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DEFORMATION QUANTIZATION IN THE TEACHING OF LIE GROUP REPRESENTATIONS

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Abstract. We present straightforward and concrete computations of the unitary irreducible representations of the Euclidean motion group $\mathrm{M}(2)$ employing the methods of deformation quantization. Deformation quantization is a quantization method of classical mechanics and is an autonomous approach to quantum mechanics, arising from the Wigner quasiprobability distributions and Weyl correspondence. We advertise the utility and power of deformation theory in Lie group representations. In implementing this idea, many aspects of the method of orbits are also learned, thus further adding to the mathematical toolkit of the beginning graduate student of physics. Furthermore, the essential unity of many topics in mathematics and physics (such as Lie theory, quantization, functional analysis and symplectic geometry) is witnessed, an aspect seldom encountered in textbooks, in an elementary way.

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1. Introduction

In this paper, the deformation quantization approach to the representation theory of Lie groups is discussed via the example of the Euclidean motion group $\mathrm{M}(2)$, which is the group of rigid motions of the plane. The development of quantum theory in the mid-1920s greatly influenced the theory of unitary representations of groups in infinite dimensional Hilbert space. The basic idea is as follows. Suppose a Lie group G acts on a set X, denoted by $(g, x) \mapsto g \cdot x$. Let V be a vector space of complex-valued functions on X which is invariant under the action of G, that is, the function $x \mapsto f(g \cdot x)$ is in V, whenever f is in V. Then, the mapping $T_g : f \mapsto T_g f$, where $(T_g f)(x) = f(g \cdot x)$ is a linear transformation on V and is invertible. The mapping $g \mapsto T_g$ from G into the group $\mathrm{GL}(V)$ of invertible linear transformations of V is called a linear representation of G in V.

Mechanics provides basic examples of group representations. In classical mechanics, an observable is a function on phase space M, which is a Poisson manifold, while in quantum mechanics, observables are self-adjoint operators on a Hilbert space. Quantization, as generally understood, maps classical observables to quantum observables, where this mapping satisfies certain conditions first laid out formally by Dirac. In the simplest case of the free particle, the canonical quantization of phase space variables turns out to be a representation of the Heisenberg Lie algebra, and the exponentiation of this representation gives the representation of the Heisenberg Lie group. This basic example already illustrates the deep and beautiful connections between quantization and representations of Lie groups. More generally, in the above definition of a linear representation, the classical observables are the functions f on M and G acts on M. This induces an action of the Lie algebra of G on the classical observables via vector fields. Modulo many technical difficulties, resolved in many general cases by the orbit method [23] or geometric quantization of Kirillov, Kostant and Souriau, the Lie algebra representations give the quantum observables and its exponentiation give rise to the corresponding Lie group representations.

There are, currently, three accepted quantization procedures in quantum theory [19]. There is the Hilbert space-based quantization developed earliest by Heisenberg, Schrödinger, Dirac and others in the 1920s, the path integral method by Feynman, and the phase space formulation of quantum mechanics or deformation quantization, which this work focuses on.

Phase space quantum mechanics is based on Wigner's quasiprobability distribution [33] and the Weyl correspondence [32] between self-adjoint operators in Hilbert space and ordinary functions, called the symbols of the operators. It turns out that the Weyl symbol of the projection onto a state is the Wigner function corresponding to the state. The Wigner function, which is a function on phase space, allows for the computation of quantum averages by classical like formulas. Moreover, its marginal distributions produce the correct probability distributions for the position and momentum of the system. Not least of its utility is that it is the approach that gives most insight into the connection between classical mechanics and quantum mechanics. It was Groenewold [15] and Moyal [25] who first gave the formulas for the symbols of the composition and commutators of two quantum observables, now known as the Moyal star-product. In the early 1970s, Bayen *et al* [6] elevated this formula as a definition of deformation of functions on Poisson manifolds and proposed deformation quantization as an autonomous quantum theory.

The central idea of deformation quantization is the deformation of the usual pointwise commutative product of functions on Poisson manifolds into a noncommutative and associative star-product or $\star$-product, and the deformation of the Poisson bracket arising from the associativity of the $\star$-product. In their seminal work, Bayen *et al* suggested that quantization should be "a deformation of the structure of the algebra of classical observables and not as a radical change in the nature of the observables" [6, p. 62]. Deformation quantization is a synthesis of works due to Weyl, Wigner, Moyal, Groenewold, Gerstenhaber, and others. In 1997, Kontsevich [24] proved the existence of deformation quantization of regular Poisson manifolds. Previous to this, Fedosov, in the early 1980's, gave a very nice geometric proof of the existence of deformation quantization of symplectic manifolds [11] and started the great interest on deformation quantization among mathematicians.

The approach taken in the papers by Várilly, Gracia-Bondía and coworkers, in some sense, is the reverse to the general procedure presented in this work. The seminal work here is due to Stratonovich [27] but detailed attention only began in the late 1980s [8, 30]. The aforementioned works uses the so-called Stratonovich-Weyl quantizers to construct on the space of phase-space functions a noncommutative 'twisted' product. This product is induced from the operator product of unitary operators coming from the projective representations of the invariance group of the quantum system. Moreover, the idea has been developed so far as to implement harmonic analysis on phase space and to derive special function identities [12, 29]. In the most famous example, the Moyal product on phase-space functions on $\mathbb{R}^{2n}$ is induced from the composition of operators on $L^2(\mathbb{R}^n)$ via the Weyl-Wigner correspondence. Here, $\mathbb{R}^{2n}$ appear as polarization of the orbits $\mathbb{R}^{2n}$ of the coadjoint action of the Heisenberg Lie group on the dual space of the Heisenberg Lie algebra [23]. Thus, in the Stratonovich-Weyl quantizer method, the representation theory of the Lie group is used to obtain the noncommutative 'twisted' product on phase-space functions. In the method presented here, the noncommutative $\star$-product of functions on orbits is employed to obtain the unitary irreducible representations of the Lie group. In both cases, the orbits of the coadjoint Lie group action and its symplectic geometry play crucial roles, that is why, in this work, we took pains to give as complete computations as possible regarding this object.

As far as the Euclidean Motion group $\mathrm{M}(2)$ is concerned, it has found fundamental application in the quantization of the conjugate pair angle and orbital angular momentum (and the physical content of such quantization), as well as the construction of Wigner functions associated to it, in the works of Kastrup [21, 22]. Its role in quantum theory, the higher rank Euclidean Motion groups, $\mathrm{M}(3)$ for instance, are thus well worth be given fuller attention.

As a quantization theory, it is inevitable that deformation quantization found use into the representation theory of Lie groups. This has already been strongly hinted in [6]. Subsequent developments in the works [1–3,5] have shown that deformation theory, together with the orbit method, is very useful in representation theory. As the beautiful paper [20], from which we copied our title, has the aim of introducing deformation quantization and phase space methods in physics instruction, particularly in quantum mechanics, we also deemed it worthwhile to teach Lie group representations via the method of deformation quantization. In as much as [1–3] have already attempted to use star-products in the representation theory of various classes of Lie groups, these papers assume many deep mathematical results and large gaps in the computations make them very difficult reading materials for beginning graduate students.

In this article, we present fairly complete and concrete computations in obtaining the unitary irreducible representations of a particular Lie group using deformation quantization. Works similar to our own are [9, 10, 17, 26]. In Section 2 important concepts about unitary representations will be discussed, in particular, its construction via the method of induced representation and illustrate the said method with Euclidean motion group $\mathrm{M}(2)$ as our example. We will briefly discuss quantization in Section 3. The non-Hilbert space-based quantization, deformation quantization, the concept of $\star$-product and its connection to unitary representation theory will be discussed in Section 4. In Section 5, our main contribution is the concrete computation of the unitary representations of $\mathrm{M}(2)$ via deformation quantization. Finally, we summarize our results in Section 6.

2. Unitary Representations

A representation of a group G on a vector space V over a field K is a homomorphism

$$\mathcal{U} : G \longrightarrow \mathrm{GL}(V)$$

of G into the group $\mathrm{GL}(V)$ of invertible linear transformations on the representation space V. The dimension of V is the degree of the representation $\mathcal{U}$. If G is a topological group and $\mathbb{U}(\mathcal{H})$ is the group of unitary operators on the Hilbert space $\mathcal{H}$, it is required that the homomorphism $\mathcal{U} : G \longrightarrow \mathbb{U}(\mathcal{H})$ be strongly continuous, and differentiable in the case of G a Lie group. We call $\mathcal{U}$ a unitary representation. A subspace $\mathcal{A}$ of $\mathcal{H}$ is said to be invariant under the unitary representation $\mathcal{U}$ if $\mathcal{U}_g\mathcal{A} \subset \mathcal{A}$ for all $g \in G$. If the trivial subspace $\{0\}$ and $\mathcal{H}$ are the only invariant closed subspaces of $\mathcal{H}$ under $\mathcal{U}$, then $\mathcal{U}$ is irreducible. The irreducible unitary representations are the "atoms" of the unitary representations of G.

Two unitary representations of G, say $\mathcal{U} : G \to \mathbb{U}(\mathcal{H})$ and $\mathcal{U}' : G \to \mathbb{U}(\mathcal{H}')$, are equivalent when there is an isometry $A : \mathcal{H} \to \mathcal{H}'$ satisfying $A \circ \mathcal{U}_g = \mathcal{U}'_g \circ A$, for all $g \in G$. So, the set of all unitary irreducible representations (UIRs) of G can be partitioned into disjoint classes of UIRs. A basic problem of representation theory of Lie groups is the construction and classification of all UIRs, up to equivalence. In many cases the UIRs are sufficient to decompose L^2-functions on G into their Fourier series or Fourier integral. In the compact group case, for example, the Peter-Weyl Theorem states that the matrix elements of the UIRs form a complete orthonormal set in $L^2(G)$.

A good resource for a comprehensive list of representations of Lie groups is the three-volume set survey work of Vilenkin and Klimyk in [31]. For the Euclidean motion group $\mathrm{M}(2)$ in our discussion, we compared our construction with that of Sugiura in [28, Chapter 4].

A more or less procedural way of constructing representations is the method of induced representations by Frobenius and Mackey [7]. This is a method of constructing a representation of a group from a representation of its subgroup.

Let $\mathcal{H}$ be the space of functions $f : G \to \mathcal{H}_0$ satisfying $f(gh) = \tau(h)^{-1}f(g)$ where $h \in H$- a closed subgroup of G, and τ is a representation H on $\mathcal{H}_0$. The representation T of G on $\mathcal{H}$, induced by τ, is defined by $(T_gf)(g_0) = f(g^{-1}g_0)$, for all $g \in G$ and $f \in \mathcal{H}$. We denote this as $T = \mathrm{Ind}_H^G\tau$.

The construction of a unitary irreducible representation of a semidirect product of a compact Lie group and an abelian group is nicely outlined in [7, Theorem 7.7]. To illustrate this theorem, one has to determine the subgroup H and its representation τ in order to construct $\mathrm{Ind}_H^G\tau$.

The motion group $\mathrm{M}(2)$ of the two-dimensional Euclidean plane is the semidirect product $\mathrm{SO}(2) \ltimes \mathbb{R}^2$. The dual of the abelian subgroup $\mathbb{R}^2$ are the one-dimensional representations $\widehat{\mathbb{R}^2} = \{\chi_a = \mathrm{e}^{\mathrm{i}a}; a \in \mathbb{R}^2\} \simeq \mathbb{R}^2$. The action of the rotation group $\mathrm{SO}(2)$ on $\widehat{\mathbb{R}^2}$, defined by $R \cdot \chi_a(r) = \chi_a(R^{-1}r) = \chi_{Ra}(r)$, generates the $\mathrm{SO}(2)$-orbits of the form $S^1_{\|a\|} \subset \mathbb{R}^2$. However, the stabilizer group $\mathrm{SO}(2)_{\chi_a}$ is composed only of the identity element. Thus, $H = \{\mathrm{Id}\,\} \ltimes \mathbb{R}^2 \simeq \mathbb{R}^2$. Hence, the unitary irreducible representation of $\mathrm{M}(2)$ is induced by the representation χ_a of $\mathbb{R}^2$ on $\mathbb{C}$. So

$$\begin{aligned}\left((\mathrm{Ind}_{\mathbb{R}^2}^{\mathrm{M}(2)}\chi_a)_{(R,r)}f\right)(R_0,r_0) &= f((R,r)^{-1}(R_0,r_0)) \\ &= f((R^{-1}R_0,0)(1,R_0^{-1}(r_0-r))) \\ &= \chi_a^{-1}(R_0^{-1}(r_0-r))f(R^{-1}R_0,0)\end{aligned}$$

where $f : \mathrm{M}(2) \to \mathbb{C} \in \mathcal{H}$. But $\mathcal{H}$ is identified with $L^2(\mathrm{SO}(2))$ where $\mathrm{SO}(2) \simeq \mathrm{M}(2)/\mathbb{R}^2$. We let $\mathcal{U}^a = \mathrm{Ind}_{\mathbb{R}^2}^{\mathrm{M}(2)}\chi_a$ and put $r_0 = 0$. Therefore, the unitary irreducible representation $\mathcal{U}^a$ of $\mathrm{M}(2)$ on $L^2(\mathrm{SO}(2))$ [28, p. 157] is defined by

$$(\mathcal{U}_g^a f)(R_\theta) = \mathrm{e}^{\mathrm{i}(r,R_\theta a)}f(R_\phi^{-1}R_\theta) \tag{1}$$

where $g = (R_\phi, r) \in \mathrm{M}(2)$, $f \in L^2(\mathrm{SO}(2))$ and $a \in \mathbb{C}$. Since $\mathcal{U}^a$ is equivalent to $\mathcal{U}^b$ if and only if $|a| = |b|$ [28, Chapter IV Theorem 1.3], an equivalence class of UIRs of $\mathrm{M}(2)$ can be represented by $\mathcal{U}^a$ where $a > 0$. Since $\mathrm{SO}(2) \simeq S^1 \ni (\cos\theta, \sin\theta)$, letting $r = (r_1, r_2)$, expression (1) becomes

$$(\mathcal{U}_g^a f)(\theta) = \mathrm{e}^{\mathrm{i}a(r_1\cos\theta + r_2\sin\theta)}f(\theta - \phi). \tag{2}$$

The set $P = \{\mathcal{U}^a; a > 0\}$ of infinite-dimensional UIRs is called the principal series of UIRs of $\mathrm{M}(2)$.

There is another set of UIRs other than the set P. These representations are the one-dimensional unitary representations $\chi_n, n \in \mathbb{Z}$ of $\mathrm{SO}(2)$ via the natural projection $p : \mathrm{M}(2) \to \mathrm{SO}(2)$, defining the operators

$$(\chi_n \circ p)(R_\phi, r) = \mathrm{e}^{\mathrm{i}n\phi}. \tag{3}$$

Hence, the complete set of representatives of the set of classes of UIRs of $\mathrm{M}(2)$ [28, Chapter IV Theorem 2.1] is

$$\widehat{\mathrm{M}(2)} = \{\mathcal{U}^a;\, a > 0\} \cup \{\chi_n \circ p;\, n \in \mathbb{Z}\}. \tag{4}$$

At this point, consider the infinite-dimensional UIR $\mathcal{U}^a$. Let U be an element of the Lie algebra $\mathfrak{m}(2) = \mathrm{span}\{X, E_1, E_2\}$ of the Euclidean motion group $\mathrm{M}(2)$ where X spans the Lie algebra of $\mathrm{SO}(2)$, E_1, E_2 are the canonical base elements that span $\mathbb{R}^2$ and the Lie brackets of these spanning elements are $[X, E_1] = -E_2$, $[X, E_2] = E_1$ and $[E_1, E_2] = 0$. Given by the one-parameter subgroup

$$\exp tU = \begin{cases} \Big(R_{-tc_1}, \Big(\dfrac{c_2}{c_1}\sin tc_1 + \dfrac{c_3}{c_1}(1-\cos tc_1), \\ \quad \dfrac{c_2}{c_1}(-1+\cos tc_1) + \dfrac{c_3}{c_1}\sin tc_1\Big)\Big) & \text{if } c_1 \neq 0 \\ (1, (tc_2, tc_3)) & \text{if } c_1 = 0 \end{cases} \tag{5}$$

of $\mathrm{M}(2)$ where $U = c_1X + c_2E_1 + c_3E_2$, expression (2) becomes

$$(\mathcal{U}^a_{\exp tU} f)(\theta) = \begin{cases} \mathrm{e}^{\mathrm{i}a\left[\frac{c_2}{c_1}(\sin(tc_1+\theta)-\sin\theta)-\frac{c_3}{c_1}(\cos(tc_1+\theta)-\cos\theta)\right]} \\ \times f(tc_1+\theta) & \text{if } c_1 \neq 0 \\ \mathrm{e}^{\mathrm{i}at(c_2\cos\theta + c_3\sin\theta)} f(\theta) & \text{if } c_1 = 0 \end{cases} \tag{6}$$

and the derivative of expression (6) with respect to t is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}^a_{\exp tU} f(\theta) = \begin{cases} \mathrm{e}^{\mathrm{i}a\left[\frac{c_2}{c_1}(\sin(tc_1+\theta)-\sin\theta)-\frac{c_3}{c_1}(\cos(tc_1+\theta)-\cos\theta)\right]} \\ \times\big[\mathrm{i}a(c_2\cos(tc_1+\theta) + c_3\sin(tc_1+\theta)) \\ \times f(tc_1+\theta) + c_1\frac{\partial}{\partial(tc_1+\theta)} f(tc_1+\theta)\big] & \text{if } c_1 \neq 0 \\ \mathrm{i}a(c_2\cos\theta + c_3\sin\theta)(\mathcal{U}^a_{\exp tU} f)(\theta) & \text{if } c_1 = 0 \end{cases} \tag{7}$$

and when $t = 0$

$$(\mathrm{d}\mathcal{U}^a(U)f)(\theta) = \mathrm{i}a(c_2\cos\theta + c_3\sin\theta)f(\theta) + c_1 f'(\theta) \tag{8}$$

where $\mathrm{d}\mathcal{U}^a(U) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}^a_{\exp tU}|_{t=0}$. The representation $\mathrm{d}\mathcal{U}^a$, defined by expression (8), is called the differential representation of $\mathcal{U}^a$.

3. Quantization

Quantization is the process of forming a quantum mechanical system from a given classical system where these two systems, classical mechanics (in the Hamiltonian formalism) and quantum mechanics (in the Heisenberg picture), are modeled by the space of C^∞-functions on a symplectic manifold M and the family of self-adjoint operators on a Hilbert space $\mathcal{H}$, respectively. Canonically, this is done by associating a classical observable f on M to a self-adjoint operator $Q(f)$ on $\mathcal{H}$, where Q is a linear map, $Q(1)$ is the identity operator and satisfies the correspondence

$$Q(\{f,g\}) = -\frac{\mathrm{i}}{\hbar}[Q(f), Q(g)] \tag{9}$$

where expression (9) was the result of Dirac's analogy of Heisenberg commutator bracket $[\cdot,\cdot]$ with the Poisson bracket $\{\cdot,\cdot\}$, which endow the two respective mechanical systems their Lie algebra structures.

When $M = T^*N$, where N is an n-dimensional smooth manifold and $\mathcal{H} = L^2(N)$, the quantization is said to be full if the operators $Q(q^i)$ and $Q(p_j)$ act irreducibly on $\mathcal{H}$. That is, the operators above are the position and momentum operators: $Q(q^i)$ is the multiplication of q^i and $Q(p_j) = -\mathrm{i}\hbar\partial_{q_j}$. By the theorem of Stone and von Neumann, it is unitarily equivalent to the Schrödinger representation.

It is known that the algebra of inhomogenous quadratic polynomials on $\mathbb{R}^{2n}$ is a maximal Lie subalgebra of the space of polynomials under the Poisson bracket and this subalgebra is identified with the Lie algebra of the Jacobi group. A representation of this group, known as the Schrödinger-Weil representation, gives rise to a quantization map. However, by the Groenewold-Van Hove theorem, it is impossible to extend this map to the whole $C^\infty(\mathbb{R}^{2n})$.

Independently, the geometric quantization of Konstant and Souriau is another Hilbert space-based quantization where the goal is the construction of quantum objects from the geometry of the classical ones [23]. This quantization procedure is the physical counterpart of Kirillov's orbit method. An orbit of a Lie group G in the coadjoint representation, also known as coadjoint orbit, is the orbit of the coadjoint action of G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$, through the point $F \in \mathfrak{g}^*$. It is given by the set

$$\Omega = \{K(g)F\,;\, g \in G\} \tag{10}$$

defined by the dual pairing $\langle K(g)F, U\rangle = \left\langle F, \mathrm{Ad}_{g^{-1}}U\right\rangle$ of the Lie algebra with its dual. It is known that the coadjoint orbit Ω is a homogeneous symplectic G-manifold and its symplectic form ω is called the Kirillov symplectic form. This method's particular interest is the correspondence between the finite-dimensional coadjoint orbits and the infinite-dimensional unitary representations of G. It first appeared in its application to nilpotent Lie groups and further extended to other classes of Lie groups.

Classical mechanics is ought to be a limiting case ($\hbar \to 0$) of quantum mechanics, and Dirac's corresponding principle (9) is indeed a strong requirement. Not only shown in $\mathbb{R}^{2n}$, but evidences of "no-go" results are found in S^1 [13] and T^*S^1 [14]. Furthermore, associating functions on a symplectic manifold to self-adjoint operators on a Hilbert space is quite a radical transition.

To go around this problem, the quantum system is, somehow, described by the same entities that were used to described the classical system, and expression (9) must be understood as an equality up to order two in $\hbar$ and to study what should correspond with the bracket of operators [16]. This is the idea behind deformation quantization.

4. Deformation Quantization

The model of quantum mechanics is described as a deformed space of classical observables. In this deformed structure, a noncommutative but associative product is introduced, called the $\star$-product.

Let $f, g \in C^\infty(M)$ where M is a Poisson manifold. This formal associative $\star$-product [16], here we denote this as $\star_\lambda$, is a bilinear map

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[\lambda]]$$

defined by

$$f \star_\lambda g = \sum_{r=0}^{\infty} \lambda^r C^r(f, g)$$

where λ is a formal parameter, C^r is a bidifferential operator with $C^r(f, g) = (-1)^r C^r(g, f)$ for all $f, g \in C^\infty(M)$ and satisfies the following properties

1. $C^0(f, g) = fg$
2. $C^1(f, g) = \{f, g\}$ and
3. $C^r(1, f) = C^r(f, 1) = 0$ for $r \geq 1$.

Property 1 shows that the noncommutative product $\star_\lambda$ is a deformation of the commutative pointwise multiplication of functions in $C^\infty(M)$. Property 2 satisfies the correspondence principle

$$f \star_\lambda g - g \star_\lambda f = 2\lambda\{f, g\} + \cdots$$

where the dots mean higher-order terms with respect to λ and if we let

$$[f, g]_\lambda = \frac{1}{2\lambda}(f \star_\lambda g - g \star_\lambda f)$$

the bracket $[\cdot,\cdot]_\lambda$ is the deformed Poisson bracket in $C^\infty(M)$. Property 3 implies $1\star_\lambda f = f\star_\lambda 1 = f$. Hence, the algebra $(C^\infty(M)[[\lambda]],\star_\lambda,[\cdot,\cdot]_\lambda)$ is the quantum analogue of the classical model $(C^\infty(M),\cdot,\{\cdot,\cdot\})$. The questions of existence and classification of these $\star$-products have already been settled.

The $\star$-product for the symplectic flat manifold $M = \mathbb{R}^{2n}$ has long been known [15, 25] and is the most important. We discuss it at length. Suppose ω is the canonical symplectic form of M in the (q,p) coordinates on some open set $O\subset M$, the Moyal $\star$-product of the algebra $(C^\infty(M)[[\lambda]],\star)$ with $\lambda = \frac{1}{2\mathrm{i}}$ is the product

$$f\star g = fg + \sum_{r=1}^{\infty}\frac{1}{r!}\left(\frac{1}{2\mathrm{i}}\right)^r P^r(f,g) \tag{11}$$

where

$$P^r(f,g) = \Lambda^{i_1 j_1}\Lambda^{i_2 j_2}\cdots\Lambda^{i_r j_r}\partial_{i_1 i_2\cdots i_r} f\,\partial_{j_1 j_2\cdots j_r} g$$

with the multi-index notation

$$\partial_{i_1 i_2\cdots i_r} = \frac{\partial}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_r}}, \qquad x := (q^1,\dots,q^n,p_1,\dots,p_n)$$

and Λ^{ij} are the ij-entries of the matrix associated to the symplectic form ω.

This $\star$-product has an integral formula [18], from which many of its important properties follow directly. Let f,g be functions in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$. By defining the symplectic Fourier transform $F:\mathcal{S}(\mathbb{R}^{2n})\to\mathcal{S}(\mathbb{R}^{2n})$ by

$$(Ff)(x) = \int_{\mathbb{R}^{2n}} f(y)\mathrm{e}^{\mathrm{i}\omega(x,y)}\frac{\mathrm{d}y}{(2\pi)^n}$$

and the symplectic convolution $\times_\omega$ as

$$(f\times_\omega g)(x) = \int_{\mathbb{R}^{2n}} f(y)g(x-y)\mathrm{e}^{\mathrm{i}\omega(y,x)}\frac{\mathrm{d}y}{(2\pi)^n}$$

the product

$$f\star g = F(Ff\times_\omega Fg)$$

admits the development of the Moyal $\star$-product defined in (11) when f,g have compactly supported Fourier transforms, converge to a function in $\mathcal{S}(\mathbb{R}^{2n})$ and has the following integral formula

$$(f\star g)(x) = \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} f(y)g(z)\mathrm{e}^{\mathrm{i}(\omega(x,z)+\omega(z,y)+\omega(y,x))}\frac{\mathrm{d}y\mathrm{d}z}{(2\pi)^{2n}}.$$

This $\star$-product on $\mathcal{S}(\mathbb{R}^{2n})$ has the following properties

1. $(\mathcal{S}(\mathbb{R}^{2n}), \star)$ is a generalized Hilbert algebra in $L^2(\mathbb{R}^{2n})$
2. $\displaystyle\int_{\mathbb{R}^{2n}} (f \star g)(x) \frac{\mathrm{d}x}{(2\pi)^n} = \int_{\mathbb{R}^{2n}} (fg)(x) \frac{\mathrm{d}x}{(2\pi)^n}$
3. $\overline{f \star g} = \bar{g} \star \bar{f}$ and
4. the operator $l_f : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}(\mathbb{R}^{2n})$ defined by $l_f(g) = f \star g$, can be extended to a bounded operator on $L^2(\mathbb{R}^{2n})$.

With Kostant and Souriau's partial success on the geometrical approach to group representations and Kirillov's orbit method, Bayen *et al* [6, p.124] is confident that $\star$-products have a promising future in representation theory. Arnal and his coworkers tested the method of quantization by deformation on the problem of construction and classification of UIRs of nilpotent Lie groups [1, 2] and exponential Lie groups [3]. The covariant Moyal $\star$-product made these computations possible [5].

For a unitary representation of a connected Lie group G corresponding to a homogeneous symplectic orbit $\Omega \simeq G/G_F$, where G_F is the stabilizer subgroup of G, the Lie algebra $\mathfrak{g}$ is identified with the Lie subalgebra $\mathfrak{g}_\Omega = \{\tilde{U} \in C^\infty(\Omega)\,; U \in \mathfrak{g}\}$ of $C^\infty(\Omega)$ where the function $\tilde{U} : \Omega \to \mathbb{R}$ is defined by

$$\tilde{U}(F) = \langle F, U \rangle \tag{12}$$

for all $F \in \Omega$ and one has to show that the Moyal $\star$-product satisfies

$$\frac{1}{2\lambda}(\tilde{U} \star \tilde{T} - \tilde{T} \star \tilde{U}) = \widetilde{[U, T]} \tag{13}$$

for any $U, T \in \mathfrak{g}$. A $\star$-product that satisfies expression (13) is a $\mathfrak{g}_\Omega$-relative quantization. Each quantization relative to a Lie algebra $\mathfrak{g}$ is a G-covariant $\star$-product and a G-covariant $\star$-product gives rise to a representation $\mathcal{U}$ of G on $C^\infty(\Omega)[[\lambda]]$ by automorphisms, which also gives rise to a differential representation $\mathrm{d}\mathcal{U}$ of $\mathcal{U}$, defined by $\mathrm{d}\mathcal{U}(U) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(\exp tU)\big|_{t=0}$. That is, we obtain a representation of $\mathfrak{g}$ on $C^\infty(\Omega)[[\lambda]]$ by endomorphisms.

The function $\tilde{U}$ on Ω is called the Hamiltonian function associated to the Hamiltonian vector field ξ_U, defined by $\xi_U f = \{\tilde{U}, f\}$. We remark that the computations above depend on the parameterization of the orbit Ω.

The techniques that were outlined in the construction of representations of nilpotent [1, 2] and exponential [3] Lie groups have led to concrete computations of representations for particular Lie groups, some of which were neither nilpotent nor exponential. Among these are the works of Diep and his students: the group of affine transformation of the real and complex plane [9, 10], the real rotation groups [26] and the MD_4-groups [17].

These papers have provided us an outline to construct and classify unitary representations of concrete Lie groups. As in the method of obtaining representations via induction, we have a more or less procedural way of the construction. Our main contribution is the development of the UIRs of $\mathrm{M}(2)$ via deformation quantization, hence an alternative to the method of induced representation. The construction in the next section is outlined as follows

1. compute the coadjoint orbit Ω_F of $\mathrm{M}(2)$ through the point $F \in \mathfrak{m}(2)^*$
2. define a chart on Ω_F and consider the Hamiltonian system $(\Omega_F, \omega, \xi_U)$ where the Hamiltonian function $\tilde{U}$ is defined in (12), ξ_U is its associated vector field and ω is the Kirillov symplectic form
3. the Moyal $\star$-product is $\mathrm{M}(2)$-covariant which will give rise to a representation l of $\mathfrak{m}(2)$ on $C^\infty(\Omega_F)[[\lambda]]$
4. the representation $\hat{l}$, defined by the operators $\hat{l}_U = \mathcal{F}_x \circ l_U \circ \mathcal{F}_x^{-1}$, is a differential representation of the UIR of $\mathrm{M}(2)$ where the operator $\mathcal{F}_x$ is a partial Fourier transform on the momentum variable x, and
5. classify these constructed representations via the coadjoint orbits.

We remark that these steps are quite straightforward to implement and provide concrete computations suitable for the learning by graduate students in Physics and Mathematics of many important mathematical concepts and objects.

5. The UIRs of $\mathrm{M}(2)$

5.1. Coadjoint Orbits

In matrix form, the Lie algebra $\mathfrak{m}(2)$ of $\mathrm{M}(2)$ is spanned by the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and these matrices satisfy the Lie brackets $[X, E_1] = -E_2, [X, E_2] = E_1$ and $[E_1, E_2] = 0$. The one-parameter subgroup of $\mathrm{M}(2)$ in (5) is computed using these matrices. Hence, $\mathfrak{m}(2)$ is identified with $\mathbb{R} \times \mathbb{R}^2$ and the elements are written as $U = c_1 X + c_2 E_1 + c_3 E_2$. The dual $\mathfrak{m}(2)^*$ is also identified with $\mathbb{R} \times \mathbb{R}^2$.

Let $g = \exp U \in \mathrm{M}(2)$ and fix $F = (\mu, \alpha) = \mu X^* + \alpha_1 E_1^* + \alpha_2 E_2^* \in \mathfrak{m}(2)^*$. The coadjoint orbit Ω_F of $\mathrm{M}(2)$ through F, given by expression (10), is the set

$$\Omega_F = \{K(\exp U)F\,; U \in \mathfrak{m}(2)\} \subset \mathfrak{m}(2)^*$$

satisfying

$$\langle K(\exp U)F, T\rangle = \langle F, \mathrm{Ad}(-\exp U)T\rangle\,.$$

Since $\mathrm{Ad}(\exp)U = \exp(\mathrm{ad}_U)$ for all $U \in \mathfrak{m}(2)$, we write

$$\begin{aligned} K(\exp U)F = \langle F, \exp(-\mathrm{ad}_U)X\rangle\, X^* \ &+\ \langle F, \exp(-\mathrm{ad}_U)E_1\rangle\, E_1^* \\ &+\ \langle F, \exp(-\mathrm{ad}_U)E_2\rangle\, E_2^*. \end{aligned} \tag{14}$$

When the operator ad_U is evaluated by basis of $\mathfrak{m}(2)$ and exponentiated, the operator $\exp(-\mathrm{ad}_U)$ in its Taylor series expression will become

$$\begin{aligned} \exp(-\mathrm{ad}_U) \ &= \sum_{r\geq 0} \frac{1}{r!} \begin{pmatrix} 0 & 0 & 0 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}^r \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{c_2}{c_1}(1-\cos c_1) + \frac{c_3}{c_1}\sin c_1 & \cos c_1 & -\sin c_1 \\ -\frac{c_2}{c_1}\sin c_1 + \frac{c_3}{c_1}(1-\cos c_1) & \sin c_1 & \cos c_1 \end{pmatrix}. \end{aligned}$$

Let

$$\begin{aligned} R_{c_1} \ &= \begin{pmatrix} \cos c_1 & -\sin c_1 \\ \sin c_1 & \cos c_1 \end{pmatrix} \\ \frac{1-R_{c_1}}{c_1}\begin{pmatrix} c_2 \\ c_3 \end{pmatrix} \ &= \begin{pmatrix} \frac{c_2}{c_1}(1-\cos c_1) + \frac{c_3}{c_1}\sin c_1 \\ \frac{c_3}{c_1}(1-\cos c_1) - \frac{c_2}{c_1}\sin c_1 \end{pmatrix}. \end{aligned}$$

So, the dual pairing of F with $\exp(-\mathrm{ad}_U)B$ where $B = X, E_1, E_2$ on each term of (14) will result to

$$K(\exp U)F = \left(\mu + \alpha \cdot \frac{1-R_{c_1}}{c_1}\begin{pmatrix} c_2 \\ c_3 \end{pmatrix}\right) X^* + \alpha R_{c_1}\begin{pmatrix} E_1^* \\ E_2^* \end{pmatrix}.$$

The coadjoint orbit of $\mathrm{M}(2)$ through F is

$$\Omega_F = \left\{\left(\mu + \alpha \cdot \frac{1-R_{c_1}}{c_1}\begin{pmatrix} c_2 \\ c_3 \end{pmatrix}, \alpha R_{c_1}\right)\,;\, U \in \mathfrak{m}(2)\right\}.$$

There are two types of orbits. If $\alpha = 0$, the orbit $\Omega_F = \{(\mu, 0)\}$ is a point- the trivial orbit. If $\alpha \neq 0$, the orbit Ω_F is the two-dimensional infinite cylinder of radius $\|\alpha\|$ which we denote $\Omega_F = T^*S^1_{\|\alpha\|}$. We first work on the nontrivial orbits, then later the trivial ones.

5.2. Hamiltonian System on the Cylinder

Fix F where $\alpha \neq 0$. The map

$$\psi : \mathbb{R}^2 \to \Omega_F = T^* S^1_{\|\alpha\|} \tag{15}$$

where $\psi(x, \theta) = xX^* + \|\alpha\| \cos\theta E_1^* + \|\alpha\| \sin\theta E_2^*$ defines a global chart on Ω_F. So each function f in $C^\infty(\Omega_F)$ is written as $f \circ \psi$ and we describe the Hamiltonian system $(\Omega_F, \omega, \xi_U)$ with respect to the chart (15) as follows

1. the Hamiltonian function associated to $U \in \mathfrak{m}(2)$ is
$$\tilde{U} = c_1 x + \|\alpha\|(c_2 + \mathrm{i}c_3, \mathrm{e}^{\mathrm{i}\theta}) \tag{16}$$
where $(\cdot, \cdot)$ is the inner product and the associated Hamiltonian vector field is
$$\xi_U = c_1 \partial_\theta - \|\alpha\|(c_2 + \mathrm{i}c_3, \mathrm{i}\mathrm{e}^{\mathrm{i}\theta})\partial_x \tag{17}$$
2. the map ψ gives rise to a symplectomorphism where the Kirillov symplectic form is the canonical form $\omega = \mathrm{d}x \wedge \mathrm{d}\theta$.

Since $U = c_1 X + c_2 E_1 + c_3 E_2 \in \mathfrak{m}(2)$, the value of the functional $\tilde{U}$ at the point $F' = xX^* + \|\alpha\| \cos\theta E_1^* + \|\alpha\| \sin\theta E_2^* \in \Omega_F$ is the value of the dual pairing

$$\langle F', U \rangle = c_1 x + c_2(\|\alpha\| \cos\theta) + c_3(\|\alpha\| \sin\theta) \tag{18}$$

and since $\xi_U f = \partial_x \tilde{U} \partial_\theta f - \partial_\theta \tilde{U} \partial_x f$ in (x, θ)-coordinates, it follows that

$$\xi_U = c_1 \partial_\theta - \|\alpha\|(-c_2 \sin\theta + c_3 \cos\theta)\partial_x. \tag{19}$$

Expressing the terms with sines and cosines in (18) and (19) as an inner product of $c_2 + \mathrm{i}c_3$ with $\mathrm{e}^{\mathrm{i}\theta}$ will result to (16) and (17), respectively. The restriction of ψ to the domain $\mathbb{R} \times \mathbb{T}$ gives rise to a diffeomorphism. Let $U = c_1 X + c_2 E_1 + c_3 E_2$ and $T = c_1' X + c_2' E_1 + c_3' E_2$. Since $[U, T] = (c_1 c_3' - c_1' c_3)E_1 + (c_1' c_2 - c_1 c_2')E_2$, so for any $F' \in \Omega_F$ the value of the Kirillov symplectic form is

$$\langle F', [U, T] \rangle = \|\alpha\| \cos\theta(c_1 c_3' - c_1' c_3) + \|\alpha\| \sin\theta(c_1' c_2 - c_1 c_2'). \tag{20}$$

But

$$\begin{aligned} \omega(\xi_U, \xi_T) &= \det \begin{pmatrix} \mathrm{d}x(\xi_U) & \mathrm{d}x(\xi_T) \\ \mathrm{d}\theta(\xi_U) & \mathrm{d}\theta(\xi_T) \end{pmatrix} \\ &= \|\alpha\| \cos\theta(c_1 c_3' - c_1' c_3) + \|\alpha\| \sin\theta(c_1' c_2 - c_1 c_2') \end{aligned}$$

is exactly (20) when $\omega = \mathrm{d}x \wedge \mathrm{d}\theta$. Hence, $\psi|_{\mathbb{R}\times\mathbb{T}}$ is a symplectomorphism. We define the variables x and θ on $T^* S^1_{\|\alpha\|}$ as the momentum and position variables, respectively.

5.3. Covariance of the Moyal $\star$-Product

The matrix associated to the canonical form $\omega = \mathrm{d}x \wedge \mathrm{d}\theta$ is

$$\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Moyal $\star$-product is defined by expression (11). Since $P^0(\tilde{U}, \tilde{T}) = \tilde{U}\tilde{T}$, $P^1(f,g) = \partial_x \tilde{U} \partial_\theta \tilde{T} - \partial_\theta \tilde{U} \partial_x \tilde{T} = \|\alpha\| \cos\theta(c_1 c_3' - c_1' c_3) + \|\alpha\| \sin\theta(c_1' c_2 - c_1 c_2')$ and $P^r(\tilde{U}, \tilde{T}) = 0$ for $r \geq 2$, the product of two Hamiltonian functions associated to $U, T \in \mathfrak{m}(2)$ is

$$\tilde{U} \star \tilde{T} = \tilde{U}\tilde{T} + \frac{1}{2\mathrm{i}}(\|\alpha\| \cos\theta(c_1 c_3' - c_1' c_3) + \|\alpha\| \sin\theta(c_1' c_2 - c_1 c_2')). \quad (21)$$

So from (21), the equality

$$\mathrm{i}\tilde{U} \star \mathrm{i}\tilde{T} - \mathrm{i}\tilde{T} \star \mathrm{i}\tilde{U} = \mathrm{i}\widetilde{[U,T]} \quad (22)$$

can be easily shown where $\widetilde{[U,T]}$ is expression (20).

Expression (13) is exactly (22) when $\lambda = \frac{1}{2\mathrm{i}}$. Thus, the Moyal $\star$-product is $\mathrm{M}(2)$-covariant. It gives rise to a representation of $\mathfrak{m}(2)$ on $C^\infty(\Omega_F)[[\lambda]]$ by endomorphism of the Moyal $\star$-product.

This representation of $\mathfrak{m}(2)$ is defined by the operators

$$l_U : C^\infty(\Omega_F)[[\lambda]] \to C^\infty(\Omega_F)[[\lambda]]$$

given by the left $\star$-product multiplication

$$l_U f = \frac{1}{2\lambda} \tilde{U} \star f.$$

The product converges in $\mathcal{S}(\Omega_F)$ and these operators extend to $L^2(\Omega_F)$. We still denote this extension as l_U for all $U \in \mathfrak{m}(2)$.

5.4. Convergence of the Operators $\hat{l}_U$

We now work on $\hat{l}$, a representation equivalent to l where its operators are intertwined by the partial Fourier transform $\mathcal{F}_x$. Hence we study the convergence of the operator

$$\hat{l}_U = \mathcal{F}_x \circ l_U \circ \mathcal{F}_x^{-1} \quad (23)$$

for all $U \in \mathfrak{m}(2)$. In the case of exponential Lie groups, $\hat{l}$ is the differential representation of the UIR of the said group associated to the orbit method of Kostant-Kirillov [3, Proposition 2.6]. Both the exponential Lie groups and $\mathrm{M}(2)$ are solvable, but the latter is non-exponential. However, we will show in Section 5.5 that our computed $\hat{l}$ is also the differential representation of the UIR of $\mathrm{M}(2)$.

Let $f \in \mathcal{S}(\Omega_F)$. The partial Fourier transform $\mathcal{F}_x$ of the function f on Ω_F is defined by

$$(\mathcal{F}_x f)(\eta, \theta) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\eta x} f(x, \theta) \frac{\mathrm{d}x}{\sqrt{2\pi}}$$

and its inverse transform $\mathcal{F}_x^{-1}$ by

$$(\mathcal{F}_x^{-1} f)(x, \theta) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\eta x} f(\eta, \theta) \frac{\mathrm{d}\eta}{\sqrt{2\pi}}.$$

The transform $\mathcal{F}_x$ is actually a Fourier transform along the momentum variable.

The derivatives

$$\partial_x \mathcal{F}_x^{-1}(f) = \mathrm{i}\mathcal{F}_x^{-1}(\eta f) \tag{24}$$

and

$$\mathcal{F}_x(xf) = \mathrm{i}\partial_\eta \mathcal{F}_x(f) \tag{25}$$

are easily computed while (24) can be generalized as

$$\frac{\partial^r}{\partial x^r} \mathcal{F}_x^{-1}(f) = \mathrm{i}^r \mathcal{F}_x^{-1}(\eta^r f). \tag{26}$$

On the other hand, the partial derivative of $\tilde{U}$ in (16) of order $r \geq 2$ with respect to the momentum or with respect to both the momentum and position is zero. So, the bidifferential $P^r(\tilde{U}, \mathcal{F}_x^{-1} f)$ is left with the nonzero term

$$\Lambda^{21}\Lambda^{21} \ldots \Lambda^{21} \partial_{\theta^r} \tilde{U} \partial_{x^r} \mathcal{F}_x^{-1}(f) \tag{27}$$

where $\Lambda^{21}\Lambda^{21} \ldots \Lambda^{21} = (-1)^r$. The rth partial derivative of $\tilde{U}$ in (16) and the generalized derivative of the inverse Fourier transform $\mathcal{F}_x^{-1}$ (26), together applied in (27), will result to

$$P^r(\tilde{U}, \mathcal{F}_x^{-1} f) = (-1)^r \|\alpha\| (c_2 + \mathrm{i}c_3, \mathrm{i}^r \mathrm{e}^{\mathrm{i}\theta})(\mathrm{i}^r \mathcal{F}_x^{-1})(\eta^r f) \tag{28}$$

for $r \geq 2$ and all functions f on Ω_F.

Now the operator (23) can be expressed as $\hat{l}_U(f) = \mathrm{i}\mathcal{F}_x(\tilde{U} \star \mathcal{F}_x^{-1}(f))$. Applying (24) and (28), we have the product

$$\begin{aligned} \tilde{U} \star \mathcal{F}_x^{-1}(f) \;&=\; c_1 x \mathcal{F}_x^{-1}(f) + \frac{c_1}{2\mathrm{i}} \partial_\theta \mathcal{F}_x^{-1}(f) \\ &\quad + \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{1}{2}\right)^r \|\alpha\| (c_2 + \mathrm{i}c_3, \mathrm{i}^r \mathrm{e}^{\mathrm{i}\theta}) \mathcal{F}_x^{-1}(\eta^r \cdot f). \end{aligned}$$

Together with (25)

$$\begin{aligned}\hat{l}_U(f) &= -c_1\partial_\eta f + \frac{c_1}{2}\partial_\theta f + \mathrm{i}\|\alpha\|\sum_{r=0}^{\infty}\frac{1}{r!}\left(-\frac{\eta}{2}\right)^r(c_2+\mathrm{i}c_3, \mathrm{i}^r\mathrm{e}^{\mathrm{i}\theta})f\\ &= c_1\left(\frac{1}{2}\partial_\theta - \partial_\eta\right)f + \mathrm{i}\|\alpha\|\left(c_2+\mathrm{i}c_3, \mathrm{e}^{\mathrm{i}\theta}\sum_{r=0}^{\infty}\frac{1}{r!}\left(-\frac{\mathrm{i}\eta}{2}\right)^r\right)f\\ &= c_1\left(\frac{1}{2}\partial_\theta - \partial_\eta\right)f + \mathrm{i}\|\alpha\|\left(c_2+\mathrm{i}c_3, \mathrm{e}^{\mathrm{i}\left(\theta-\frac{\eta}{2}\right)}\right)f.\end{aligned}$$

Let $s = \theta - \frac{\eta}{2}$. By the change of variables, the operator $\hat{l}_U$ will finally converge to

$$\hat{l}_U = c_1\frac{\partial}{\partial s} + \mathrm{i}\|\alpha\|(c_2\cos s + c_3\sin s). \tag{29}$$

5.5. Representations Associated to the Nontrivial Orbits

The representation space $L^2(\Omega_F)$ to which $\hat{l}_U$ in (29) is defined on is too big. The Lie subalgebra $\mathfrak{h} = \mathbb{R}^2$ is a real algebraic polarization of $\mathfrak{m}(2)$. By Remark 6 in [23, p.29], the leaves of the $\mathrm{M}(2)-$invariant foliation of Ω_F are the tangent lines $K(H)F = T^*_F S^1_{\|\alpha\|}$ that coincides with the momentum variable. This means that the subalgebra of functions on Ω_F which are constant along these leaves is a maximal abelian subalgebra of $C^\infty(\Omega_F)$. We choose those functions which clearly depend on the position variable but constant along the momentum, hence reducing $L^2(\Omega_F)$ into $L^2\left(S^1_{\|\alpha\|}\right)$. Furthermore, $L^2\left(S^1_{\|\alpha\|}\right)$ is isomorphic to $L^2(S^1)$ given by the map $f \mapsto f|_{S^1}$, where $(s_1, s_2) \in S^1_{\|\alpha\|}$ is identified with $\left(\frac{s_1}{\|\alpha\|}, \frac{s_2}{\|\alpha\|}\right) \in S^1$. So, $\hat{l}$ is a representation of $\mathfrak{m}(2)$ in $L^2(S^1)$. We are left to show that $\hat{l}$ is the differential of the unitary representation of $\mathrm{M}(2)$ defined in (2).

Set $\|\alpha\| = a$ and $s = \theta$ in (29). But this is exactly (8). To show uniqueness, we apply the differential operator $\hat{l}_U$ to expression (6) for the case $c_1 \neq 0$, so

$$\begin{aligned}\hat{l}_U(\mathcal{U}^a_{\exp tU}f)(\theta) &= \mathrm{i}a(c_2\cos\theta + c_3\sin\theta)(\mathcal{U}^a_{\exp tU}f)(\theta)\\ &\quad + c_1\frac{\partial}{\partial\theta}(\mathcal{U}^a_{\exp tU}f)(\theta).\end{aligned} \tag{30}$$

The second term in (30) is computed as

$$\begin{aligned}c_1\frac{\partial}{\partial\theta}(\mathcal{U}^a_{\exp tU}f)(\theta) &= \mathrm{e}^{\mathrm{i}a[\frac{c_2}{c_1}(\sin(tc_1+\theta)-\sin\theta) - \frac{c_3}{c_1}(\cos(tc_1+\theta)-\cos\theta)]}\\ &\quad \big(\mathrm{i}a[c_2(\cos(tc_1+\theta)-\cos\theta) + c_3(\sin(tc_1+\theta)\\ &\quad -\sin\theta)]f(tc_1+\theta) + c_1 f'(tc_1+\theta)\big).\end{aligned} \tag{31}$$

When (31) replaces the second term in (30)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{U}^a_{\exp tU} f)(\theta) = \hat{l}_U(\mathcal{U}^a_{\exp tU} f)(\theta)$$

where the left-hand side is the derivative of $\mathcal{U}^a$ expressed in (7).

For the case $c_1 = 0$, expression (7) and the application of the operator $\hat{l}_U$ to expression (6) are equal, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{U}^a_{\exp tU} f)(\theta) = \mathrm{i}a(c_2 \cos\theta + c_3 \sin\theta)(\mathcal{U}^a_{\exp tU} f)(\theta) = \hat{l}_U(\mathcal{U}^a_{\exp tU} f)(\theta).$$

Both cases have shown that the derivative with respect to t and the application $\hat{l}_U$ to $(\mathcal{U}^a_{\exp tU} f)(\theta)$ are equal, for all $f \in L^2(S^1)$. Moreover, $(\mathcal{U}^a_{\exp tU} f)(\theta) = f(\theta)$ when $t = 0$. Hence, $(\mathcal{U}^a_{\exp tU} f)(\theta)$ is the unique solution to the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t,\theta) = \hat{l}_U S(t,\theta), \qquad S(0,\theta) = \mathrm{Id}\,.$$

This means that $\exp(\hat{l}_U) f(\theta) = (\mathcal{U}^a_{\exp U} f)(\theta)$.

5.6. Representations Associated to the Trivial Orbits

When $F = (\mu, 0)$, the coadjoint orbit of $\mathrm{M}(2)$ is the zero-dimensional

$$\Omega_F = \{(\mu, 0)\}$$

which is a point. The set of C^∞-functions on this orbit can be described as

$$C^\infty(\Omega_F) = \{f : \Omega_F \to \mathbb{C};\ f(\mu, 0) = z\} \simeq \mathbb{C}.$$

The Hamiltonian function $\tilde{U} : \Omega_F \to \mathbb{R}$ is the constant function $\tilde{U}(F) = c_1\mu$. Obviously, the vector field ξ_U associated to this function is the zero vector field. The Kirillov form is computed as $\langle F, [U,T]\rangle = 0$ for any $U, T \in \mathfrak{m}(2)$.

The Moyal $\star$-product on the space $C^\infty(\Omega_F)$ is

$$f \star g = fg$$

for any functions $f, g \in C^\infty(\Omega_F)$. Hence, this $\star$-product is trivially covariant satisfying

$$\mathrm{i}\tilde{U} \star \mathrm{i}\tilde{T} - \mathrm{i}\tilde{T} \star \mathrm{i}\tilde{U} = \mathrm{i}\widetilde{[U,T]} = 0$$

for any $U, T \in \mathfrak{m}(2)$. So, there exists a one-dimensional representation l of $\mathfrak{m}(2)$ on $C^\infty(\Omega_F))[[\lambda]]$ defined by

$$(l_U)(f) = \mathrm{i}\tilde{U} \star f = (\mathrm{i}c_1\mu)f. \tag{32}$$

The operator $l_U = 0$ when $U \in \operatorname{span}\{E_1, E_2\}$.

The one-parameter subgroup $U = tX, t \in \mathbb{R}$ is identified with $\mathfrak{so}(2) \simeq \mathbb{R}$. So, the unitary operator $\chi_\mu(\exp tX) = \mathrm{e}^{\mathrm{i}t\mu}$ is the unique solution to the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t,x) = l_X S(t,x), \qquad S(0,x) = \mathrm{Id}$$

satisfying $\exp(tl_X) = \chi_\mu(\exp tX)$.

Since the set

$$\{\chi_n \circ p;\, n \in \mathbb{Z}\}$$

are the one-dimensional UIRs of $\mathrm{M}(2)$, the set of orbits

$$\{\Omega_F = \{(\mu, 0)\};\, \mu \in \mathbb{Z}\}$$

corresponds with these one-dimensional UIRs and the rest of the non-integer orbits correspond with

$$\{\chi_\mu \circ p;\, \mu \in \mathbb{R}/\mathbb{Z}\}.$$

6. Conclusion

This article has aimed to introduce deformation quantization as a powerful tool in constructing and classifying Lie group representations. The covariance property of the Moyal $\star$-product and its convergence in the Schwartz space are the key properties that made these constructions and classifications possible. This is an alternative to the traditional method of induced representation where its main weakness is the construction of a representation of a group induced from an unknown representation of its subgroup. For example, the construction of unitary representations of $\mathrm{M}(n), n > 2$ will be induced from the representations of $\mathrm{SO}(n-1) \ltimes \mathbb{R}^n$, determined in [7, Theorem 7.7].

The main result of this work is the construction of the complete set of unitary representations of the solvable $\mathrm{M}(2)$. The exponentiation of the representations defined by of the operators in (29) and in (32) describes the complete set of representatives of UIRs of $\mathrm{M}(2)$ in (4). We have tested Arnal and Cortet's program in [1–3], despite the original design for nilpotent and exponential Lie groups. The orbits $\{T^*S^1_a;\, a > 0\}$ correspond uniquely with the set $\{\mathcal{U}^a;\, a > 0\}$ of infinite-dimensional UIRs of $\mathrm{M}(2)$. This one-to-one correspondence is a similar result to what Kirillov observed with the nilpotent Lie groups. However, for the trivial orbits, only the integer-valued points correspond with the one-dimensional UIRs of $\mathrm{M}(2)$.

Though the computations in [9, 10, 17, 26] has provided a better understanding of the implementation of the program, this paper implemented it on a cylinder, different from the computations presented in [4], and on trivial orbits which was neglected in [9]. While the program has been effectively implemented on a flat orbit generated by the coadjoint action of a solvable Lie group, we wish to extend the implementation of this program in other low dimensional Lie groups, such as the Heisenberg and the Jacobi groups; the space-time groups $\mathrm{M}(3)$, the Galilean and Poincáre groups, and the generalization on $\mathrm{M}(n), n > 2$.

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