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Berry-Esseen's Central Limit Theorem for Non-causal Linear Processes in Hilbert Space

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Abstract

Let H be a real separable Hilbert space and $(a_k)_{k\in\mathbb{Z}}$ a sequence of bounded linear operators from H to H. We consider the linear process X defined for any k in $\mathbb{Z}$ by $X_k = \sum_{j\in\mathbb{Z}} a_j(\varepsilon_{k-j})$ where $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a sequence of i.i.d. centered H-valued random variables. We investigate the rate of convergence in the CLT for X and in particular we obtain the usual Berry-Esseen's bound provided that $\sum_{j\in\mathbb{Z}} |j|\|a_j\|_{\mathcal{L}(H)} < +\infty$ and ε_0 belongs to L_H^∞.

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1 Introduction and notations

Let $(H, \|.\|_H)$ be a separable real Hilbert space and $(\mathcal{L}, \|.\|_{\mathcal{L}(H)})$ be the class of bounded linear operators from H to H with its usual uniform norm. Consider a sequence $(\varepsilon_k)_{k\in\mathbb{Z}}$ of i.i.d. centered random variables, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with values in H. If $(a_k)_{k\in\mathbb{Z}}$ is a sequence in $\mathcal{L}$, we define the (non-causal) linear process $X = (X_k)_{k\in\mathbb{Z}}$ in H by

$$X_k = \sum_{j\in\mathbb{Z}} a_j(\varepsilon_{k-j}), \qquad k \in \mathbb{Z}. \tag{1.1}$$

If $\sum_{j\in\mathbb{Z}} \|a_j\|_{\mathcal{L}(H)} < \infty$ and $E\|\varepsilon_0\|_H < +\infty$ then the series in (1.1) converges almost surely and in $L_H^1(\Omega, \mathcal{A}, \mathbb{P})$ (see Bosq [2]). The condition $\sum_{j\in\mathbb{Z}} \|a_j\|_{\mathcal{L}(H)} < \infty$ is know to be sharp for the $\sqrt{n}$-normalized partial sums of X to satisfies a CLT provided that $(\varepsilon_k)_{k\in\mathbb{Z}}$ are i.i.d. centered having finite second moments (see Merlevede et al. [6]). In this work, we investigate the rate of convergence in the CLT for X under the condition

$$\sum_{j\in\mathbb{Z}} |j|^\tau \|a_j\|_{\mathcal{L}(H)} < \infty \tag{1.2}$$

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with $\tau = 1$ when $(\varepsilon_k)_{k\in\mathbb{Z}}$ are assumed to be i.i.d. centered and such that ε_0 belongs to L^∞_H and $\tau = 1/2$ when $(\varepsilon_k)_{k\in\mathbb{Z}}$ are i.i.d. centered and such that ε_0 belongs to some Orlicz space $L_{H,\psi}$ (see section 2). This problem was previously studied (with $\tau = 1$ in Condition (1.2)) by Bosq [3] for (causal) Hilbert linear processes but a mistake in his proof was pointed out by V. Paulauskas [7]. However, in the particular case of Hilbertian autoregressive processes of order 1, Bosq [4] obtained the usual Berry-Esseen inequality provided that $(\varepsilon_k)_{k\in\mathbb{Z}}$ are i.i.d. centered with ε_0 in L^∞_H.

2 Main result

In the sequel, C_{ε_0} is the autocovariance operator of ε_0, $A := \sum_{j\in\mathbb{Z}} a_j$ and A^* is the adjoint of A. For any sequence $Z = (Z_k)_{k\in\mathbb{Z}}$ of random variables with values in H we denote

$$\Delta_n(Z) = \sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\left\| \frac{1}{\sqrt{n}} \sum_{k=1}^n Z_k \right\|_H \leq t \right) - \mathbb{P}\left(\|N\|_H \leq t \right) \right|$$

where $N \sim \mathcal{N}(0, AC_{\varepsilon_0}A^*)$.

For any $j \in \mathbb{Z}$, denote $c_{j,n} = \sum_{i=1}^n b_{i-j}$ where $b_i = a_i$ for any $i \neq 0$ and $b_0 = a_0 - A$.

Lemma 2.1. *For any positive integer n,*

$$\sum_{k=1}^n X_k = A\left(\sum_{k=1}^n \varepsilon_k\right) + Q_n + R_n$$

where $Q_n = \sum_{k=1}^n \sum_{|j|>n} a_{k-j}(\varepsilon_j)$ *and* $R_n = \sum_{|j|\leq n} c_{j,n}(\varepsilon_j)$.

Recall that a Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to+\infty}\psi(t) = +\infty$ and $\psi(0) = 0$. We define the Orlicz space $L_{H,\psi}$ as the space of H-valued random variables Z defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $E[\psi(\|Z\|_H/c)] < +\infty$ for some $c > 0$. The Orlicz space $L_{H,\psi}$ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any H-valued random variable Z by

$$\|Z\|_\psi = \inf\{\, c > 0\,;\, E[\psi(\|Z\|_H/c)] \leq 1 \,\}$$

is a Banach space. In the sequel, $c(N)$ denotes a bound of the density of $\mathcal{N}(0, AC_{\varepsilon_0}A^*)$ (see Davydov et al. [5]). Our main result is the following.

Theorem 2.2. *Let* $(\varepsilon_k)_{k\in\mathbb{Z}}$ *be a sequence of i.i.d. centered H-valued random variables and let X be the Hilbertian linear process defined by* (1.1).

i) If ε_0 *belongs to* L^∞_H *and* $\sum_{j\in\mathbb{Z}} |j| \|a_j\|_{\mathcal{L}(H)} < \infty$ *then*

$$\Delta_n(X) \leq \frac{c_1}{\sqrt{n}} \tag{2.1}$$

where $c_1 = c_2 + 14c(N)\|\varepsilon_0\|_\infty \sum_{j\in\mathbb{Z}} |j| \|a_j\|_{\mathcal{L}(H)}$ *and* c_2 *is a positive constant which depend only on the distribution of* ε_0.

ii) If ψ is a Young function then

$$\Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \varphi\left(\frac{c(N)\|Q_n + R_n\|_\psi}{\sqrt{n}}\right) \tag{2.2}$$

where $\varphi(x) = xh^{-1}(1/x)$ and $h(x) = x\psi(x)$ for any real $x > 0$.

The inequality (2.2) ensures a rate of convergence to zero for $\Delta_n(X)$ as n goes to infinity provided that $\Delta_n(A(\varepsilon_0))$ goes to zero as n goes to infinity and a bound for $\|Q_n + R_n\|_\psi$ exists. As just an illustration, we have the following corollary.

Corollary 2.3. *Assume that $(\varepsilon_k)_{k\in\mathbb{Z}}$ are i.i.d. centered H-valued random variables and that the condition* (1.2) *holds with $\tau = 1/2$.*

i) If ε_0 belongs to L_{H,ψ_1} then $\Delta_n(X) = O\left(\frac{\log n}{\sqrt{n}}\right)$ where ψ_1 is the Young function defined by $\psi_1(x) = \exp(x) - 1$.

ii) If ε_0 belongs to L_H^r for $r \geq 3$ then $\Delta_n(X) = O\left(n^{-\frac{r}{2(r+1)}}\right)$.

3 Proofs

Proof of Lemma 2.1. For any positive integer n, we have

$$\begin{aligned}
R_n = \sum_{j=-n}^{n} c_{j,n}(\varepsilon_j) &= \sum_{k=1}^{n}\sum_{j=-n}^{n} b_{k-j}(\varepsilon_j) \\
&= \sum_{k=1}^{n}\sum_{j\in[-n,n]\setminus\{k\}} a_{k-j}(\varepsilon_j) + (a_0 - A)\left(\sum_{k=1}^{n}\varepsilon_k\right) \\
&= \sum_{k=1}^{n}\sum_{j=-n}^{n} a_{k-j}(\varepsilon_j) - A\left(\sum_{k=1}^{n}\varepsilon_k\right) \\
&= -Q_n + \sum_{k=1}^{n} X_k - A\left(\sum_{k=1}^{n}\varepsilon_k\right).
\end{aligned}$$

The proof of Lemma 2.1 is complete.

Proof of Theorem 2.2. Let $\lambda > 0$ and $t > 0$ be fixed and denote $U = A\left(\sum_{k=1}^{n}\varepsilon_k/\sqrt{n}\right)$ and $V = (Q_n + R_n)/\sqrt{n}$. So $U + V = \sum_{k=1}^{n} X_k/\sqrt{n}$ and

$$\mathbb{P}(\|U+V\|_H \leq t) \leq \mathbb{P}(\|U\|_H \leq t+\lambda) + \mathbb{P}(\|V\|_H \geq \lambda) \tag{3.1}$$

For $\lambda_0 = 2\|V\|_\infty$, we obtain

$$\mathbb{P}(\|U+V\|_H \leq t) - \mathbb{P}(\|N\|_H \leq t) \leq \mathbb{P}(\|U\|_H \leq t+\lambda_0) - \mathbb{P}(\|N\|_H \leq t).$$

If $c(N)$ denotes a bound for the density of $\|N\|_H$ (see Davydov et al. [5]) then

$$\Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \frac{2c(N)\|Q_n + R_n\|_\infty}{\sqrt{n}}.$$

Noting that

$$Q_n = \sum_{j\geq n+2} a_j \left(\sum_{k=1-j}^{-n-1} \varepsilon_k\right) + \sum_{j<0} a_j \left(\sum_{k=n+1}^{n-j} \varepsilon_k\right) \tag{3.2}$$

and

$$R_n = R_n^{'} + R_n^{''} \tag{3.3}$$

where

$$R_n^{'} = -\sum_{j=-n}^{-1} a_j \left(\sum_{k=1}^{-j} \varepsilon_k\right) - \sum_{j<-n} a_j \left(\sum_{k=1}^{n} \varepsilon_k\right) - \sum_{j>0} a_j \left(\sum_{k=n-j+1}^{n} \varepsilon_k\right)$$

and

$$R_n^{''} = \sum_{j=1}^{n} a_j \left(\sum_{k=-j+1}^{0} \varepsilon_k\right) + \sum_{j=n+1}^{2n} a_j \left(\sum_{k=-n}^{n-j} \varepsilon_k\right),$$

we derive that $\|Q_n + R_n\|_\infty \leq 7\|\varepsilon_0\|_\infty \sum_{j\in\mathbb{Z}} |j| \|a_j\|_{\mathcal{L}(H)}$ and consequently

$$\Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \frac{14c(N)\|\varepsilon_0\|_\infty \sum_{j\in\mathbb{Z}} |j| \|a_j\|_{\mathcal{L}(H)}}{\sqrt{n}}.$$

Combining the last inequality with the Berry-Esseen inequality for i.i.d. centered H-valued random variables (see Yurinski [11] or Bosq [2], Theorem 2.9) we obtain (2.1).

In the other part, if ψ is a Young function we have $\mathbb{P}(\|V\|_H \geq \lambda) \leq \frac{1}{\psi(\lambda/\|V\|_\psi)}$ and keeping in mind inequality (3.1), we derive

$$\Delta_n(X) \leq \Delta_n(A(\varepsilon)) + c(N)\lambda + \frac{1}{\psi\left(\lambda/\|V\|_\psi\right)}.$$

Noting that $c(N)\lambda = \frac{1}{\psi\left(\lambda/\|V\|_\psi\right)}$ if and only if $\lambda = \frac{\varphi\left(c(N)\|V\|_\psi\right)}{c(N)}$ where φ is defined by $\varphi(x) = xh^{-1}(1/x)$ and h by $h(x) = x\psi(x)$, we conclude

$$\Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \varphi\left(\frac{c(N)\|Q_n + R_n\|_\psi}{\sqrt{n}}\right).$$

The proof of Theorem 2.2 is complete.

Proof of Corollary 2.3. Assume that $\|\varepsilon_0\|_{\psi_1} < \infty$ where ψ_1 is the Young function defined by $\psi_1(x) = \exp(x) - 1$. There exists $a > 0$ such that $E\left(\exp(a\|\varepsilon_0\|_H)\right) \leq 2$. So, there exist (see Arak and Zaizsev [1]) constants B and L such that

$$E\|\varepsilon_0\|_H^m \leq \frac{m!}{2} B^2 L^{m-2}, \quad m = 2, 3, 4, \ldots$$

Applying Pinelis-Sakhanenko inequality (see Pinelis and Sakhanenko [9] or Bosq [2]), we obtain

$$\mathbb{P}\left(\left\|\sum_{k=p}^{q} \varepsilon_k\right\|_H \geq x\right) \leq \exp\left(-\frac{x^2}{2(q-p+1)B^2 + 2xL}\right), \quad x > 0$$

and using Lemma 2.2.10 in van der Vaart and Wellner [10], there exists a universal constant K such that

$$\left\| \sum_{k=p}^{q} \varepsilon_k \right\|_{\psi_1} \leq K \left(L + B\sqrt{q-p+1} \right) \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we derive $\|Q_n + R_n\|_{\psi_1} \leq C \sum_{j\in\mathbb{Z}} \sqrt{|j|} \|a_j\|_{\mathcal{L}(H)}$ where the constant C does not depend on n. Keeping in mind the Berry-Esseen's central limit theorem for i.i.d. centered H-valued random variables (see Yurinski [11] or Bosq [2], Theorem 2.9), we apply Theorem 2.2 with the Young function ψ_1. Since the function φ defined by $\varphi(x) = xh^{-1}(1/x)$ with $h(x) = x\psi_1(x)$ satisfies

$$\lim_{x\to 0} \frac{\varphi(x)}{x \log(1+\frac{1}{x})} = 0,$$

we derive $\Delta_n(X) = O\left(\frac{\log n}{\sqrt{n}}\right)$.

Now, assume that $\|\varepsilon_0\|_r < \infty$ for some $r \geq 3$. Applying Pinelis inequality (see Pinelis [8]), there exists a universal constant K such that

$$\left\| \sum_{k=p}^{q} \varepsilon_k \right\|_r \leq K \left(r \left(\sum_{k=p}^{q} E\|\varepsilon_k\|_H^r \right)^{1/r} + \sqrt{r} \left(\sum_{k=p}^{q} E\|\varepsilon_k\|_H^2 \right)^{1/2} \right)$$

and consequently

$$\left\| \sum_{k=p}^{q} \varepsilon_k \right\|_r \leq 2Kr\|\varepsilon_0\|_r \sqrt{q-p+1}. \tag{3.5}$$

Combining (3.2), (3.3) and (3.5), we derive $\|Q_n + R_n\|_r \leq C \sum_{j\in\mathbb{Z}} \sqrt{|j|} \|a_j\|_{\mathcal{L}(H)}$ where the constant C does not depend on n. Again, applying Berry-Esseen's CLT (see Yurinski [11] or Bosq [2], Theorem 2.9) and Theorem 2.2 with the Young function $\psi(x) = x^r$ and the function φ given by $\varphi(x) = x^{r/(r+1)}$, we obtain $\Delta_n(X) = O\left(n^{-\frac{r}{2(r+1)}}\right)$. The proof of Corollary 2.3 is complete.

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