Functiones et Approximatio
48.1 (2013), 123–131
doi: 10.7169/facm/2013.48.1.10

A REMARK ON THE GOLDBACH-VINOGRADOV THEOREM

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Abstract: Let N denote a sufficiently large odd integer. In this paper it is proved that N can be represented as the sum of three primes, one of which is $\leqslant N^{\frac{11}{400}+\varepsilon}$ for any $\varepsilon > 0$. This result constitutes an improvement upon that of K. C. Wong, who obtained the exponent $\frac{7}{216}$.
Keywords: prime, sieve, mean value theorem.

1. Introduction

In 1937 Vinogradov [9] proved that any sufficiently large odd integer can be represented as the sum of three primes, and this result was named the Goldbach–Vinogradov theorem. Afterwards, some authors engaged in the refinement of it. One result in this aspect is due to Pan [8], in 1959 he showed that for any sufficiently large odd integer, the equation

$$N = p_1 + p_2 + p_3, \qquad p_1 \leqslant U \tag{1.1}$$

is solvable in primes p_1, p_2, p_3, where $U = N^{\frac{2c}{2c+1}+\varepsilon}$ and c is determined by $\zeta(\frac{1}{2} + it) \ll (|t| + 1)^c$. The classical result $c = \frac{1}{6}$ then provides us with $U = N^{\frac{1}{4}+\varepsilon}$.

In 1995 Zhan [11] improved Pan's result by showing that the equation (1.1) is solvable in primes p_1, p_2, p_3 with $U = N^{\frac{7}{120}+\varepsilon}$. To explain the method in [11] let us put $y = N^{\theta_1}$, $U = y^{\theta_2}$ and $\mathcal{I} = (0.5U, U]$, $\mathcal{J} = (0.5y, y]$, $\mathcal{K} = (N - y, N]$. Then the arguments in [11] shows that a sieving process for almost all short intervals of the form $(x, x + x^{\theta_2}]$ enable us to prove that

$$\sum_{\substack{p_1+p_2+p_3=N \\ p_1\in\mathcal{I},p_2\in\mathcal{J},p_3\in\mathcal{K}}} 1 \gg \frac{yU}{\log^3 N} \tag{1.2}$$

with $U = N^{\frac{7}{12}\theta_2}$, and Zhan's exponent $\frac{7}{120}$ follows from Harman's sieving process with $\theta_2 = \frac{1}{10} + \varepsilon$ in [4].

Project supported by The National Natural Science Foundation of China (grant no.11071186)
2010 Mathematics Subject Classification: primary: 11N36; secondary: 11P32

In 1996, Wong [10] introduced a sieving process with exponent $\theta_2 = \frac{1}{18} + \varepsilon$ and he got $U = N^{\frac{7}{216}+\varepsilon}$ in (1.1) upon replacing Harman's sieving process in the arguments in [11] by this one.

In 1996, Jia [6] constructed a sieving process with the exponent $\theta_2 = \frac{1}{20} + \varepsilon$, and we can get $U = N^{\frac{7}{240}+\varepsilon}$ in (1.1) immediately by applying this sieving process in the arguments in [11] instead of that of Harman's.

The arguments in [11] for (1.2) „sieved $\mathcal{J}$ but not $\mathcal{K}$". In [1] Baker, Harman and Pintz developed a vector sieve to show that for almost all even integers $2n$ in the short interval $(x, x + x^{2\theta_1\theta_2}]$ we have

$$\sum_{\substack{p_1+p_2=2n \\ p_1\in\mathcal{J}',p_2\in\mathcal{K}'}} 1 > 0, \tag{1.3}$$

where $\mathcal{J}' = (x^{\theta_1}, 2x^{\theta_1}], \mathcal{K}' = (x - 2x^{\theta_1}, x], \theta_1 = \frac{11}{20} + \varepsilon, \theta_2 = \frac{1}{16} + \varepsilon$. In their arguments they sieved both $\mathcal{J}'$ and $\mathcal{K}'$.

In this paper, we shall show that the sieve process in [1] can be used to investigate the equation (1.1) and obtain the following sharper result

Theorem. *The equation (1.1) is solvable in primes* p_1, p_2, p_3 *with* $U = N^{\frac{11}{400}+\varepsilon}$.

For comparison, we have

$$\frac{7}{120} = 0.058333\cdots; \qquad \frac{7}{216} = 0.032407\cdots;$$
$$\frac{7}{240} = 0.029166\cdots; \qquad \frac{11}{400} = 0.0275.$$

2. Some preliminary lemmas

In this paper, N always denotes a sufficiently large odd integer. Let $\varepsilon \in (0, 10^{-10})$ and A denote a sufficiently large constant. The constants in O-term and $\ll$-symbol depend at most on ε and A. The letter p, with or without subscript, is reserved for a prime number. As usual, $\varphi(n)$ denotes the Euler's function, and $\mu(n)$ denotes the Mőbius function. By $\rho(n)$ we denote the characteristic function of the set of prime numbers. We denote by $\pi(x)$ the number of primes up to x. We use $e(\alpha)$ to denote $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. We denote by $\sum_{x(q)*}$ a sum with x running over a reduced system of residues modulo q. Let $\mathbb{N}$ denote the set of positive integers. Put

$$y = N^{\frac{11}{20}+\varepsilon}, \qquad U = y^{\frac{1}{20}+\varepsilon}, \qquad \tau = U\log^{-8A} N, \qquad Q = \log^{8A} N,$$
$$\mathcal{I} = (0.5U, U] \cap \mathbb{N}, \qquad \mathcal{J} = (0.5y, y] \cap \mathbb{N}, \qquad \mathcal{K} = (N - y, N] \cap \mathbb{N},$$
$$f(\alpha) = \sum_{p\in\mathcal{I}} e(\alpha p), \qquad S(\alpha, x) = \sum_{\frac{x}{2}<n\leqslant x} \frac{e(\alpha n)}{\log n}, \qquad T(\alpha, x) = \sum_{\frac{x}{2}<n\leqslant x} \frac{e(\alpha n)}{\log x},$$
$$C_q(n) = \sum_{a(q)*} e_q(an), \qquad \mathfrak{S}(N) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi^3(q)} C_q(-N).$$

Lemma 1. *There exist sequences $a_j(n)(j=0,1)$ such that*

(i) $a_j(n)=0$ *unless* $p|n \Rightarrow p>Q$;
(ii) $a_0(n) \leqslant \rho(n) \leqslant a_1(n)$ *for* $n \in \mathcal{J}$;
(iii) $\sum_{n\in\mathcal{J}} |a_j(n)|^2 \ll y\log^{2A} N$;
(iv) $\int_{\frac{y}{2}}^{y} \left|\sum_{\substack{t<n\leqslant t+\tau \\ n\equiv l(\bmod q)}} a_j(n) - \frac{u_j\tau}{\varphi(q)\log y}\right|^2 dt \ll \frac{\tau^2 y}{\log^{40A} N}$, $(l,q)=1$, $q\leqslant Q$;
(v) $0.01456 < u_0 < 1 < u_1 < 2.70918$.

Proof. Let X be a sufficiently large real number and for $x\in(X,2X)$ set

$$\mathcal{A}^{(x)} = \{n|x<n\leqslant x+\eta x\}, \qquad \eta = \frac{1}{2}X^{-\frac{19}{20}+\varepsilon}, \qquad \delta=\varepsilon^3,$$
$$P(z)=\prod_{p<z} p, \qquad S(\mathcal{A}^{(x)},z) = \sum_{\substack{n\in\mathcal{A}^{(x)} \\ (n,P(z))=1}} 1, \qquad \mathcal{A}_d^{(x)} = \{a|a\in\mathcal{A}_d^{(x)}\}.$$

Then we have

$$\pi(x+\eta x)-\pi(x) = \sum_{x<n\leqslant x+\eta x} \rho(n) = S(\mathcal{A}^{(x)},(2X)^{\frac{1}{2}}). \tag{2.1}$$

By Buchstab's identity we obtain

$$\begin{aligned} S(\mathcal{A}^{(x)},(2X)^{\frac{1}{2}}) &\geqslant S(\mathcal{A}^{(x)},X^{\frac{8}{95}}) - \sum_{X^{\frac{8}{95}}<p\leqslant(2X)^{\frac{1}{2}}} S(\mathcal{A}_p^{(x)},X^{\frac{8}{95}}) \\ &\quad + \sum_{j=1}^{94} \sum_{(p_1,p_2)\in D_j} S(\mathcal{A}_{p_1p_2}^{(x)},p_1), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} S(\mathcal{A}^{(x)},(2X)^{\frac{1}{2}}) &\leqslant S(\mathcal{A}^{(x)},X^{\frac{8}{95}}) - \sum_{X^{\frac{8}{95}}<p\leqslant X^{\frac{1}{4}}} S(\mathcal{A}_p^{(x)},X^{\frac{8}{95}}) \\ &\quad + \sum_{X^{\frac{8}{95}}<p_1<p_2\leqslant X^{\frac{1}{4}}} S(\mathcal{A}_{p_1p_2}^{(x)},p_1), \end{aligned} \tag{2.3}$$

where and below $D_j(1\leqslant j\leqslant 94)$, defined in [6], are disjoint sub-domains of the domain $\{X^{\frac{8}{95}}<p_1<p_2\leqslant(2X)^{\frac{1}{2}}\}$.

By the arguments in [6], except for a subset of $(X,2X)$ the measure of which is $O(X\log^{-40A}X)$, we have,

$$S(\mathcal{A}^{(x)},(2X)^{\frac{1}{2}}) > 0.01456\frac{\eta x}{\log x}. \tag{2.4}$$

From (23) and Lemma 20 in [6] we get

$$\begin{aligned} S(\mathcal{A}^{(x)},X^{\frac{8}{95}}) &= \frac{85}{9}w\left(\frac{85}{9}\right)\frac{\eta x}{\log x} + O\left(\frac{\delta\eta x}{\log x}\right) \\ &< 5.30495\frac{\eta x}{\log x}, \end{aligned} \tag{2.5}$$

where and below $w(x)$ denotes the Buchstab's function.

By the arguments in the proof of Lemma 23 in [6] we obtain

$$\sum_{X^{\frac{8}{95}}<p\leqslant X^{\frac{1}{4}}} S(\mathcal{A}_p^{(x)}, X^{\frac{8}{95}}) = \frac{85}{9}\frac{\eta x}{\log x}\int_{\frac{9}{85}}^{\frac{1}{4}} \frac{1}{t} w\left(\frac{85}{9}(1-t)\right) dt + O\left(\frac{\delta\eta x}{\log x}\right)$$
$$> 4.55359\frac{\eta x}{\log x}. \tag{2.6}$$

The methods used from (29) to (32) in [6] provide us with

$$\sum_{X^{\frac{8}{95}}<p_1<p_2\leqslant X^{\frac{1}{4}}} S(\mathcal{A}_{p_1p_2}^{(x)}, p_1)$$
$$\leqslant \frac{85}{9}\frac{\eta x}{\log x}\int_{\frac{9}{85}}^{\frac{1}{4}} \frac{dt}{t}\int_{\frac{9}{85}}^{t} \frac{1}{u} w\left(\frac{85}{9}(1-t-u)\right) du + O\left(\frac{\delta\eta x}{\log x}\right)$$
$$< 1.95782\frac{\eta x}{\log x}. \tag{2.7}$$

From (2.3) and (2.5)–(2.7) we get

$$S(\mathcal{A}^{(x)}, (2X)^{\frac{1}{2}}) < 2.70918\frac{\eta x}{\log x}. \tag{2.8}$$

Let

$$\rho(n,z) = \begin{cases} 1, & \text{if } n\in\mathbb{N},\ p|n \Rightarrow p\geqslant z;\\ 0, & \text{otherwise .}\end{cases}$$

Then we have the Buchstab's identity

$$\rho(n,z) = \rho(n,w) - \sum_{w\leqslant p<z} \rho\left(\frac{n}{p}, p\right), \qquad 2\leqslant w<z. \tag{2.9}$$

For $n\in\mathcal{J}$, by (2.9) we have

$$\rho(n) \geqslant \rho(n, y^{\frac{9}{85}}) - \sum_{y^{\frac{9}{85}}\leqslant p<(2y)^{\frac{1}{2}}} \rho\left(\frac{n}{p}, y^{\frac{9}{85}}\right) + \sum_{j=1}^{94}\sum_{(p_1,p_2)\in D_j} \rho\left(\frac{n}{p_1p_2}, p_1\right)$$
$$= a_0(n) \tag{2.10}$$

and

$$\rho(n) \leqslant \rho(n, y^{\frac{9}{85}}) - \sum_{y^{\frac{9}{85}}\leqslant p<y^{\frac{1}{4}}} \rho\left(\frac{n}{p}, y^{\frac{9}{85}}\right) + \sum_{y^{\frac{8}{95}}<p_1<p_2\leqslant y^{\frac{1}{4}}} \rho\left(\frac{n}{p_1p_2}, p_1\right)$$
$$= a_1(n), \tag{2.11}$$

which correspond to (2.2) and (2.3) respectively. Then it is easy to see that $a_j(n)$ satisfy the properties (i)-(iii).

By the arguments in [6] we know that

$$\sum_{t<n\leqslant t+\tau} a_0(n)-\frac{u_0\tau}{\log y} \tag{2.12}$$

are actually the error terms in the sieve estimations in [6] and it was proved there that

$$\int_{\frac{y}{2}}^{y}\left|\sum_{t<n\leqslant t+\tau} a_0(n)-\frac{u_0\tau}{\log y}\right|^2 dt \ll \tau^2 y\log^{-40A} N. \tag{2.13}$$

By essentially the same method we can show that the inequality (iv) holds for $j=0$. By the same reason we have (iv) for $j=1$ also. At last, property (v) follows from (2.1)-(2.4) and (2.8) and the definitions of $a_j(n)$. ■

Lemma 2. *There exist sequences $b_j(n)(j=0,1)$ such that*

(i) $b_j(n)=0$ *unless* $p|n \Rightarrow p>Q$;
(ii) $b_0(n)\leqslant \rho(n)\leqslant b_1(n)$ *for* $n\in\mathcal{K}$;
(iii) $\sum_{n\in\mathcal{K}}|b_j(n)|^2 \ll y\log^{2A} N$;
(iv) $\left|\sum_{\substack{n\in\mathcal{K}\\ n\equiv l(\bmod q)}} b_j(n)-\frac{v_j y}{\varphi(q)\log N}\right| \ll \frac{y}{\log^{40A} N},\ (l,q)=1,\ q\leqslant Q$;
(v) $0.9953<v_0<1<v_1<1.0001.$

Proof. In the case $Q=1$, the required sequences are constructed in [1] which satisfy the properties (i)-(iii) and (v). It can be showed that they satisfy property (iv) by essentially the same methods as those used in [1], see also [5, 7]. ■

For $(a,q)=1, 1\leqslant a\leqslant q$ let

$$\mathfrak{M}(q,a)=\left(\frac{a}{q}-\frac{1}{\tau},\frac{a}{q}+\frac{1}{\tau}\right],$$
$$\mathfrak{M}=\bigcup_{1\leqslant q\leqslant Q}\ \bigcup_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}}\mathfrak{M}(q,a),$$
$$\mathfrak{m}=\left(-\frac{1}{\tau},1-\frac{1}{\tau}\right]\Big\backslash\mathfrak{M}.$$

Then we have the Farey dissection

$$\left(-\frac{1}{\tau},1-\frac{1}{\tau}\right]=\mathfrak{M}\cup\mathfrak{m}. \tag{2.14}$$

Lemma 3 ([2]). *We have*

(i) $f(\alpha)\ll U\log^{-3A} N,\quad \alpha\in\mathfrak{m}$,
(ii) $f(\alpha)=\frac{\mu(q)}{\varphi(q)}S(\lambda,U)+O(U\exp(-\log^{\frac{1}{3}} N)),\ \alpha=\frac{a}{q}+\lambda\in\mathfrak{M}(q,a).$

3. Proof of Theorem

For the proof of the theorem let us consider the sum

$$S(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_1\in\mathcal{I}, p_2\in\mathcal{J}, p_3\in\mathcal{K}}} 1 = \sum_{\substack{p+n_1+n_2=N \\ p\in\mathcal{I}, n_1\in\mathcal{J}, n_2\in\mathcal{K}}} \rho(n_1)\rho(n_2).$$

Let the sequences $a_j(n), b_j(n) (j = 0, 1)$ be those provided by Lemma 1 and Lemma 2 respectively and

$$g_j(\alpha) = \sum_{n\in\mathcal{J}} a_j(n)e(\alpha n), \qquad h_k(\alpha) = \sum_{n\in\mathcal{K}} b_k(n)e(\alpha n).$$

By the inequality

$$\rho(n_1)\rho(n_2) \geqslant a_0(n_1)b_1(n_2) + a_1(n_1)b_0(n_2) - a_1(n_1)b_1(n_2)$$

for which see Lemma 10.1 in [5], we have

$$S(N) \geqslant S_{0,1}(N) + S_{1,0}(N) - S_{1,1}(N), \tag{3.1}$$

where

$$S_{j,k}(N) = \sum_{\substack{p+n_1+n_2=N \\ p\in\mathcal{I}, n_1\in\mathcal{J}, n_2\in\mathcal{K}}} a_j(n_1)b_k(n_2). \tag{3.2}$$

By the Farey dissection (2.14) we have

$$S_{j,k}(N) = \int_{-\frac{1}{\tau}}^{1-\frac{1}{\tau}} f(\alpha)g_j(\alpha)h_k(\alpha)e(-\alpha N)d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}. \tag{3.3}$$

It follows from Lemma 3(i) and Cauchy's inequality that

$$\begin{aligned} \int_{\mathfrak{m}} &\ll \frac{U}{\log^{3A} N} \left(\int_0^1 |g_j(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \left(\int_0^1 |h_k(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\ &\ll \frac{U}{\log^{3A} N} \left(\sum_{n\in\mathcal{J}} |a_j(n)|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathcal{K}} |b_k(n)|^2\right)^{\frac{1}{2}} \\ &\ll \frac{Uy}{\log^A N}, \end{aligned} \tag{3.4}$$

where the bounds in Lemma 1(iii) and Lemma 2(iii) are used.

By Lemma 3(ii) we obtain

$$\begin{aligned}
\int_{\mathfrak{M}(q,a)} &= \frac{\mu(q)}{\varphi(q)} \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} S(\lambda, U) g_j\left(\frac{a}{q}+\lambda\right) h_k\left(\frac{a}{q}+\lambda\right) e\left(-\left(\frac{a}{q}+\lambda\right)N\right) d\lambda \\
&\quad + O\left(U \exp(-\log^{\frac{1}{3}} N) \int_0^1 |g_j(\alpha)||h_k(\alpha)| d\alpha\right) \\
&= \frac{\mu^2(q) u_j}{\varphi^2(q)} \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} S(\lambda, U) T(\lambda, y) h_k\left(\frac{a}{q}+\lambda\right) e\left(-\left(\frac{a}{q}+\lambda\right)N\right) d\lambda \\
&\quad + O\left(\frac{1}{\varphi(q)} \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} |S(\lambda, U)| \left| g_j\left(\frac{a}{q}+\lambda\right) - \frac{\mu(q) u_j}{\varphi(q)} T(\lambda, y)\right| \right. \\
&\quad \times \left. \left| h_k\left(\frac{a}{q}+\lambda\right)\right| d\lambda\right) + O\left(\frac{Uy}{\log^{20A} N}\right) \\
&= I_1(q,a) + O(I_2(q,a)) + O\left(\frac{Uy}{\log^{20A} N}\right), \qquad (3.5)
\end{aligned}$$

where the arguments which lead to (3.4) are applied.

From the trivial bound $S(\lambda, U) \ll U$, Cauchy's inequality and Gallagher's inequality in [3] we obtain

$$\begin{aligned}
I_2(q,a) &\ll \frac{U}{\varphi(q)} \left(\int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} \left| g_j\left(\frac{a}{q}+\lambda\right) - \frac{\mu(q) u_j}{\varphi(q)} T(\lambda, y)\right|^2 d\lambda\right)^{\frac{1}{2}} \left(\int_0^1 |h_k(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\
&\ll \frac{U y^{\frac{1}{2}} \log^A N}{\varphi(q)\tau} \left(\int_{\frac{y}{2}}^{y} \left| \sum_{t<n\leqslant t+\tau} a_j(n) e_q(an) - \frac{\mu(q) u_j \tau}{\varphi(q) \log y}\right|^2 dt\right)^{\frac{1}{2}} \\
&\ll \frac{U y^{\frac{1}{2}} \log^A N}{\tau} \left(\max_{(l,q)=1} \int_{\frac{y}{2}}^{y} \left| \sum_{\substack{t<n\leqslant t+\tau \\ n\equiv l (\bmod q)}} a_j(n) - \frac{u_j \tau}{\varphi(q) \log y}\right|^2 dt\right)^{\frac{1}{2}} \\
&\ll \frac{Uy}{\log^{18A} N}. \qquad (3.6)
\end{aligned}$$

Now

$$\begin{aligned}
I_1(q,a) &= \frac{\mu^2(q) u_j}{\varphi^2(q)} \int_{-\frac{1}{2}}^{\frac{1}{2}} S(\lambda, U) T(\lambda, y) h_k\left(\frac{a}{q}+\lambda\right) e\left(-\left(\frac{a}{q}+\lambda\right)N\right) d\lambda \\
&\quad + O\left(\frac{1}{\varphi^2(q)} \int_{\frac{1}{\tau}}^{\frac{1}{2}} \left| S(\lambda, U) T(\lambda, y) h_k\left(\frac{a}{q}+\lambda\right)\right| d\lambda\right) \\
&= I_1^{(1)}(q,a) + O(I_1^{(2)}(q,a)). \qquad (3.7)
\end{aligned}$$

By the trivial bound $S(\lambda, U) \ll \lambda^{-1}$ we get

$$\begin{aligned} I_1^{(2)}(q,a) &\ll \frac{\tau}{\varphi^2(q)} \left(\int_0^1 |T(\lambda,y)|^2 d\lambda \right)^{\frac{1}{2}} \left(\int_0^1 |h_k(\lambda)|^2 d\lambda \right)^{\frac{1}{2}} \\ &\ll \frac{Uy}{\varphi^2(q)\log^{6A} N}. \end{aligned} \tag{3.8}$$

It is easy to see that

$$\begin{aligned} I_1^{(1)}(q,a) &= \frac{\mu^2(q)u_j}{\varphi^2(q)} \sum_{\substack{n_1+n_2+n_3=N \\ n_1\in\mathcal{I}, n_2\in\mathcal{J}, n_3\in\mathcal{K}}} \frac{b_k(n_3)}{\log n_1 \log y} e_q(a(n_3-N)) \\ &= \frac{\mu^2(q)u_j}{\varphi^2(q)} \sum_{n_3\in\mathcal{K}} b_k(n_3) e_q(a(n_3-N)) \sum_{\substack{n_1+n_2=N-n_3 \\ n_1\in\mathcal{I}, n_2\in\mathcal{J}}} \frac{1}{\log n_1 \log y} \\ &= \left(1+O\left(\frac{1}{\log N}\right)\right) \frac{\mu^2(q)u_j U}{\varphi^2(q)\log U \log y} \sum_{n\in\mathcal{K}} b_k(n) e_q(a(n-N)) \\ &= \left(1+O\left(\frac{1}{\log N}\right)\right) \frac{\mu^2(q)u_j U}{\varphi^2(q)\log U \log y} \sum_{l(q)*} e_q(a(l-N)) \sum_{\substack{n\in\mathcal{K} \\ n\equiv l(\bmod q)}} b_k(n) \\ &= \left(1+O\left(\frac{1}{\log N}\right)\right) \frac{\mu(q)u_j v_k U y}{\varphi^3(q)\log U \log y \log N} e_q(-aN) \\ &\quad + O\left(\frac{Uy}{\log^{20A} N}\right), \end{aligned} \tag{3.9}$$

where Lemma 2(iv) is used.

From (3.5)-(3.9) we have

$$\begin{aligned} \int_{\mathfrak{M}} &= \sum_{1\leqslant q\leqslant Q} \sum_{\substack{1\leqslant a\leqslant q \\ (a,q)=1}} \int_{\mathfrak{M}(q,a)} \\ &= \left(1+O\left(\frac{1}{\log N}\right)\right) \frac{u_j v_k U y}{\log U \log y \log N} \sum_{1\leqslant q\leqslant Q} \frac{\mu(q)C_q(-N)}{\varphi^3(q)} + O\left(\frac{Uy}{\log^A N}\right) \\ &= \left(1+O\left(\frac{1}{\log N}\right)\right) \frac{u_j v_k U y \mathfrak{S}(N)}{\log U \log y \log N} + O\left(\frac{Uy}{Q}\right) + O\left(\frac{Uy}{\log^A N}\right) \\ &= \frac{u_j v_k U y \mathfrak{S}(N)}{\log U \log y \log N} + O\left(\frac{Uy}{\log^4 N}\right). \end{aligned} \tag{3.10}$$

It follows from (3.3)-(3.4) and (3.10) that

$$S_{j,k}(N) = \frac{u_j v_k U y \mathfrak{S}(N)}{\log U \log y \log N} + O\left(\frac{Uy}{\log^4 N}\right). \tag{3.11}$$

By (3.1), (3.11), Lemma 1(v) and Lemma 2(v), we have

$$\begin{aligned} S(N) &\geqslant S_{0,1}(N) + S_{1,0}(N) - S_{1,1}(N) \\ &= (u_0v_1 + u_1v_0 - u_1v_1)\frac{Uy\mathfrak{S}(N)}{\log U \log y \log N} + O\left(\frac{Uy}{\log^4 N}\right) \\ &> \frac{0.0015Uy\mathfrak{S}(N)}{\log U \log y \log N} \gg \frac{Uy\mathfrak{S}(N)}{\log U \log y \log N}, \end{aligned} \tag{3.12}$$

where the well-known fact $\mathfrak{S}(N) > \frac{1}{2}$ is used. Now by (3.12) the proof of the theorem is completed.

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Received: 27 December 2011