $X, Y \in T\bar{M}$. □

On a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ the Nijenhuis tensor is

$$N(X, Y) = (\bar{\nabla}_\phi X\phi)Y - (\bar{\nabla}_\phi Y\phi)X - \phi(\bar{\nabla}_X\phi)Y + \phi(\bar{\nabla}_Y\phi)X \tag{5.4}$$

for all $X, Y \in T\bar{M}$. From (5.2) and (5.4), we get

$$N(X, Y) = 4\alpha(g(\phi X, Y)\xi) + (\alpha+\beta+1)(\eta(X)Y - \eta(Y)X) - \eta(X)\bar{\nabla}_Y\xi \tag{5.5}$$

$$+2(\alpha+\beta+1)\eta(X)\eta(Y)\xi + \eta(\phi(\bar{\nabla}_Y\phi)X + \eta(\bar{\nabla}_Y X)\xi.$$

Thus using (2.15) in (5.5), we find that the Nijenhuis tensor of nearly trans-hyperbolic Sasakian manifold with a semi-symmetric non-metric connection which is given by

$$N(X, Y) = 4\alpha(g(\phi X, Y)\xi) - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi - \eta[X, Y]\xi \tag{5.6}$$

$$+(\alpha+\beta+1)\left[3\eta(X)Y + \eta(Y)X - 2\eta(X)\eta(Y)\xi\right] + 4\phi(\bar{\nabla}_Y\phi)X$$

for all $X, Y \in T\bar{M}$. Now, we prove the following theorem.

Theorem 5.3. *Let M be a ξ-vertical CR-submanifold of nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then the distribution D is integrable if the following conditions are satisfied:*

$$S(X, Z) \in D, \quad h(X, \phi Z) = h(\phi X, Z) \tag{5.7}$$

for any $X,\ Z \in D$.

Proof. The torsion tensor $S(X, Y)$ of the almost contact metric structure (ϕ, ξ, η, g) is given by

$$S(X,Y) = N(X,Y) + 2d\eta(X,Y)\xi = N(X,Y) + 2g(\phi X, Y)\xi. \tag{5.8}$$

Thus, we have

$$S(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2g(\phi X, Y)\xi \tag{5.9}$$

for $X,\ Y \in TM$.
The distribution D is integrable if and only if for all $X,\ Y \in D$ and $\eta([X,Y]) = 0$ as $\xi \in D^{\perp}$.
If $S(X, Y) \in D$, then from (5.6) and (5.8) we have

$$2(\alpha + 1)g(\phi X, Y) + \eta([X,Y])\xi \tag{5.10}$$

$$+4(\phi Q\nabla_Y \phi X + \phi h(Y, \phi X) + Q\nabla_Y X + h(X,Y)) \in D.$$

or

$$2(\alpha + 1)g(\phi X, Y) + \eta([X,Y])\xi \tag{5.11}$$

$$+4(\phi Q\nabla_Y \phi X + \phi h(Y, phiX) + Q\nabla_Y X + h(X,Y)) = 0$$

for $X,\ Y \in D$ and $\xi \in D^{\perp}$.
Replacing Y by ϕZ for $Z \in D$ in the above equation, we get

$$2(\alpha + 1)g(\phi X, \phi Z)Q\xi \tag{5.12}$$

$$+4(\phi Q\nabla_{\phi} Y\phi X + \phi h(\phi Z, \phi X) + Q\nabla_{\phi} ZX + h(X, \phi Z) = 0.$$

Interchanging X and Z in (5.12) and subtracting these relations, we obtain

$$\phi Q[\phi X, \phi Z] + Q[X, \phi Z] + h(X, \phi Z) - h(Z, \phi X) = 0. \tag{5.13}$$

Consequently, from (5.13), we get

$$h(X, \phi Z) = h(Z, \phi X)$$

for any $X,\ Y \in D$. □

Theorem 5.4. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

$$A_{\phi}YZ - A_{\phi}ZY = \frac{1}{3}\phi P[Y,Z] + 2\alpha(\eta(Y)Z - \eta(Z)Y) + (\alpha + \beta + 1)(\eta(Y)\phi Z - \eta(Z)\phi Y)$$

for any $Y,\ Z \in D^{\perp}$.

Proof. For $Y,\ Z \in D^{\perp}$ and $X \in T(M)$, we get

$$2g(A_{\phi}ZY, X) = 2g(h(X,Y), \phi Z) = g(h(X,Y), \phi Z) + g(h(X,Y), \phi Z)$$

$$= g(\bar{\nabla}_X Y + \bar{\nabla}_Y X, \phi Z) = -g(\phi(\bar{\nabla}_X Y + \bar{\nabla}_Y X), Z)$$

$$= -g\left[(\bar{\nabla}_Y \phi X + \bar{\nabla}_Y \phi X, Z) - 2(\alpha g(X,Y)\xi - (\alpha + \beta + 1)(\eta(X)\phi Y) - \eta(Y)\phi X), Z\right]$$

$$= -g(\bar{\nabla}_Y \phi X, Z) - g(\bar{\nabla}_X \phi Y, Z) + 2\alpha\eta(Z)g(X,Y)$$

$$-(\alpha + \beta + 1)\left[g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X)\right].$$

$$2g(A_{\phi Z}Y,X) = g(\bar{\nabla}_Y Z,\phi X) + g(A_\phi YZ,X) + 2\alpha\eta(Z)g(X,Y)$$
$$-(\alpha+\beta+1)g(\phi X,Z)\eta(Y) - (\alpha+\beta+1)g(\phi Y,Z)\eta(X).$$

The above equation is true for all $X \in T(M)$, therefore, transvecting the vector field X both sides, we obtain

$$2A_\phi ZY = A_\phi YZ - \phi\bar{\nabla}_Y Z + 2\alpha\eta(Z)Y + (\alpha+\beta+1)\eta(Y)\phi Z - (\alpha+\beta+1)g(\phi Y,Z)\xi$$

or

$$2A_\phi ZY = A_\phi YZ - \phi\bar{\nabla}_Y Z + 2\alpha\eta(Z)Y + (\alpha+\beta+1)\eta(Y)\phi Z \tag{5.14}$$

for any $Y,\ Z \in D^\perp$. Interchanging the vector fields Y and Z, we get

$$2A_\phi YZ = A_\phi ZY - \phi\bar{\nabla}_Z Y + 2\alpha\eta(Y)Z + (\alpha+\beta+1)\eta(Z)\phi Y. \tag{5.15}$$

Subtracting (5.14) and (5.15), we find

$$A_\phi YZ - A_\phi ZY = \frac{1}{3}\phi P[Y,Z] + 2\alpha(\eta(Y)Z - \eta(Z)Y) \tag{5.16}$$
$$+(\alpha+\beta+1)(\eta(Z)\phi Y - \eta(Y)\phi Z)$$

for any $Y,\ Z \in D^\perp$. □

Theorem 5.5. *Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then the distribution $D^\perp$ is integrable if and only if*

$$A_\phi YZ - A_\phi ZY = 2\alpha(\eta(Y)Z - \eta(Z)Y) + (\alpha+\beta+1)(\eta(Z)\phi Y - \eta(Y)\phi Z). \tag{5.17}$$

for any Y, $Z \in D^\perp$.

Proof. From (5.16), the proof of the theorem is obvious. □

Corollary 5.6. *Let M be a horizontal CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then the distribution $D^\perp$ is integrable if and only if*

$$A_\phi YZ - A_{\phi Z}Y = 0 \tag{5.18}$$

for any $Y,\ Z \in D^\perp$.

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